UNDER PEER REVIEW

A Study On Dual Hyperbolic Generalized Pandita Numbers

Abstract. In this paper, we introduce the generalized dual hyperbolic Pandita numbers. As special cases, we deal with dual hyperbolic Pandita and dual hyperbolic Pandita-Lucas numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Moreover, we give Catalan's, Cassini's, d'Ocagne's, Gelin-Cesàro's, Melham's identities and present matrices related with these sequences.

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1. Introduction

The hypercomplex numbers systems, [8], are extensions of real numbers. Some commutative examples of hypercomplex number systems are complex numbers,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers, [6],

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

and dual numbers, [14],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Some non-commutative examples of hypercomplex number systems are quaternions, [24],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [9] and sedenions [15] are part of a sequence of real algebras constructed through a recursive method known as the Cayley–Dickson process. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are all derived from the real numbers $\mathbb{R}$ via this doubling procedure. The process can be extended beyond sedenions to generate higher-dimensional algebras known as 2^n-ions (see for example [3], [12], [5]).

Quaternions were introduced by the Irish mathematician W. R. Hamilton (1805–1865) as an extension of the complex numbers [24]. Hyperbolic numbers with complex coefficients were first studied by J. Cockle in 1848 [10]. Later, H. H. Cheng and S. Thompson [7] introduced dual numbers with complex coefficients, which they termed complex dual numbers. Dual hyperbolic numbers were subsequently introduced by Akar, Yüce, and Şahin [13].

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and addition of dual hyperbolic numbers is defined as componentwise.

The set of dual hyperbolic numbers constitutes a commutative ring, a real vector space, and an algebra. However, H_D does not form a field, as not every dual hyperbolic number possesses a multiplicative inverse. For further details on the algebraic structure and properties of dual hyperbolic numbers, see [13].

We now recall the definition of generalized Pandita numbers.

A generalized Pandita sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-2} + W_{n-3} - W_{n-4} \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (1.1) holds for all integer n. Soykan has conducted a study on this particular sequence, for more details, see [17].

The first few generalized Pandita numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Pandita numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$W_0 - W_1 + 2W_2 - W_3$
2	W_2	$W_1 + W_2 - W_3$
3	W_3	$W_0 + W_1 - W_2$
4	$W_1 - W_0 - W_2 + 2W_3$	$2W_0 - 2W_1 + 2W_2 - W_3$
5	$W_1 - 2W_0 - W_2 + 3W_3$	$3W_2 - 2W_3$
6	$W_1 - 3W_0 - 2W_2 + 5W_3$	$3W_1 - 2W_2$
7	$2W_1 - 5W_0 - 4W_2 + 8W_3$	$3W_0 - 2W_1$
8	$3W_1 - 8W_0 - 6W_2 + 12W_3$	$W_0 - 3W_1 + 6W_2 - 3W_3$
9	$4W_1 - 12W_0 - 9W_2 + 18W_3$	$5W_1 - 2W_0 - W_2 - W_3$
10	$6W_1 - 18W_0 - 14W_2 + 27W_3$	$3W_0 + W_1 - 5W_2 + 2W_3$
11	$9W_1 - 27W_0 - 21W_2 + 40W_3$	$4W_0 - 8W_1 + 8W_2 - 3W_3$
12	$13W_1 - 40W_0 - 31W_2 + 59W_3$	$4W_1 - 4W_0 + 5W_2 - 4W_3$
13	$19W_1 - 59W_0 - 46W_2 + 87W_3$	$9W_1 - 12W_2 + 4W_3$

If we set $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 3$ then $\{W_n\}$ is the well-known Pandita sequence and if we set $W_0 = 4, W_1 = 2, W_2 = 2, W_3 = 5$ then $\{W_n\}$ is the well-known Pandita-Lucas sequence. In other words, Pandita sequence $\{P_n\}_{n\geq 0}$ and Pandita-Lucas sequence $\{S_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \qquad n \geq 4, \tag{1.2}$$

and

$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}, \qquad S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5, \qquad n \geq 4. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$S_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

We can list some important properties of generalized Pandita numbers that are needed.

- Binet formula of generalized Pandita sequence can be calculated using its characteristic equation which is given as

$$x^4 - 2x^3 + x^2 - x + 1 = (x^3 - x^2 - 1)(x - 1) = 0$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet formula can be given as

$$\begin{aligned}
W_n &= \frac{z_1\alpha^n}{3\alpha - 2} + \frac{z_2\beta^n}{3\beta - 2} + \frac{z_3\gamma^n}{3\gamma - 2} + z_4 \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4,
\end{aligned}$$

where z_1, z_2 and z_3 are given below

$$\begin{aligned}
z_1 &= (\alpha W_3 - \alpha(2 - \alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0), \\
z_2 &= (\beta W_3 - \beta(2 - \beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0), \\
z_3 &= (\gamma W_3 - \gamma(2 - \gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0), \\
z_4 &= -W_3 + W_2 + W_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{z_1}{3\alpha - 2}, \\
A_2 &= \frac{z_2}{3\beta - 2}, \\
A_3 &= \frac{z_3}{3\gamma - 2}, \\
A_4 &= z_4.
\end{aligned} \tag{1.4}$$

Binet formula of Pandita and Pandita-Lucas sequences are

$$P_n = \frac{\alpha^{n+3}}{3\alpha - 2} + \frac{\beta^{n+3}}{3\beta - 2} + \frac{\gamma^{n+3}}{3\gamma - 2} - 1,$$

and

$$S_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

- The generating function for generalized Pandita numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1 + W_0)x^2 + (W_3 - 2W_2 + W_1 - W_0)x^3}{1 - 2x + x^2 - x^3 + x^4}.$$

For more details about generalized Pandita numbers, see [17].

