A fractional version of the Stochastic Alpha–Beta–Rho (SABR) model within a fast-varying stochastic environment (FVSE)

Abstract

This paper proposes a fractional extension of the SABR model that more accurately captures the irregular behavior of volatility, including its long-memory and rough characteristics. Leveraging empirical evidence that log-volatility exhibits fractional Brownian motion dynamics with a low Hurst exponent, and embedding the volatility within a fast-varying stochastic environment, the model is formulated via a fractional stochastic differential equation combining fractional dynamics with high-frequency stochastic drivers. This approach enhances the modeling of implied volatility surfaces (smile and skew) while preserving analytical tractability for practical tasks such as calibration and variance swap pricing. The proposed fractional SABR model thus offers a robust and realistic framework with promising applications in market calibration, volatility forecasting, and further theoretical or numerical development.

Keywords: *SABR model, Bessel process, stochastic volatility, fractional Brownian motion, volatility smile.*

1 Introduction

The study of financial asset volatility is a very important topic in the field of finance. This importance stems largely from the fact that it allows measuring the uncertainty regarding the evolution of the return of a given asset (such as a stock or a stock index). This is why investors wish to choose a level of risk "exposure" compatible with their risk tolerance. Consequently, they seek to hedge against adverse events that may affect financial markets. Thus, knowledge of the volatility of such contingent assets plays an essential role in the evaluation and hedging of any investment, especially when it is risky. Therefore, forecasting this volatility is indispensable to control risk (and when possible, reduce it). Hence, anticipating potential variations in price behavior in markets allows us to protect against possible losses.

The options market emerged with the advent of the famous Black-Scholes formula. In their model published in 1973, volatility was considered constant. This is not always the case, which is why many stochastic models were created to overcome the shortcomings of the Black-Scholes model. The Black-Scholes model does not always reflect market realities. The volatility surface is the graphical representation of volatility as a function of strike price and option maturity for the same underlying asset. It is very difficult to simulate due to the occurrence of sudden movements taking extreme values. This is why modeling asset prices that account for these volatility surfaces remains a concern.

The volatility smile represents volatility as a function of strike price for the same underlying asset. The term "smile" is used because the graph resembles a smile on a face (see figure 1(a) at the end of the introduction). To better explain market behavior, volatility smiles appeared in the market just after the 1987 stock market crash. They have become crucial for most options markets.

We have two categories of volatility models: local volatility models and stochastic volatility models. Local volatility models were extensively studied by Dupire [10] and Derman and Kani [9],[8], where they considered volatility as a deterministic function of the asset price and time. These local volatility models do not allow effective hedging of volatility risk since they do not introduce an additional risk factor for volatility. Empirical observation shows that for the same underlying asset, multiple volatilities can coexist. The importance of the volatility smile is that it allows measuring the level of volatility. To solve the problem of constant volatility in the Black-Scholes model, Hagan, Kumar, Lesniewski, and Woodward [15] developed the SABR model in 2002.

The SABR (Stochastic Alpha Beta Rho) model is a stochastic volatility model derived from the CEV (Constant Elasticity of Variance) model. It was proposed to address the limitations of the CEV model, which also considers volatility as constant. The SABR model, long favored by practitioners, has limitations since it uses standard Brownian motions to model the underlying dynamics and volatility process.

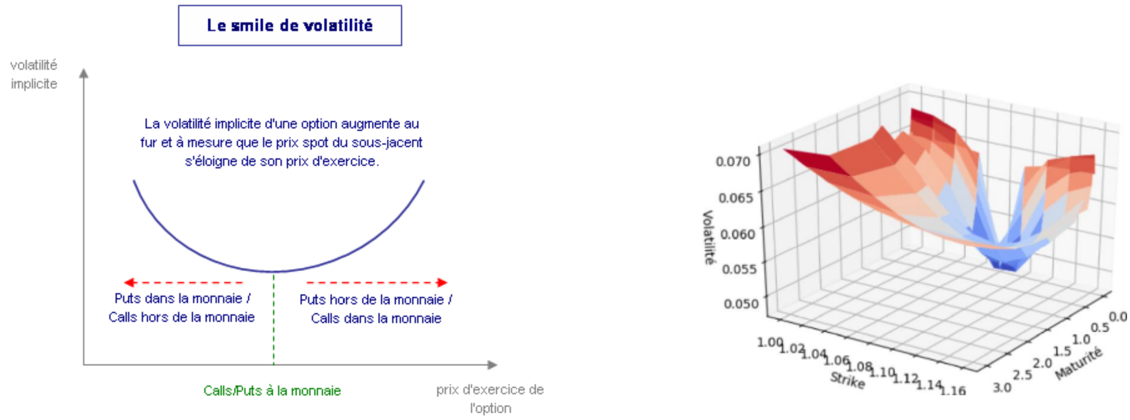
In the work of Huy N. and Miklos Rasonyi (2018) [5], the authors showed that the fractional volatility process has long or positive memory for $H > \frac{1}{2}$ and negative or short memory for $H < \frac{1}{2}$. Similarly, Jean-Pierre Fouque and Ruimeng Hu (2018) [11] demonstrated long-term dependence for $H \in (0, \frac{1}{2})$ in the fractional stochastic volatility process. Recently, an exact simulation of the SABR model was proposed by [4]. Zhenyu Cui et al. [7] and [2] proposed an efficient simulation of the generalized SABR model and stochastic local volatility models. In [12], the authors proposed pricing under high volatility conditions, and in [23] they used implied volatility to calibrate SABR parameters from market data.

Our objective is to propose a fractional SABR model that uses fractional Brownian motion to model the volatility process. This volatility process is a function of a fast-varying process driven by fractional Brownian motion. Fractional processes account for jumps in the underlying price. This may enable better option pricing and reproduce a more accurate volatility smile.

In the SABR model, the price of an underlying asset is governed by the following system of Stochastic Differential Equations (SDEs):

$$\begin{cases} dS_t = \sigma_t S_t^{\beta} dW_t \\ d\sigma_t = \alpha \sigma dZ_t \\ S(0) = S_0 e^{(rT)} \\ \sigma(0) = \sigma_0 \\ d\langle W_t, Z_t \rangle = \rho dt \end{cases} \tag{1}$$

Schroder [21] and Islah [16] proposed an approximation of the cumulative distribution function of the conditional SABR process. In 2017, Alvaro et al. [19] proposed an efficient multi-step simulation of the SABR model using the Stochastic Collocation Monte Carlo (SCMC) technique as in [13] to accelerate computation time. One limitation of the standard model is that volatility is governed by a stochastic differential equation driven by a standard Brownian motion, which is sensitive to extreme volatility movements. The SABR model is not more robust than other stochastic models (the Black-Scholes model, the CEV model (which is a special case of the SABR model), the Heston model, the Dupire model, etc.) in modeling extreme movements of volatility or the underlying, but has the advantage of having easily calibratable parameters (see [1] for more details). This leads us to propose a fractional version of the SABR model that allows for better estimation of option prices despite the highly irregular behavior of volatility. We used fractional Brownian motions in a fast-varying stochastic environment that accounts for jumps in the underlying price. The paper is organized as follows: Section 2 presents a fractional version of the SABR model. Section 3 is devoted to numerical simulation.



(a) Volatility smile (b) Volatility surface

Figure 1: Illustration of the volatility smile (1(a)) and the volatility surface for $\rho = 0.75$ (1(b))

2 Formulation of the Stochastic Alpha Beta Rho-Fractional Stochastic Volatility (SABR-FSV) Model

The SABR-FSV model we propose here is an extension of the standard SABR model described in the SDE system (1), defined as follows:

- **i)** β and ρ are respectively the elasticity and correlation. They remain unchanged as in equation (1);
- **ii)** The SDE for the underlying dynamics is still driven by a standard Brownian motion;
- **iii)** In this new model, we introduced a Fast-Varying Stochastic Environment (FVSE) driven by a fractional Brownian motion;
- **iv)** The volatility defined in this way is governed by the fast-varying stochastic environment to account for the irregular behavior of volatility observed in practice;
- **v)** The volatility is now driven by a fractional Brownian motion;
- **vi)** We introduced two parameters ε and θ where ε allows for fast mean reversion of the FVSE and θ is the correlation between the Brownian motion of the FVSE and that of the volatility.

The SABR-FSV model is defined by the following system of stochastic differential equations:

$$
\begin{cases}
dS_t = \sigma_t^H S_t^\beta \left(\rho_t dB_\sigma^H + \sqrt{1-\rho_t^2} dB_s\right) \qquad (A) \\
d\langle B_\sigma^H, B_s\rangle = \rho dt \text{ with } |\rho| < 1 \\
d\sigma_t^H = \alpha\sigma_t^H \left[\mu(X_t^{\epsilon,H})dt + \nu(X_t^{\epsilon,H}) \left(\theta dB_X^H + \sqrt{1-\theta^2} dB_\sigma^H\right)\right] \qquad (B) \\
dX_t^{\epsilon,H} = \frac{c}{\epsilon} X_t^{\epsilon,H} dt + \frac{1}{\epsilon^H} dB_X^{h_t} \qquad (C) \\
d\langle B_\sigma^H, B_X^H\rangle = \theta dt \text{ with } |\theta| < 1 \\
S(0) = S_0 \exp(rT) \\
\sigma(0) = \sigma_0
\end{cases}
\tag{2}
$$

S_t is the underlying asset price at time t; σ_t^H is the stochastic volatility process; B_s is a standard Brownian motion; B_σ^H and B_X^H are fractional Brownian motions (fBm); ρ is the correlation between B_σ^H and B_s such that $|\rho| \leq 1$; θ is the correlation between B_σ^H and B_X^H such that $|\theta| \leq 1$; β is the variance elasticity with $\beta \in [0, 1]$; α is the volatility of the volatility process σ_t^H with $\alpha > 0$; H is the Hurst parameter with $H \in (0, 1)$; $X_t^{\epsilon,H}$ is a multi-fractional process; ϵ is a parameter that allows for fast mean reversion of the multi-fractional process $X_t^{\epsilon,H}$; S_0 and σ_0 are the initial conditions; r and T are respectively the interest rate and maturity.
To guarantee the existence and uniqueness of (2) (B), we formulate the following hypothesis:
(H): The functions $\mu(\cdot)$ and $\nu(\cdot)$ are measurable, bounded and satisfy k-Lipschitz conditions.
Solving the dynamics of this model relies on the same techniques as for the SABR model found in the works of Islah [16], Schroder [21]. It consists in establishing a link between the proposed SABR-FSV process and the squared Bessel process, and since the density of the latter is known, this allows us to develop an explicit formula.

2.1 Existence and Uniqueness

See [24] and [14] for all details regarding existence and uniqueness for an SDE driven by fractional Brownian motion. Consider the following stochastic differential equations, extracted from equation (**??**):

$$(E_1) : dX_t^{\epsilon,H} = \frac{c}{\epsilon} X_t^{\epsilon,H} dt + \frac{1}{\epsilon^H} dB_X^H$$

$$(E_2) : d\sigma_t^H = \alpha\sigma_t^H \left[\mu(X_t^{\epsilon,H})dt + \nu(X_t^{\epsilon,H}) \left(\theta dB_X^H + \sqrt{1-\theta^2} dB_\sigma^H\right)\right]$$

$$(E_3) : dS_t = \sigma_t^H S_t^\beta \left(\rho_t dB_\sigma^H + \sqrt{1-\rho_t^2} dB_s\right)$$

We state the following theorems that guarantee existence and uniqueness of solutions after defining some concepts.

Definition 2.1. *Let $B^H = (B_t^H)$ be a $\mathcal{F}_t$-fractional Brownian motion, Y_0 a random vector on $(\Omega, \mathcal{F}, \mathcal{F}_t)$ independent of B^H, and let b and σ be locally bounded measurable functions. A strong solution of the homogeneous SDE*

$$Y_t = Y_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s^H \tag{3}$$

or of the nonhomogeneous SDE

$$Y_t = Y_0 + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dB_s^H \tag{4}$$

is the triplet $(Y, B^H, \mathcal{F}_t)$ where

i) Y is $\mathcal{F}_t^{B^H}$-adapted, where $\mathcal{F}_t^{B^H}$ is the Brownian filtration of B^H;

ii) (Y, B^H) satisfies (3) or (4).

Definition 2.2. *Let b and σ be locally bounded measurable functions. A weak solution of the SDE (3) or (4) is a triplet $(Y, B^H, \mathcal{F}_t)$ on a space $(\Omega, \mathcal{F}, \mathbb{P})$ where*

i) B^H is a $\mathcal{F}_t$-fractional Brownian motion

ii) (Y, B) satisfies (3) or (4)

with $\mathcal{F}_t^{B^H, Y}$ the filtration generated by (Y, B^H).

Theorem 2.1. *(Itô's Existence and Uniqueness, homogeneous case)*
Assume the coefficients of (3) satisfy the following conditions: there exists $K < \infty$ such that, $\forall x, y \in \mathbb{R}^d$,
$|b_i(x) - b_i(y)| \leq K\|x - y\|$;
$|\sigma_{ij}(x) - \sigma_{ij}(y)| \leq K\|x - y\|$;
then the SDE

$$Y_t = Y_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s^H$$

admits a unique strong solution (see [22] for details).