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

LEMMA 1. *[11, Lemma 1.4].Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Pandita sequence $\{W_n\}$.*

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0)}{3\alpha - 2} e^{\alpha x} \\
&+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0)}{3\beta - 2} e^{\beta x} \\
&+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0)}{3\gamma - 2} e^{\gamma x} \\
&+ (-W_3 + W_2 + W_0)e^x.
\end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

COROLLARY 2. *Exponential generating function of Pandita and Pandita-Lucas numbers*

a): $\sum\limits_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{\alpha^{n+3}}{3\alpha - 2} + \frac{\beta^{n+3}}{3\beta - 2} + \frac{\gamma^{n+3}}{3\gamma - 2} - 1)\frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}}{3\alpha - 2} + \frac{\beta^3 e^{\beta x}}{3\beta - 2} + \frac{\gamma^3 e^{\gamma x}}{3\gamma - 2} - e^x.$

b): $\sum\limits_{n=0}^{\infty} S_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1)\frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

Next, we give some information on published papers related to hyperbplic and dual hyperbolic numbers in literature.

- Cockle [10] presented the hyperbolic numbers with complex coefficients.
- Akar at al [13] introduced the dual hyperbolic numbers.
- Cheng and Thompson[7] studied dual numbers with complex coefficients.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan at al [19] introduced dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a, V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Cihan at al [21] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$

$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}.$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1, L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1$.

- Soykan at al [22] introduced dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J_n} = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}, J_0 = a, J_1 = b$.

- Bród at al [1] studied dual hyperbolic generalized Balancing numbers are

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}, B_0 = 0, B_1 = 1$.

- Yılmaz and Soykan [23] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T_0} = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$.

- Dikmen [2] introduced dual hyperbolic generalised Leonardo numbers given by

$$\widehat{l_0} = l_0 + jl_1 + \varepsilon l_2 + j\varepsilon l_3$$

$l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3$.

- Eren and Soykan [4] introduced dual hyperbolic generalized Woodall numbers given by

$$\widehat{R_0} = R_0 + jR_1 + \varepsilon R_2 + j\varepsilon R_3$$

where $R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3},\ R_0 = -1, R_1 = 1, R_2 = 7$.

In this paper, we define the dual hyperbolic generalized Pandita numbers in the next section and give some properties of them.

2. Dual Hyperbolic Generalized Pandita Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual hyperbolic generalized Pandita numbers and present generating functions and Binet formulas for them. We now define dual hyperbolic generalized Pandita numbers over $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Pandita number is

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{2.1}$$

The sequence $\{\widehat{W}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n. Note that

$$\begin{aligned}
\widehat{W}_0 &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3 \\
\widehat{W}_1 &= W_1 + jW_2 + \varepsilon W_3 + j\varepsilon W_4 \\
\widehat{W}_2 &= W_2 + jW_3 + \varepsilon W_4 + j\varepsilon W_5
\end{aligned}$$

It can be easily shown that

$$\widehat{W}_n = 2\widehat{W}_{n-1} - \widehat{W}_{n-2} + \widehat{W}_{n-3} - \widehat{W}_{n-4} \tag{2.2}$$

and

$$\widehat{W}_{-n} = \widehat{W}_{-(n-1)} - \widehat{W}_{-(n-2)} + 2\widehat{W}_{-(n-3)} - \widehat{W}_{-(n-4)}$$

The first few dual hyperbolic generalized Pandita numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few dual hyperbolic generalized Pandita numbers

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$\widehat{W}_0 - \widehat{W}_1 + 2\widehat{W}_2 - \widehat{W}_3$
2	$\widehat{W}_2$	$\widehat{W}_1 + \widehat{W}_2 - \widehat{W}_3$
3	$\widehat{W}_3$	$W_0 + W_1 - W_2$
4	$\widehat{W}_1 - \widehat{W}_0 - \widehat{W}_2 + 2\widehat{W}_3$	$2\widehat{W}_0 - 2\widehat{W}_1 + 2\widehat{W}_2 - \widehat{W}_3$
5	$\widehat{W}_1 - 2\widehat{W}_0 - \widehat{W}_2 + 3\widehat{W}_3$	$3\widehat{W}_2 - 2\widehat{W}_3$
6	$\widehat{W}_1 - 3\widehat{W}_0 - 2\widehat{W}_2 + 5\widehat{W}_3$	$3\widehat{W}_1 - 2\widehat{W}_2$
7	$2\widehat{W}_1 - 5\widehat{W}_0 - 4\widehat{W}_2 + 8\widehat{W}_3$	$3\widehat{W}_0 - 2\widehat{W}_1$
8	$3\widehat{W}_1 - 8\widehat{W}_0 - 6\widehat{W}_2 + 12\widehat{W}_3$	$\widehat{W}_0 - 3\widehat{W}_1 + 6\widehat{W}_2 - 3\widehat{W}_3$
9	$4\widehat{W}_1 - 12\widehat{W}_0 - 9\widehat{W}_2 + 18\widehat{W}_3$	$5\widehat{W}_1 - 2\widehat{W}_0 - \widehat{W}_2 - \widehat{W}_3$
10	$6\widehat{W}_1 - 18\widehat{W}_0 - 14\widehat{W}_2 + 27\widehat{W}_3$	$3\widehat{W}_0 + \widehat{W}_1 - 5\widehat{W}_2 + 2\widehat{W}_3$
11	$9\widehat{W}_1 - 27\widehat{W}_0 - 21\widehat{W}_2 + 40\widehat{W}_3$	$4\widehat{W}_0 - 8\widehat{W}_1 + 8\widehat{W}_2 - 3\widehat{W}_3$
12	$13\widehat{W}_1 - 40\widehat{W}_0 - 31\widehat{W}_2 + 59\widehat{W}_3$	$4\widehat{W}_1 - 4\widehat{W}_0 + 5\widehat{W}_2 - 4\widehat{W}_3$
13	$19\widehat{W}_1 - 59\widehat{W}_0 - 46\widehat{W}_2 + 87\widehat{W}_3$	$9\widehat{W}_1 - 12\widehat{W}_2 + 4\widehat{W}_3$