Theorem 2.2. *(Itô's Existence and Uniqueness, nonhomogeneous case)*
Assume the coefficients of 4 satisfy the following conditions: for all $i \in \{1, \cdots, d\}, j \in \{1, \cdots, m\}, x, y \in \mathbb{R}^d$ and $0 < T < \infty$;
$\exists K_t > 0, \forall t \in [0, T]$,
$\max(|b_i(t, 0)|, |\sigma_{ij}(t, 0)|) \leq K_T$
$|b_i(t, x) - b_i(t, y)| \leq K_T\|x - y\|$
$\|\sigma_{ij}(t, x) - \sigma_{ij}(t, x)\| \leq K_T\|x - y\|$
then the SDE

$$Y_t = Y_0 + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dB_s^H$$

admits a unique solution (see [22] for details).

We used the above theorems to show existence and uniqueness of solutions.
For existence and uniqueness of SDE (E_1), we set

$$dX_t^H = \frac{c}{\varepsilon}\mu(X_t^H)dt + \sigma(X_t^H)dB_t^H, \text{ with } \mu(t, x) = x \text{ and } \sigma(t, x) = 1$$

The functions μ and σ satisfy the conditions of theorem (2.2) and according to definition (2.2), the SDE $dX_t^H = \frac{c}{\varepsilon}X_t^H dt + \frac{1}{\varepsilon^H}dB_t^H$ admits a unique solution (see [20] and [17] for details). For existence and uniqueness of SDE (E_2): $d\sigma_t^H = \alpha\sigma_t^H \left[\mu(X_t^{\epsilon,H})dt + \nu(X_t^{\epsilon,H})\left(\theta dB_X^H + \sqrt{1-\theta^2}dB_\sigma^H\right)\right]$, if we set $dZ_t^h = \theta dB_X^H + \sqrt{1-\theta^2}dB_\sigma^H$, we obtain $d\sigma_t^H = \alpha\sigma_t^H\left(\mu(X_t^H)dt + \nu(X_t^H)dZ_t^H\right)$, with Z_t^H a fractional Brownian motion. By hypothesis, the functions $\mu(\cdot)$ and $\nu(\cdot)$ being Lipschitz and locally bounded thus satisfy the conditions of theorem (2.3), then according to definition (2.2) SDE E_2 admits a unique solution.

For SDE (E_3), we will consider different cases depending on β.

* For $\beta = 0$, SDE (E_3) takes the form:

$$dS_t = \sigma_t^H\left(\rho dB_\sigma^H + \sqrt{1-\rho^2}dB_S\right) \qquad (E_3) \tag{5}$$

  $B_S(t)$ is a standard Brownian motion and $B_\sigma^H(t)$ is a fractional Brownian motion, and $\sigma_H(t)$ is the fractional volatility process.
  Setting $W_t^H = \rho dB_\sigma^H + \sqrt{1-\rho^2}dB_S$, SDE ($E_3$) takes the form: $dS_t = \sigma_t^H dW_t^H$ since $S_t^0 = 1$, and according to theorem (2.3), SDE (E_3) admits a unique exact solution of the form:

$$S_t = \rho\int_0^t \sigma^H(t)dB_\sigma^H + \sqrt{1-\rho^2}\int_0^t \sigma^H(t)dB_S \tag{6}$$

* For $\beta = 1$ SDE (E_3) takes the form:

$$dS_t = \sigma_t^H S_t\left(\rho dB_\sigma^H + \sqrt{1-\rho^2}dB_S\right) \qquad (E_3) \tag{7}$$

  Setting $dW_t^H = \rho dB_\sigma^H + \sqrt{1-\rho^2}dB_S$ with $|\rho| < 1$, we obtain $dS_t = \sigma_t^H S_t dW_t^H$ and according to theorem (2.3) SDE (E_3) admits a unique strong solution of the form

$$\begin{aligned} S(t) \simeq S(s)\exp\Bigg\{&-\frac{1}{2}\left(1+2\rho^2\sqrt{1-\rho^2}\right)\int_s^t \sigma_H^2(x)dx \\ &+\rho\int_s^t \sigma^H(x)dB^\sigma(x) + \sqrt{1-\rho^2}\int_s^t \sigma^H(x)dB^S(x)\Bigg\} \end{aligned}$$

* For $\beta \in (0,1)$ we use the Bessel equation and establish a link with (E_3), we have the relation: $dZ_t = dW_t + \dfrac{n-1}{2Z_t}dt$ where W_t is a Brownian motion.
  The drift of this expression is $b(x) = \dfrac{n-1}{2x}$ and we note that this function is neither Lipschitz nor locally bounded near 0, so we cannot use Itô's existence theorem to show solution uniqueness. To show existence and uniqueness of this solution, we define in general the Bessel process $(Y_t)_{t\geq 0}$ which is the unique solution of the SDE $dY_t = \dfrac{\alpha}{2Y_t}dt + dB_t$ with B_t a Brownian motion, $(Y_t)_{t\geq 0}$ the Bessel process, $\alpha = n-1$ the Bessel index starting at $Y_0 = y$. The strong uniqueness of the Bessel process thus guarantees uniqueness of SDE (E_3) when β takes values in $(0,1)$ (see [18] for details).

Lemma 2.1. *The solution of SDE* $dX_t^{\epsilon,H} = \frac{c}{\epsilon}X_t^{\epsilon,H}dt + \frac{1}{\epsilon^H}dB_X^H$ *is of the form*

$$X_t^{\epsilon,H} = \epsilon^{-H}\int_{-\infty}^t \exp\left(-\frac{c(t-s)}{\epsilon}\right)dB_X^H \tag{8}$$

Proof. We have $dX_t^{\epsilon,H} = \frac{c}{\epsilon}X_t^{\epsilon,H}dt + \frac{1}{\epsilon^H}dB_X^H$.
Let $f(t,x) = x\exp(-\frac{c}{\epsilon}t)$ be a continuous function, twice differentiable with respect to x as the product of two continuous differentiable functions. Applying Itô's theorem to the function f, we establish that:

$$X_t^{\epsilon,H} = \frac{1}{\epsilon^H}\int_{-\infty}^{t}\exp\left(\frac{c}{\epsilon}(t-s)\right)dB_X^H \quad \blacksquare$$

□

Lemma 2.2. *The solution of the SDE* $d\sigma_t^H = \alpha\sigma_t^H\left[\mu(X_t^{\epsilon,H})dt + \nu(X_t^{\epsilon,H})\left(\theta dB_X^H + \sqrt{1-\theta^2}dB_\sigma^H\right)\right]$ *is of the form:*

$$\begin{aligned}\sigma(t)^H \quad &\simeq \quad \left(\sigma(s)\exp\left\{\textstyle\int_0^t \alpha^2\left[\mu\left(X_t^{\epsilon,H}\right) - \frac{1}{2}\alpha\nu^2\left(X_t^{\epsilon,H}\right)\left(2\theta^2\sqrt{1-\theta^2}+1\right)\right]dt\right.\right.\\ &+ \quad \left.\left.\textstyle\int_0^t \alpha^2\nu\left(X_t^{\epsilon,H}\right)dB_\sigma^H\right\}\right)^{\frac{1}{\alpha}}\end{aligned} \tag{9}$$

Proof. Let $f(x) = \ln(x)$. Applying Itô's theorem to the function f allows us to establish the relation:

$$\begin{aligned}\sigma_t^H = &\left(\sigma_0\exp\left\{\int_0^t \alpha^2\left[\mu\left(X_t^{\epsilon,H}\right) - \frac{1}{2}\alpha\nu^2\left(X_t^{\epsilon,H}\right)\left(2\theta^2\sqrt{1-\theta^2}+1\right)\right]dt\right.\right.\\ &+ \left.\left.\int_0^t \alpha^2\nu\left(X_t^{\epsilon,H}\right)dB_\sigma^H\right\}\right)^{\frac{1}{\alpha}}\end{aligned} \tag{10}$$

■ □

Now we present the solution for the SABR-FSV dynamics (SDE (A)). The explicit solution formula varies depending on the elasticity. We state some important lemmas.

Definition 2.3. ***(Euclidean norm of Brownian motion)***
Let $n > 1$ and $B = (B_1, B_2, \cdots, B_n)$ be an n-dimensional Brownian motion. Let $X_t = \|B_t\|$ meaning $X_t^2 = \sum_{i=1}^n (B_i)^2(t)$. Generally, we can define a process $(W)_{t\geq 0}$ by

$$dW_t = \frac{1}{X_t}B_t \cdot dB_t = \frac{1}{\|B_t\|}\sum_{i=1}^n B_i(t)dB_i(t),\ W_0 = 0, \tag{11}$$

The process $(W)_{t\geq 0}$ is a continuous martingale as a sum of martingales, and the quadratic variation of W at t (the process $W_t^2 - t, t \geq 0$ is a martingale). Therefore $(W)_{t\geq 0}$ is a Brownian motion, and the equality $d(X_t^2) = 2B_t \cdot dB_t + ndt$ can be written as $d(X_t^2) = 2X_t dW_t + ndt$. Applying Itô's formula, we obtain $dX_t = dW_t + \frac{n-1}{2}\frac{dt}{X_t}$ where $(W_t)_{t\geq 0}$ is a Brownian motion. Setting $V_t = X_t^2$, we get
$dV_t = 2\sqrt{V_t}dW_t + ndt$.

The process $(X_t)_{t\geq 0}$ is a Bessel process (BES) of dimension n, and $(V_t)_{t\geq 0}$ is a squared Bessel process (BESQ) of dimension n.

The SABR model is a modification of the CEV (Constant Elasticity of Variance) process. In the CEV model, volatility is constant. It is described by:

$$dS_t = \sigma_t^H S_t^\beta dB_s \tag{12}$$

with $S(0) = S_0$ being $\mathcal{F}_0$-measurable.

Lemma 2.3. *Let* $W_t = \dfrac{S_t^{1-\beta}}{1-\beta}$ *(as in [21]). For* $\beta \neq 1$*, applying Itô's theorem to* W_t *establishes:*

$$dW_t = \sigma_t^H dB_t - \frac{\beta(\sigma_t^H)^2}{(2-2\beta)W_t}dt \tag{13}$$

Proof. Let $f(t,x) = \dfrac{x^{1-\beta}}{1-\beta}$ with $\beta \neq 1$, and $f \in C^{1,2}([0,+\infty] \times \mathbb{R})$. Setting $W_t = \dfrac{S_t^{1-\beta}}{1-\beta}$ (with $\beta \neq 1$) (an inverse transformation of S_t^{β}), we can write using Itô's formula: $dW_t = (1-\beta)\dfrac{S_t^{-\beta}}{1-\beta}\sigma_t^H S_t^{\beta} dB_t - \dfrac{1}{2}\beta(1-\beta)\dfrac{S_t^{-1-\beta}}{1-\beta}(\sigma_t^H)^2 S_t^{2\beta} dt$. After simplification we obtain $dW_t = \sigma_t^H dB_t - \dfrac{\beta(\sigma_t^H)^2}{2S_t^{1-\beta}}dt \quad (*)$. Since $W_t = \dfrac{S_t^{1-\beta}}{1-\beta} \Rightarrow S_t^{1-\beta} = (1-\beta)W_t$, substituting this expression into (*) gives:

$$dW_t = \sigma_t^H dB_t - \frac{\beta(\sigma_t^H)^2}{(2-2\beta)W_t}dt$$

■ □

Theorem 2.3. *The solution of the fractional SABR dynamics for* $\beta = 0$*,* $\beta = 1$*, and* $\beta \in (0,1)$ *is given by:*

i. For $\beta = 0$*:*

$$S(t) \simeq S(s) + \rho \int_s^t \sigma^H(x) dB^{\sigma}(x) + \sqrt{1-\rho^2} \int_s^t \sigma(x) dB^S(x) \tag{14}$$

ii. For $\beta = 1$*:*

$$\begin{aligned} S(t) \simeq S(s) \exp\Bigg\{ -\frac{1}{2}\left(1 + 2\rho^2\sqrt{1-\rho^2}\right) \int_s^t \sigma_H^2(x) dx \\ + \rho \int_s^t \sigma^H(x) dB^{\sigma}(x) + \sqrt{1-\rho^2} \int_s^t \sigma^H(x) dB^S(x) \Bigg\} \end{aligned} \tag{15}$$