As special cases, the nth dual hyperbolic Pandita numbers and the nth dual hyperbolic Pandita-Lucas numbers are given as

$$\widehat{P}_n = P_n + jP_{n+1} + \varepsilon P_{n+2} + j\varepsilon P_{n+3} \tag{2.3}$$

and

$$\widehat{S}_n = S_n + jS_{n+1} + \varepsilon S_{n+2} + j\varepsilon S_{n+3} \tag{2.4}$$

respectively. The sequences $\{\widehat{P}_n\}_{n\geq 0}$ and $\{\widehat{S}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{P}_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$\widehat{S}_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.3) and (2.4) holds for all integer n

For dual hyperbolic Pandita numbers (taking $W_n = P_n,\ \ P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3,$) we get

$$\begin{aligned}
\widehat{P}_0 &= j + 2\varepsilon + 3j\varepsilon, \\
\widehat{P}_1 &= 2j + 3\varepsilon + 5j\varepsilon + 1, \\
\widehat{P}_2 &= 3j + 5\varepsilon + 8j\varepsilon + 2,
\end{aligned}$$

and for dual hyperbolic Pandita-Lucas numbers (taking $W_n = S_n, \ \ S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5,$) we get

$$\begin{aligned} \widehat{S}_0 &= 2j + 2\varepsilon + 5j\varepsilon + 4, \\ \widehat{S}_1 &= 2j + 5\varepsilon + 6j\varepsilon + 2. \\ \widehat{S}_2 &= 5j + 6\varepsilon + 7j\varepsilon + 2 \end{aligned}$$

A few dual hyperbolic Pandita numbers and dual hyperbolic Pandita-Lucas numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4.

Table 3. Dual hyperbolic Pandita numbers

n	$\widehat{P}_n$	$\widehat{P}_{-n}$
0	$j + 2\varepsilon + 3j\varepsilon$	$j + 2\varepsilon + 3j\varepsilon$
1	$2j + 3\varepsilon + 5j\varepsilon + 1$	$\varepsilon + 2j\varepsilon$
2	$3j + 5\varepsilon + 8j\varepsilon + 2$	$-j\varepsilon$
3	$5j + 8\varepsilon + 12j\varepsilon + 3$	-1
4	$8j + 12\varepsilon + 18j\varepsilon + 5$	$-j - 1$
5	$12j + 18\varepsilon + 27j\varepsilon + 8$	$-j - \varepsilon$

Table 4. Dual hyperbolic Pandita-Lucas numbers

n	$\widehat{S}_n$	$\widehat{S}_{-n}$
0	$2j + 2\varepsilon + 5j\varepsilon + 4$	$2j + 2\varepsilon + 5j\varepsilon + 4$
1	$2j + 5\varepsilon + 6j\varepsilon + 2$	$-4j + 2\varepsilon + 2j\varepsilon + 1$
2	$5j + 6\varepsilon + 7j\varepsilon + 2$	$j + 4\varepsilon + 2j\varepsilon - 1$
3	$6j + 7\varepsilon + 11j\varepsilon + 5$	$\varepsilon - j + 4j\varepsilon + 4$
4	$7j + 11\varepsilon + 16j\varepsilon + 6$	$4j - \varepsilon + j\varepsilon + 3$
5	$11j + 16\varepsilon + 22j\varepsilon + 7$	$-3j + 4\varepsilon - j\varepsilon - 4$

Now, we will state Binet's formula for the dual hyperbolic generalized Pandita numbers and in the rest of the paper, we fix the following notations:

$$\widehat{\alpha} = 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \tag{2.5}$$

$$\widehat{\beta} = 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3. \tag{2.6}$$

$$\widehat{\gamma} = 1 + j\gamma + \varepsilon\gamma^2 + j\varepsilon\gamma^3 \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + j + \varepsilon + j\varepsilon, \tag{2.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+\alpha^2+2\alpha j+2\alpha^2\left(\alpha^2+1\right)\varepsilon+4\alpha^3 j\varepsilon \\
\widehat{\beta}^2 &= 1+\beta^2+2j\beta+(2\beta^4+2\beta^2)\varepsilon+4j\varepsilon\beta^3, \\
\widehat{\alpha}\widehat{\beta} &= 1+\alpha\beta+(\alpha+\beta)j+\left(\alpha^2+\beta^2+2\alpha\beta^3+\alpha^3\beta\right)\varepsilon+(\alpha+\beta)\left(\alpha^2+\beta^2\right)j\varepsilon, \\
\widehat{\alpha}^2\widehat{\beta} &= 1+\alpha^2+\beta^2+\alpha^2\beta^2+2(\alpha\beta+1)(\alpha+\beta)j+2\left(\alpha^2+\beta^2+\alpha^2\beta^2+4\alpha\beta+1\right)\left(\alpha^2+\beta^2\right)\varepsilon \\
&\quad +4(\alpha+\beta)\left(\alpha^2+\beta^2+\alpha\beta^3\right)j\varepsilon, \\
\widehat{\alpha}\widehat{\beta}^2 &= 1+\beta^2+2\alpha\beta+\left(\alpha+2\beta+\alpha\beta^2\right)j+\left(\beta^2+2\alpha\beta+1\right)\left(\alpha^2+2\beta^2\right)\varepsilon+\left(\alpha+2\beta+\alpha\beta^2\right)\left(\alpha^2+2\beta^2\right)j\varepsilon, \\
\widehat{\alpha}^2\widehat{\beta}^2 &= 1+\beta^2+\alpha^2+\alpha^2\beta^2+4\alpha\beta+2(\alpha\beta+1)(\alpha+\beta)j+2\left(\alpha^2+\beta^2+\alpha^2\beta^2+4\alpha\beta+1\right)\left(\alpha^2+\beta^2\right)\varepsilon \\
&\quad +4(\alpha\beta+1)(\alpha+\beta)\left(\alpha^2+\beta^2\right)j\varepsilon
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Pandita number is*

$$\widehat{W}_n = A_1\alpha^n\widehat{\alpha}+A_2\beta^n\widehat{\beta}+A_3\gamma^n\widehat{\gamma}+\widehat{1}A_4. \tag{2.9}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula

$$W_n = A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4.$$

where A_1, A_2, A_3, A_4 are given in (1.4) we get

$$\begin{aligned}
\widehat{W}_n &= W_n+jW_{n+1}+\varepsilon W_{n+2}+j\varepsilon W_{n+3} \\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4+j(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1}+A_4) \\
&\quad +\varepsilon(A_1\alpha^{n+2}+A_2\beta^{n+2}+A_3\gamma^{n+2}+A_4)+j\varepsilon(A_1\alpha^{n+3}+A_2\beta^{n+3}+A_3\gamma^{n+3}+A_4) \\
&= A_1\alpha^n(1+j\alpha+\varepsilon\alpha^2+j\varepsilon\alpha^3)+A_2\beta^n(1+j\beta+\varepsilon\beta^2+j\varepsilon\beta^3) \\
&\quad +A_3\gamma^n(1+j\gamma+\varepsilon\gamma^2+j\varepsilon\gamma^3)+A_4(1+j+\varepsilon+j\varepsilon) \\
&= A_1\alpha^n\widehat{\alpha}+A_2\beta^n\widehat{\beta}+A_3\gamma^n\widehat{\gamma}+\widehat{1}A_4.
\end{aligned}$$

This proves (2.9). □

As special cases, for any integer n, the Binet's Formula of nth dual hyperbolic Pandita number is

$$\widehat{P}_n = \frac{\alpha^{n+3}\widehat{\alpha}}{3\alpha-2}+\frac{\beta^{n+3}\widehat{\beta}}{3\beta-2}+\frac{\gamma^{n+3}\widehat{\gamma}}{3\gamma-2}-\widehat{1} \tag{2.10}$$

and the Binet's Formula of nth dual hyperbolic Pandita-Lucas number is

$$\widehat{S}_n = \widehat{\alpha}\alpha^n+\widehat{\beta}\beta^n+\widehat{\gamma}\gamma^n+\widehat{1}, \tag{2.11}$$

Next, we present generating function.

Theorem 4. *Let $f_{\widehat{W}_n}(x) = \sum\limits_{n=0}^{\infty} \widehat{W}_n x^n$ donate the generating function of dual hyperbolic generalized Pandita numbers is given as follows:*

$$f_{\widehat{W}_n}(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 2\widehat{W}_0)x + (\widehat{W}_2 - 2\widehat{W}_1 + \widehat{W}_0)x^2 + (\widehat{W}_3 - 2\widehat{W}_2 + \widehat{W}_1 - \widehat{W}_0)x^3}{1 - 2x + x^2 - x^3 + x^4}.$$

Proof. Using the definition of dual hyperbolic Pandita numbers, and substracting $xf(x)$, $x^2 f(x)$ and $x^3 f(x)$ from $f(x)$ we obtain $(1 - 2x + x^2 - x^3 + x^4) f_{GW_n}(x)$