(Conditional cumulative distribution for the fractional SABR process).
For $S_0 \gg 0$*, the conditional cumulative distribution of* S_t *on a generic interval* $[s,t]$ *given* σ_t^H*,* $\int_s^t \sigma_{u,H}^2 du$ *is:*

iii.(a) For $0 < \beta < 1/2$ *with absorption at* 0 *and* $\dfrac{1}{2} < \beta < 1$*:*

$$\mathbb{P}\left(S_t \leq K \mid S_0 > 0, \sigma_t^H, \sigma_s^H, \int_s^t \sigma_{u,H}^2 du \right) = 1 - F_{\chi^2(a,b,c)} \tag{16}$$

iii.(b) For $0 < \beta < 1/2$ *with reflection at* $V = 0$*:*

$$\mathbb{P}\left(S_t \leq K \mid S_0 > 0, \sigma_t^H, \sigma_s^H, \int_s^t \sigma_{u,H}^2 du \right) = 1 - F_{\chi^2(c,2-b,a)} \tag{17}$$

where:

$$
\begin{aligned}
a &= \frac{1}{z(t)}\left(\frac{S_0^{1-\beta}}{1-\beta}+\Delta_0\right)^2;\\
b &= 2-\frac{1-2\beta-\rho^2(1-\beta)}{(1-\beta)(1-\rho^2)};\\
c &= \frac{K^{2(1-\beta)}}{(1-\beta)^2 z(t)};\\
z(t) &= (1-\rho^2)\int_s^t \sigma_{u,H}^2 du
\end{aligned}
$$

$$
\Delta_0 = \frac{\rho}{\alpha}\frac{\sigma_t^H-\sigma_0^H-\alpha\int_s^t\sigma_u^H\mu(X_u)du-\alpha\theta\int_s^t\sigma_u^H\nu(X_u)dB_u^X}{\sqrt{1-\theta^2}}
$$

and $\chi^2(i,y,\lambda)$ *is the cumulative distribution function of the non-central chi-square distribution, with:*

$$
\chi^2(i,y,\lambda)=\frac{1}{2}\exp\left(-\frac{i+\lambda}{2}\right)\left(\frac{i}{\lambda}\right)^{\frac{y}{4}-\frac{1}{2}} I_{\frac{y}{2}-1}(\sqrt{\lambda i}) \tag{18}
$$

where $I_V(z)$ *is the modified Bessel function of the first kind given by:*

$$
I_a(y)=\left(\frac{y}{2}\right)^a\sum_{j=0}^{\infty}\frac{\left(\frac{y^2}{4}\right)^j}{j!\Gamma(a+j+1)}
$$

Proof. i. $(\beta=0)$,

The equation $dS_t=\sigma_t^H S_t^\beta\left(\rho dB_t^{\sigma,H}+\sqrt{1-\rho^2}dB_t^S\right)$ becomes $dS_t=\sigma_t^H\left(\rho dB_t^{\sigma,H}+\sqrt{1-\rho^2}dB_t^S\right)$. Integrating this equation over $[s,t]$, we obtain:

$$
\begin{aligned}
S(t)-S(s) =& \rho\int_s^t\sigma^H(x)dB_\sigma^H(x)+\sqrt{1-\rho^2}\int_s^t\sigma(x)^H dB^S(x)\\
S(t) \simeq& S(s)+\rho\int_s^t\sigma^H(x)dB_\sigma^H(x)+\sqrt{1-\rho^2}\int_s^t\sigma(x)^H dB_S(x)
\end{aligned}
$$

ii. $(\beta=1)$ The equation $dS_t=\sigma_t^H S_t^\beta\left(\rho dB_t^{\sigma,H}+\sqrt{1-\rho^2}dB_t^S\right)$ becomes $dS_t=\sigma_t^H S_t\left(\rho dB_t^{\sigma,H}+\sqrt{1-\rho^2}dB_t^S\right)$. Applying Itô's theorem establishes:

$$
\begin{aligned}
S(t)\simeq S(s)\exp\Bigg\{&\rho\int_s^t\sigma^H(x)dB_t^{\sigma,H}+\sqrt{1-\rho^2}\int_s^t\sigma^H(x)dB^S(x)\\
&-\frac{1}{2}\left(1+2\rho^2\sqrt{1-\rho^2}\right)\int_s^t\sigma_H^2(x)dx\Bigg\}
\end{aligned}
$$

iii. $(\beta\in(0,1))$ Using the Cholesky transformation and substituting dB_t^S with

$$
\rho dB_t^\sigma+\sqrt{1-\rho^2}dB_t^S
$$

in the previous lemma, we establish:

$$
W_t=W_0+\rho\int_0^t\sigma_i^H dB_i^\sigma+\sqrt{1-\rho^2}\int_0^t\sigma_i^H dB_i^S-\int_0^t\frac{\beta(\sigma_s^H)^2}{(2-2\beta)W_i}di. \tag{19}
$$

To express $\rho \int_0^t \sigma_i^H dB_i^\sigma$, we need to integrate the volatility process σ_t^H. We have:

$$\int_0^t d\sigma_i^H = \alpha \int_0^t \sigma_i^H \mu(X_i)di + \alpha\theta \int_0^t \sigma_i^H \nu(X_i)dB_i^X + \alpha\sqrt{1-\theta^2} \int_0^t \sigma_i^H \nu(X_i)dB_i^\sigma$$

and we approximate the integral $\int_0^t \sigma_i^H dB_i^\sigma$ by:

$$\int_0^t \sigma_i^H dB_i^\sigma \approx \frac{\sigma_t^H - \sigma_0^H - \alpha\theta \int_0^t \sigma_i^H \mu(X_i)di - \alpha\theta \int_0^t \sigma_i^H \nu(X_i)dB_i^X}{\alpha\sqrt{1-\theta^2}} \tag{20}$$

We then set:

$$\widetilde{W}_t = \widetilde{W}_0 + \sqrt{1-\theta^2} \int_0^t \sigma_i^t dB_i^S + \int_0^t \frac{\beta(\sigma_i^H)}{(2-2\beta)\widetilde{W}_i} di \tag{21}$$

with initial condition:

$$\widetilde{W}_0 = W_0 + \rho \frac{\sigma_t^H - \sigma_0^H - \alpha\theta \int_0^t \sigma_i^H \mu(X_i)di - \alpha\theta \int_0^t \sigma_i^H \nu(X_i)dB_i^X}{\alpha\sqrt{1-\theta^2}} \tag{22}$$

The second transformation consists in setting $G_t = (\widetilde{W}_t)^2$. Applying Itô's formula or integration by parts, we get:

$$dG_t = 2\widetilde{W}_t d\widetilde{W}_t + \left[d\widetilde{W}_t\right]^2$$

Which gives:

$$dG_t = 2\widetilde{W}_t \left(\sqrt{1-\rho^2}\sigma_t^H dB_t^S + \frac{\beta(\sigma_t^H)^2}{(2-2\beta)\widetilde{W}_t} dt \right) + (1-\rho^2)(\sigma_t^H)^2 dt$$

and we obtain:

$$dG_t = 2\sqrt{G_t}\sqrt{1-\rho^2}\sigma_t^H dB_t^S + \frac{1-2\beta-\rho^2(1-\beta)}{(1-\beta)(1-\rho^2)}(1-\rho^2)(\sigma_t^H)^2 dt$$