$$\begin{aligned}
& (1 - 2x + x^2 - x^3 + x^4) f_{\widehat{W}_n}(x) \\
= \; & \sum_{n=0}^{\infty} \widehat{W}_n x^n - 2x \sum_{n=0}^{\infty} \widehat{W}_n x^n + x^2 \sum_{n=0}^{\infty} \widehat{W}_n x^n - x^3 \sum_{n=0}^{\infty} \widehat{W}_n x^n + x^4 \sum_{n=0}^{\infty} \widehat{W}_n x^n, \\
= \; & \sum_{n=0}^{\infty} \widehat{W}_n x^n - 2 \sum_{n=0}^{\infty} \widehat{W}_n x^{n+1} + \sum_{n=0}^{\infty} \widehat{W}_n x^{n+2} - \sum_{n=0}^{\infty} \widehat{W}_n x^{n+3} + \sum_{n=0}^{\infty} \widehat{W}_n x^{n+4}, \\
= \; & \sum_{n=0}^{\infty} \widehat{W}_n x^n - 2 \sum_{n=1}^{\infty} \widehat{W}_{(n-1)} x^n + \sum_{n=2}^{\infty} \widehat{W}_{(n-2)} x^n - \sum_{n=3}^{\infty} \widehat{W}_{(n-3)} x^n + \sum_{n=4}^{\infty} \widehat{W}_{(n-4)} x^n, \\
= \; & (\widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2 + \widehat{W}_3 x^3) - 2(\widehat{W}_0 x + \widehat{W}_1 x^2 + \widehat{W}_2 x^3) + (\widehat{W}_0 x^2 + \widehat{W}_1 x^3) - \widehat{W}_0 x^3 \\
& + \sum_{n=4}^{\infty} (\widehat{W}_n - 2\widehat{W}_{n-1} - \widehat{W}_{n-2} - \widehat{W}_{n-3} + \widehat{W}_{n-4}) x^n, \\
= \; & \widehat{W}_0 + (\widehat{W}_1 - 2\widehat{W}_0)x + (\widehat{W}_2 - 2\widehat{W}_1 + \widehat{W}_0)x^2 + (\widehat{W}_3 - 2\widehat{W}_2 + \widehat{W}_1 - \widehat{W}_0)x^3.
\end{aligned}$$

And rearranging above equation, we get (4). □

The following results are immediate consequences of the preceding Theorem.

Corollary 5. *For all integers n, we have following identities:*

a): $\sum_{n=0}^{\infty} \widehat{P}_n x^n = \dfrac{(j + 5\varepsilon + 4j\varepsilon) + (1 - \varepsilon - j\varepsilon)x + (\varepsilon + j\varepsilon)x^2 + (3j\varepsilon)x^3}{1 - 2x + x^2 - x^3 + x^4}.$

b): $\sum_{n=0}^{\infty} \widehat{S}_n x^n = \dfrac{(2j + 2\varepsilon + 5j\varepsilon + 4) + (\varepsilon - 2j - 6x - 4j\varepsilon)x + (3j - 2\varepsilon + 2)x^2 + (2\varepsilon - 4j + 8j\varepsilon + 7)x^3}{1 - 2x + x^2 - x^3 + x^4}.$

Theorem (4) gives the following results as special cases,

$(1 - 2x + x^2 - x^3 + x^4) f_{\widehat{P}_n}(x) = \widehat{P}_0 + (\widehat{P}_1 - 2\widehat{P}_0)x + (\widehat{P}_2 - 2\widehat{P}_1 + \widehat{P}_0)x^2 + (\widehat{P}_3 - 2\widehat{P}_2 + \widehat{P}_1 - \widehat{P}_0)x^3 = (j + 5\varepsilon + 4j\varepsilon) + (1 - \varepsilon - j\varepsilon)x + (\varepsilon + j\varepsilon)x^2 + (3j\varepsilon)x^3,$

$(1 - 2x + x^2 - x^3 + x^4) f_{\widehat{S}_n}(x) = \widehat{S}_0 + (\widehat{S}_1 - 2\widehat{S}_0)x + (\widehat{S}_2 - 2\widehat{S}_1 + \widehat{S}_0)x^2 + (\widehat{S}_3 - 2\widehat{S}_2 + \widehat{S}_1 - \widehat{S}_0)x^3 = (2j + 2\varepsilon + 5j\varepsilon + 4) + (\varepsilon - 2j - 6x - 4j\varepsilon)x + (3j - 2\varepsilon + 2)x^2 + (2\varepsilon - 4j + 8j\varepsilon + 7)x^3.$

Next, we give the exponential dual hyperbolic generating function of $\sum\limits_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!}$ of the sequence $\widehat{W}_n$.

Lemma 6. *Suppose that $f_{\widehat{W}_n}(x) = \sum\limits_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!}$ is the exponential dual hyperbolic generating function of the generalized Pandita sequence $\{W_n\}$.*

Then $\sum\limits_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}==A_1e^{\alpha x}\widehat{\alpha}+A_2e^{\beta x}\widehat{\beta}+A_3e^{\gamma x}\widehat{\gamma}+A_4e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula

$$W_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4.$$

where A_1, A_2, A_3, A_4 are given in (1.4) we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}W_n\frac{x^n}{n!}+j\sum_{n=0}^{\infty}W_{n+1}\frac{x^n}{n!}+\varepsilon\sum_{n=0}^{\infty}W_{n+2}\frac{x^n}{n!}+j\varepsilon\sum_{n=0}^{\infty}W_{n+3}\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}(A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4)\frac{x^n}{n!}+j\sum_{n=0}^{\infty}(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1}+A_4)\frac{x^n}{n!}\\
&\quad +\varepsilon\sum_{n=0}^{\infty}(A_1\alpha^{n+2}+A_2\beta^{n+2}+A_3\gamma^{n+2}+A_4)\frac{x^n}{n!}+j\varepsilon\sum_{n=0}^{\infty}(A_1\alpha^{n+3}+A_2\beta^{n+3}+A_3\gamma^{n+3}+A_4)\frac{x^n}{n!}\\
&= (A_1e^{\alpha x}+A_2e^{\beta x}+A_3e^{\gamma x}+A_4e^{x})+j(A_1\alpha e^{\alpha x}+A_2\beta e^{\beta x}+A_3\gamma e^{\gamma x}+A_4e^{x})\\
&\quad +\varepsilon(A_1\alpha^2e^{\alpha x}+A_2\beta^2e^{\beta x}+A_3\gamma^2e^{\gamma x}+A_4e^{x})+j\varepsilon(A_1\alpha^3e^{\alpha x}+A_2\beta^3e^{\beta x}+A_3\gamma^3e^{\gamma x}+A_4e^{x})\\
&= A_1e^{\alpha x}(1+j\alpha++\varepsilon\alpha^2+j\varepsilon\alpha^3)+A_2e^{\beta x}(1+j\beta++\varepsilon\beta^2+j\varepsilon\beta^3)\\
&\quad +A_3e^{\gamma x}(1+j\gamma++\varepsilon\gamma^2+j\varepsilon\gamma^3)+A_4e^{x}(1+j++\varepsilon+j\varepsilon)\\
&= A_1e^{\alpha x}\widehat{\alpha}+A_2e^{\beta x}\widehat{\beta}+A_3e^{\gamma x}\widehat{\gamma}+A_4e^{x}\widehat{1}
\end{aligned}$$

This proves (6). □

The previous Lemma 6 gives the following results as particular examples.

COROLLARY 7. *Exponential dual hyperbolic generating function of Pandita and Pandita-Lucas numbers*

a): $\sum\limits_{n=0}^{\infty}\widehat{P}_n\frac{x^n}{n!}=\dfrac{\alpha^3e^{\alpha x}\widehat{\alpha}}{3\alpha-2}+\dfrac{\beta^3e^{\beta x}\widehat{\beta}}{3\beta-2}+\dfrac{\gamma^3e^{\gamma x}\widehat{\gamma}}{3\gamma-2}-e^{x}\widehat{1}.$

b): $\sum\limits_{n=0}^{\infty}\widehat{S}_n\frac{x^n}{n!}=e^{\alpha x}\widehat{\alpha}+e^{\beta x}\widehat{\beta}+e^{\gamma x}\widehat{\gamma}+e^{x}\widehat{1}.$

3. Obtaining Binet Formula From Generating Function

We next find Binet's formula generalized dual hyperbolic Pandita number $\{\widehat{W}_n\}$ by the use of generating function for $\widehat{W}_n$.

THEOREM 8. *Binet's formula of generalized Gaussian Pandita numbers*

(3.1)
$$\widehat{W}_n=\frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.$$

where

$$\begin{aligned} q_1 &= \widehat{W}_0\alpha^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\alpha^2 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)\alpha - \widehat{W}_0 + \widehat{W}_1 - 2\widehat{W}_2 + \widehat{W}_3, \\ q_2 &= \widehat{W}_0\beta^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\beta^2 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)\beta - \widehat{W}_0 + \widehat{W}_1 - 2\widehat{W}_2 + \widehat{W}_3, \\ q_3 &= \widehat{W}_0\gamma^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\gamma^2 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)\gamma - \widehat{W}_0 + \widehat{W}_1 - 2\widehat{W}_2 + \widehat{W}_3, \\ q_4 &= \widehat{W}_0\delta^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\delta^2 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)\delta - \widehat{W}_0 + \widehat{W}_1 - 2\widehat{W}_2 + \widehat{W}_3. \end{aligned}$$

Proof. Let

$$h(x) = x^4 - x^3 + x^2 - 2x + 1.$$

Then for some α, β, γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).$$

i.e.,

$$x^4 - x^3 + x^2 - 2x + 1 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = \frac{1}{x^2} - \frac{2}{x} - \frac{1}{x^3} + \frac{1}{x^4} + 1 = 0.$$

This implies $x^4 - x^3 + x^2 - 2x + 1 = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\left(\widehat{W}_1 - \widehat{W}_0 - 2\widehat{W}_2 + \widehat{W}_3\right)x^3 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)x^2 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)x + \widehat{W}_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\begin{aligned} \frac{\left(\widehat{W}_1 - \widehat{W}_0 - 2\widehat{W}_2 + \widehat{W}_3\right)x^3 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)x^2 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)x + \widehat{W}_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} &= \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} \\ &\quad + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}. \end{aligned} \tag{3.3}$$

So

$$\begin{aligned} & \left(\widehat{W}_1 - \widehat{W}_0 - 2\widehat{W}_2 + \widehat{W}_3\right)x^3 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)x^2 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)x + \widehat{W}_0 \\ = \ & B_1(1-\beta x)(1-\gamma x)(1-\delta x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\ & + B_3(1-\alpha x)(1-\beta x)(1-\delta x) + B_3(1-\alpha x)(1-\beta x)(1-\gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get$\widehat{W}_0 + \frac{1}{\alpha^2}\left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right) - \frac{1}{\alpha^3}\left(\widehat{W}_0 - \widehat{W}_1 + 2\widehat{W}_2 - \widehat{W}_3\right) + \frac{1}{\alpha}\left(\widehat{W}_1 - 2\widehat{W}_0\right) = -B_1\left(\frac{1}{\alpha}\beta - 1\right)\left(\frac{1}{\alpha}\gamma - 1\right)\left(\frac{1}{\alpha}\delta - 1\right).$