Setting $z(t) = (1-\rho^2)\int_0^t (\sigma_i^H)^2 di$, we get:

$$dG_t = 2\sqrt{G_t}\sqrt{1-\rho^2}\sigma_t^H dB_t^S + \frac{1-2\beta-\rho^2(1-\beta)}{(1-\beta)(1-\rho^2)} dz(t) \tag{23}$$

since:

$$dz(t) = d\left[(1-\rho^2)\int_0^t (\sigma_i^H)^2 di\right] = (1-\rho^2)(\sigma_t^H)^2 dt$$

Using a time change as in [16], [6] and [18], i.e., setting $B_{z(t)}^S = \int_0^{z(t)} dB_i^S = \sqrt{1-\rho^2}\int_0^t \sigma_i^H dB_i^S$, thus $dB_{z(t)}^S = \sqrt{1-\rho^2}\sigma_t^H dB_t^S$. Substituting this into (23), we obtain:

$$dG_{z(t)} = 2\sqrt{\left|G_{z(t)}\right|} dB_{z(t)} + \frac{1-2\beta-\rho^2(1-\beta)}{(1-\beta)(1-\rho^2)} dz(t) \tag{24}$$

which is a Squared Bessel Process of dimension $\delta = \dfrac{1-2\beta-\rho^2(1-\beta)}{(1-\beta)(1-\rho^2)}$ starting at:

$$G_0 = (\widetilde{W}_0)^2 = \left(W_0 + \rho \frac{\sigma_t^H - \sigma_0^H - \alpha\theta \int_0^t \sigma_s^H \mu(X_s)ds - \alpha\theta \int_0^t \sigma_s^H \nu(X_s)dB_\sigma}{\alpha\sqrt{1-\theta^2}} \right)^2$$

The squared Bessel process is Markovian and its transition density is known (see S.Bordin [3] [p.136] for details). Based on the transition density of the squared Bessel diffusion given in the definition, we obtain the transition density for equation 12: First note:

$$S_t = \left((1-\beta)\sqrt{\left|Z_{v(t)}\right|}\right)^{\frac{1}{1-\beta}}, \beta \neq 1 \tag{25}$$

and define the field

$h : s \to [(1-\beta)\sqrt{s}]^{\frac{1}{1-\beta}}$, $s \geq 0$, $\beta \neq 1$ with inverse function $h^{-1} : x \to \frac{x^{2(1-\beta)}}{(1-\beta)^2}$, $x \geq 0, \beta \neq 1$. Thus from (24) and (25), we have: $S_t = h\left(G_{z(t)}\right)$ and $G_0 = h^{-1}(S_0) = \frac{S_0^{2(1-\beta)}}{(1-\beta)^2}$. Thus $G_{z(t)}$ has density $q^\delta(z(t), G_0, x)$, and it follows that the density of S_t is given by:

$$f(S \mid S_0) = q^\delta\left(z(t), G_0, h^{-1}(S)\right)\frac{dh^{-1}(S)}{dS}$$

where $f(S \mid S_0)$ denotes the conditional transition density for the SABR process. Using the two cases considered in proposition 2 of [16] and result 2.2 in [6] (transition density of the squared Bessel process), the transition densities for the SABR process S_t in equation (12) are of the form:

1) For $0 < \beta < \frac{1}{2}$ with absorption at 0 and for $\frac{1}{2} < \beta < 1$:

$$f(S_t \mid S_0) = \frac{1}{z(t)}\left(\frac{S_t}{S_0}\right)^{-\frac{1}{2}}\exp\left(-\frac{S_t^{2(1-\beta)} + S_0^{2(1-\beta)}}{2(1-\beta)^2 z(t)}\right) I_{\left|\frac{\delta-2}{2}\right|}\left(\frac{(S_0S_t)^{1-\beta}}{z(t)(1-\beta)^2}\right)\frac{S_t^{1-2\beta}}{1-\beta} \tag{26}$$

with $z(t) = (1-\rho^2)\int_0^t(\sigma_u^H)^2du$ and $\delta = \frac{1-2\beta-\rho^2(1-\beta)}{(1-\beta)(1-\rho^2)}$

2) For $0 < \beta < \frac{1}{2}$ with reflection at $S = 0$:

$$f(S_t \mid S_0) = \frac{1}{z(t)}\left(\frac{S_t}{S_0}\right)^{-\frac{1}{2}}\exp\left(-\frac{S_t^{2(1-\beta)} + S_0^{2(1-\beta)}}{2(1-\beta)^2 z(t)}\right) I_{\frac{\delta-2}{2}}\left(\frac{(S_0S_t)^{1-\beta}}{z(t)(1-\beta)^2}\right)\frac{S_t^{1-2\beta}}{1-\beta} \tag{27}$$

Using the relationship between the transition density of the squared Bessel process and the chi-square distribution, we obtain the cumulative distribution function of the fractional SABR process. (For details, consult Schroder [21], Islah [16], and Chen [6]). The relationship between the chi-square density and that of the squared Bessel process establishes the theorem. ■ □

3 Simulation of the SABR-FSV Process

To simulate the dynamics of the SABR-FSV process, several simulation steps are required. We outline the different simulation stages for the SABR-FSV model.
To simulate the conditional price dynamics $S(t)$, we must first simulate its components:

- Simulate the fractional stochastic volatility process σ_t^H. This requires first simulating the fractional Brownian motion (fBm) before simulating the Fast-Varying Stochastic Environment $X_t^{\varepsilon,H}$ (FVSE).
- Simulate the conditional integrated variance process $\int_s^t [\sigma_H(u)]^2 du \mid \sigma^H(t), \sigma^H(s)$.
- Simulate the SABR-FSV process for the underlying price by inverting the cumulative distribution function in equation 16.

3.1 Simulation of SABR-FSV Components

We used MATLAB (version 2024b) for simulations with varying parameters. The simulation involves generating fractional Brownian motions as vectors using MATLAB's built-in pseudo-random number generator. After simulating the Brownian motions, we sample a fractional Brownian motion random field called the Fast-Varying Stochastic Environment (FVSE). Once the FVSE is simulated, we generate the Fractional Stochastic Volatility Process (FSVP). Finally, with the FSVP generated, we simulate the fractional SABR process dynamics. The presented graphs show random trajectories of different processes.