This gives

$$\begin{aligned} B_1 &= \alpha^3(\widehat{W}_0 + \frac{1}{\alpha^2}\left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right) + \frac{1}{\alpha^3}\left(\widehat{W}_1 - 5\widehat{W}_0 - 4\widehat{W}_2 + \widehat{W}_3\right) + \frac{1}{\alpha}\left(\widehat{W}_1 - 2\widehat{W}_0\right)) \\ &= \frac{\widehat{W}_0\alpha^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\alpha^2 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)\alpha - \widehat{W}_0 + \widehat{W}_1 - 2\widehat{W}_2 + \widehat{W}_3}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{\widehat{W}_0\beta^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\beta^2 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)\beta - \widehat{W}_0 + \widehat{W}_1 - 2\widehat{W}_2 + \widehat{W}_3}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{\widehat{W}_0\gamma^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\gamma^2 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)\gamma - \widehat{W}_0 + \widehat{W}_1 - 2\widehat{W}_2 + \widehat{W}_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{\widehat{W}_0\delta^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\delta^2 + \left(\widehat{W}_0 - 2\widehat{W}_1 + \widehat{W}_2\right)\delta - \widehat{W}_0 + \widehat{W}_1 - 2\widehat{W}_2 + \widehat{W}_3}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty}\widehat{W}_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty}\widehat{W}_n x^n = B_1\sum_{n=0}^{\infty}\alpha^n x^n + B_2\sum_{n=0}^{\infty}\beta^n x^n + B_3\sum_{n=0}^{\infty}\gamma^n x^n + B_4\sum_{n=0}^{\infty}\delta^n x^n = \sum_{n=0}^{\infty}(B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$\widehat{W}_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

The following identity establishes a relationship between the dual hyperbolic Pandita numbers and the Pandita–Lucas numbers.

Corollary 9. *For all integers m, n the following identities holds:*

$$\widehat{W}_{m+n} = P_{m-2}\widehat{W}_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})\widehat{W}_{n+2} + (P_{m-3} - P_{m-4})\widehat{W}_{n+1} - \widehat{W}_n P_{m-3}.$$

Proof. First we assume that $m, n \geq 0$.The Theorem (9) can be proved by mathematical induction on m. If $m = 0$ we get

$$\widehat{W}_n = P_{-2}\widehat{W}_{n+3} + (P_{-4} - P_{-3} - P_{-5})\widehat{W}_{n+2} + (P_{-3} - P_{-4})\widehat{W}_{n+1} - \widehat{W}_n P_{-3}.$$

which is true since $P_{-2}=0, P=-1, P_{-4}=-1, P_{-5}=0$. Assume that the equality holds for $m\leq k$. For $m=k+1$, we get

$$\begin{aligned}\widehat{W}_{k+1+n} &= 2\widehat{W}_{n+k}-\widehat{W}_{n+k-1}+\widehat{W}_{n+k-2}-\widehat{W}_{n+k-3},\\ & 2(P_{m-2}\widehat{W}_{n+3}+(P_{m-4}-P_{m-3}-P_{m-5})\widehat{W}_{n+2}+(P_{m-3}-P_{m-4})\widehat{W}_{n+1}-\widehat{W}_nP_{m-3})\\ & -(P_{m-3}\widehat{W}_{n+3}+(P_{m-5}-P_{m-4}-P_{m-6})\widehat{W}_{n+2}+(P_{m-4}-P_{m-5})\widehat{W}_{n+1}-\widehat{W}_nP_{m-4})\\ & +(P_{m-4}\widehat{W}_{n+3}+(P_{m-6}-P_{m-5}-P_{m-7})\widehat{W}_{n+2}+(P_{m-5}-P_{m-6})\widehat{W}_{n+1}-\widehat{W}_nP_{m-5})\\ & -(P_{m-5}\widehat{W}_{n+3}+(P_{m-7}-P_{m-6}-P_{m-8})\widehat{W}_{n+2}+(P_{m-6}-P_{m-7})\widehat{W}_{n+1}-\widehat{W}_nP_{m-6}).\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 9.

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $\widehat{W}_n=\widehat{P}_n$ or $\widehat{W}_n=\widehat{S}_n$ in above Theorem, respectively, we get:

Corollary 10.

$$\begin{aligned}\widehat{P}_{m+n} &= P_{m-2}\widehat{P}_{n+3}+(P_{m-4}-P_{m-3}-P_{m-5})\widehat{P}_{n+2}+(P_{m-3}-P_{m-4})\widehat{P}_{n+1}-\widehat{P}_nP_{m-3},\\ \widehat{S}_{m+n} &= P_{m-2}\widehat{S}_{n+3}+(P_{m-4}-P_{m-3}-P_{m-5})\widehat{S}_{n+2}+(P_{m-3}-P_{m-4})\widehat{S}_{n+1}-\widehat{S}_nP_{m-3}.\end{aligned}$$

4. Simson's Formulas

In this section, we present Simson's formula for the dual hyperbolic generalized Pandita numbers . This is a special case of [16, Theorem 4.1].