The simulation algorithm is summarized as follows:

Algorithm

Input: $S_0, \sigma_0, \alpha, \beta, \rho, H, \epsilon, \theta$
for $p = 1, \cdots, n$ do
  Simulate $B_X^H(t)$
  Simulate $B_\sigma^H(t)$
  Simulate $X_t^{\epsilon,H}$
  Simulate $\sigma^{H,\epsilon}(t) \mid \sigma_0$
  Simulate $\int_0^T \sigma(s)^2 ds \mid \sigma(t), \sigma_0$
  Simulate $S(t) \mid S(0), \sigma(t), \int_0^T \sigma(s)^2 ds$
end for

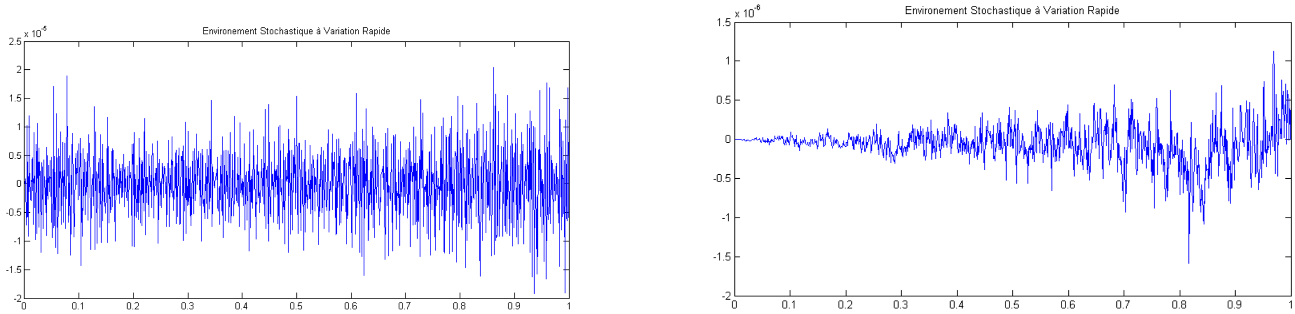
To avoid overloading our work with graphs, we omit visualizations of fractional Brownian motions here.

3.1.1 Simulation of the Fast-Varying Stochastic Environment (FVSE)

The following figures simulate the FVSE by varying parameters H and ϵ. In Figure 2(a) ($H = 0.2$, $\epsilon = 10^{-5}$), we observe high amplitudes, while in Figure 2(b) ($H = 0.8$, $\epsilon = 10^5$) we see small ridges growing over time with a slight trend.

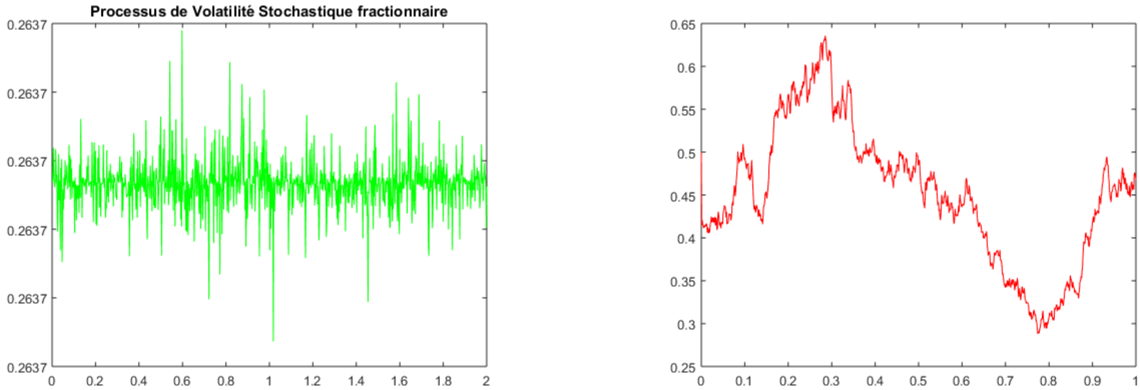
3.1.2 Simulation of the Fractional Stochastic Volatility Process (FSVP)

Below are simulations of the Fractional Stochastic Volatility Process (FSVP).

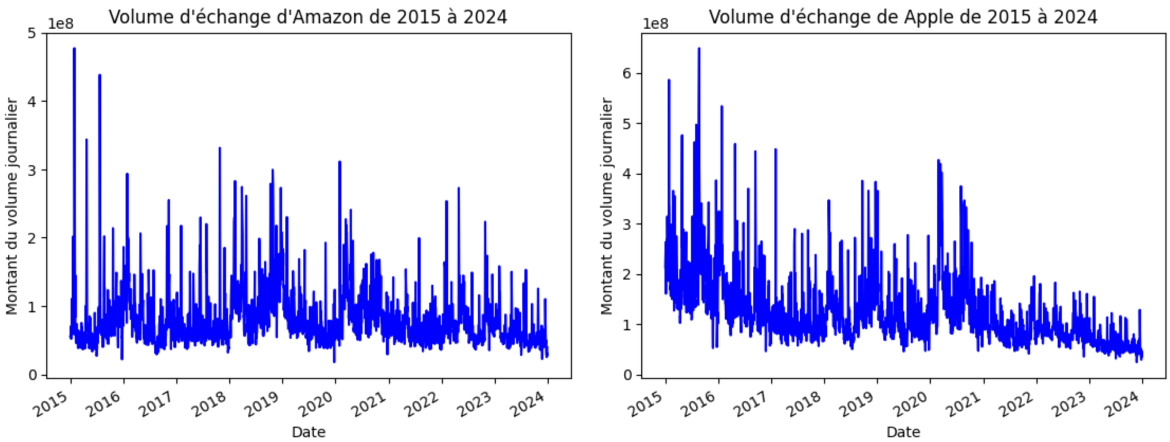


(a) FVSE for $H = 0.2$ and $\epsilon = 10^{-5}$ (b) FVSE for $H = 0.8$ and $\epsilon = 10^{5}$

Figure 2: Fast-Varying Stochastic Environment (FVSE)



(a) FSVP for $H = 0.7$, $\epsilon = 10$, $\theta = 0.1$ (b) Volatility of the SABR model



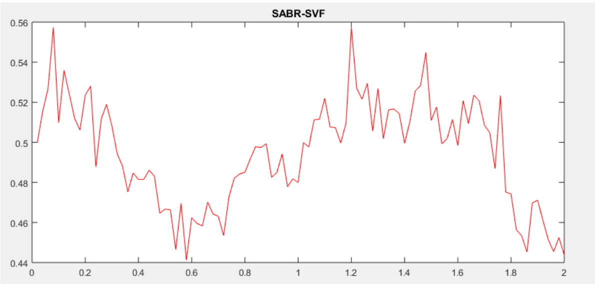
(c) Trading volume of AMAZON (d) Trading volume of APPLE

Figure 3: Comparison between FSVP, SABR volatility, and empirical data (2015-2024)

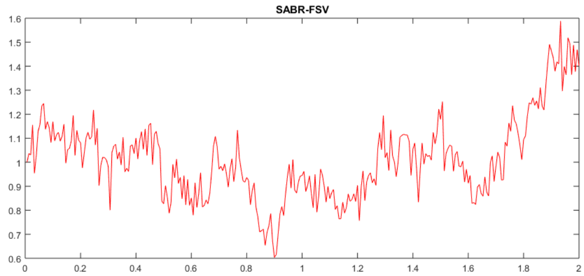
Figures 3(a) and 3(b) compare the proposed FSVP with the volatility process used in standard SABR modeling. Figures 3(d) and 3(c) show empirical trading volume data for APPLE and AMAZON. The standard SABR volatility exhibits trends similar to standard Brownian motion, but shows significant deviation from empirical data. The theoretical FSVP we simulated shows strong similarities with empirical patterns, indicating that the fractional SABR model better approximates reality.

3.2 Simulation of the Stochastic Alpha Beta Rho-Fractional Stochastic Volatility (SABR-FSV) Process

After simulating the SABR-FSV components, we present simulations using the Euler method in MATLAB.



(a) SABR-FSV for $H = 0.2$, $\theta = 0.1$

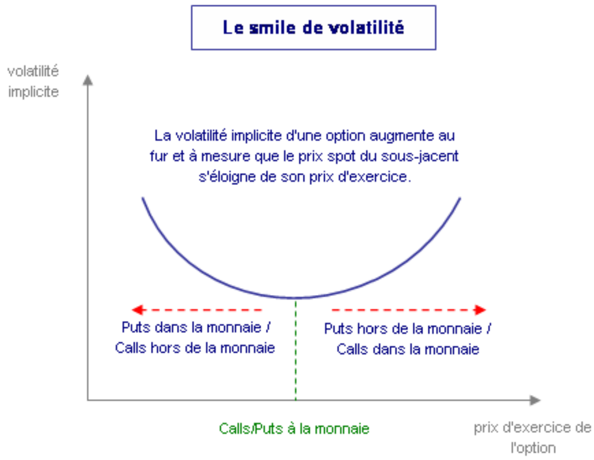


(b) SABR-FSV for $H = 0.2$, $\rho = -0.9$

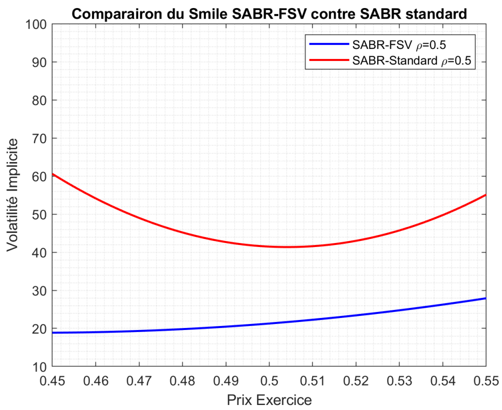
Figure 4: Simulation of two SABR-FSV trajectories

Figure 4(a) shows that SABR-FSV exhibits strong trends with smooth amplitudes for $\theta = 0.1$ and $H = 0.2$, while Figure 4(b) displays low, closely spaced amplitudes for $H = 0.2$ and $\rho = -0.9$. These results demonstrate that we can obtain either low or high implied volatilities depending on parameters, making it suitable for unstable markets. Parameter calibration enables accurate pricing for maturities ranging from three (3) months to 5 years.

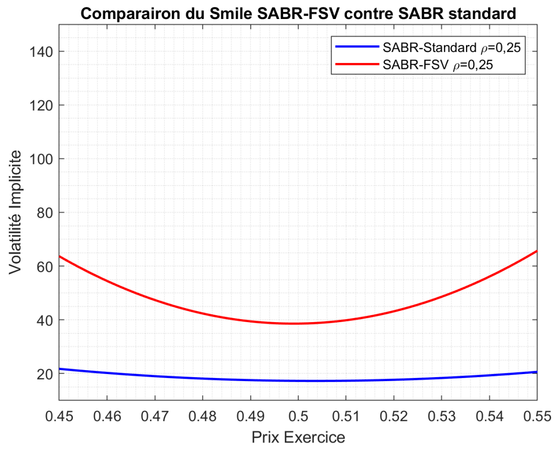
The following graphs illustrate volatility smiles and their behavior, comparing SABR-FSV with the standard SABR model.



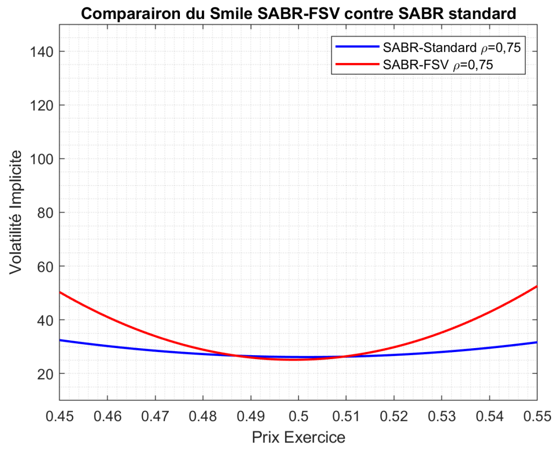
(a) Volatility smile illustration



(b) SABR-FSV vs. standard SABR for $\rho = 0.5$



(c)



(d)

Figure 5: SABR-FSV vs. standard SABR for $\rho = 0.25$ (5(c)) and $\rho = 0.75$ (5(d))

The figures above show the evolution of implied volatility as a function of strike price (volatility smile). In most simulated cases, we observe the characteristic "smile" shape of volatility versus strike price seen in practice. By significantly increasing the sample size and number of simulations, the results become closer to real market behavior. However, this requires more powerful computing resources.

4 Conclusion

In this chapter, we proposed a fractional version of the SABR model, designed to better capture the irregularity and complexity of volatility dynamics observed in financial markets. This model, called SABR-FSV, enables more accurate option pricing and better reproduction of the volatility smile.
Our results demonstrate that the smile generated by the SABR-FSV model is significantly closer to empirical observations than that from the classical SABR model. Furthermore, this model also proves suitable for pricing exotic options, such as barrier options, as well as interest rate products.
However, this improvement in accuracy comes with non-negligible computational costs due to the complexity introduced by the fractional components of the model.

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