Theorem 11. *(Simpson's formula for dual hyperbolic generalized Pandita numbers) For all integers n we have,*

$$\begin{vmatrix}\widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n\\ \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1}\\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2}\\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} & \widehat{W}_{n-3}\end{vmatrix} = \begin{vmatrix}\widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0\\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1}\\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2}\\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3}\end{vmatrix} = (\widehat{W}_3-2\widehat{W}_2+\widehat{W}_0)(\widehat{W}_3-2\widehat{W}_1+\widehat{W}_0)(\widehat{W}_3^2-\widehat{W}_2^2$$

$$+\widehat{W}_1^2-\widehat{W}_0^2-\widehat{W}_2\widehat{W}_3-2\widehat{W}_1\widehat{W}_3+\widehat{W}_1\widehat{W}_2+\widehat{W}_0\widehat{W}_3+2\widehat{W}_0\widehat{W}_2-\widehat{W}_0\widehat{W}_1).$$

Proof. Using Theorem 3 it can be proved by using induction use [16, Theorem 4.1]

From the Theorem 11 we get the following Corollary.

Corollary 12. *For all integers n, the Simson's formulas of dual hyperbolic Pandita numbers and dual hyperbolic Pandita Lucas numbers are given as respectively*

a): $$\begin{vmatrix}\widehat{P}_{n+3} & \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n\\ \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1}\\ \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2}\\ \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} & \widehat{P}_{n-3}\end{vmatrix} = 17+16j+115\varepsilon+260j\varepsilon.$$

b): $\begin{vmatrix} \widehat{S}_{n+3} & \widehat{S}_{n+2} & \widehat{S}_{n+1} & \widehat{S}_{n} \\ \widehat{S}_{n+2} & \widehat{S}_{n+1} & \widehat{S}_{n} & \widehat{S}_{n-1} \\ \widehat{S}_{n+1} & \widehat{S}_{n} & \widehat{S}_{n-1} & \widehat{S}_{n-2} \\ \widehat{S}_{n} & \widehat{S}_{n-1} & \widehat{S}_{n-2} & \widehat{S}_{n-3} \end{vmatrix} = 452 + 655j + 1125\varepsilon - 126j\varepsilon.$

5. Linear Sums

In this section, we give the summation formulas of the dual hyperbolic generalized Pandita numbers with positive and negatif subscripts.

Now, we present the summation formulas of the generalized Pandita numbers.

THEOREM 13. *For the generalized Pandita numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_k = -(n+3)W_{n+3} + (n+4)W_{n+2} + (n+4)W_n + 3W_3 - 4W_2 - 3W_0.$

(b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(-3(n+2)W_{2n+2} + (3n+8)W_{2n+1} + 2W_{2n} + (3n+7)W_{2n-1} + 7W_3 - 8W_2 - W_1 - 6W_0).$

(c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(3n+4)W_{2n+2} + (3n+8)W_{2n+1} + W_{2n} + 3(n+2)W_{2n-1} + 6W_3 - 8W_2 + W_1 - 7W_0).$

Proof. For the proof, see Soykan [18, Theorem 3.12]. □

THEOREM 14. *For the dual hyperbolic Pandita numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} \widehat{W}_k = -(n+3)\widehat{W}_{n+3} + (n+4)\widehat{W}_{n+2} + (n+4)\widehat{W}_n + 3\widehat{W}_3 - 4\widehat{W}_2 - 3\widehat{W}_0.$

(b): $\sum_{k=0}^{n} \widehat{W}_{2k} = \frac{1}{3}(-3(n+2)\widehat{W}_{2n+2} + (3n+8)\widehat{W}_{2n+1} + 2\widehat{W}_{2n} + (3n+7)\widehat{W}_{2n-1} + 7\widehat{W}_3 - 8\widehat{W}_2 - \widehat{W}_1 - 6\widehat{W}_0).$

(c): $\sum_{k=0}^{n} \widehat{W}_{2k+1} = \frac{1}{3}(-(3n+4)\widehat{W}_{2n+2} + (3n+8)\widehat{W}_{2n+1} + \widehat{W}_{2n} + 3(n+2)\widehat{W}_{2n-1} + 6\widehat{W}_3 - 8\widehat{W}_2 + \widehat{W}_1 - 7\widehat{W}_0).$

Proof. Use Theorem 13 and the definition of $\widehat{W}n$. □

As a special case of the theorem 14, we present the following Corollary.

COROLLARY 15. *For $n \geq 0$, dual hyperbolic Pandita numbers have the following properties:*

(a): $\sum_{k=0}^{n} \widehat{P}_k = -(n+3)\widehat{P}_{n+3} + (n+4)\widehat{P}_{n+2} + (n+4)\widehat{P}_n + 1 - 5j\varepsilon - 2\varepsilon.$

(b): $\sum_{k=0}^{n} \widehat{P}_{2k} = \frac{1}{3}(-3(n+2)\widehat{P}_{2n+2} + (3n+8)\widehat{P}_{2n+1} + 2\widehat{P}_{2n} + (3n+7)\widehat{P}_{2n-1} + 3j + \varepsilon - 3j\varepsilon + 4).$

(c): $\sum_{k=0}^{n} \widehat{P}_{2k+1} = \frac{1}{3}(-(3n+4)\widehat{P}_{2n+2} + (3n+8)\widehat{P}_{2n+1} + \widehat{P}_{2n} + 3(n+2)\widehat{P}_{2n-1} + j - 3\varepsilon - 8j\varepsilon + 3).$

COROLLARY 16. *For $n \geq 0$, dual hyperbolic Pandita Lucas numbers have the following properties.*

(a): $\sum_{k=0}^{n} \widehat{S}_k = -(n+3)\widehat{S}_{n+3} + (n+4)\widehat{S}_{n+2} + (n+4)\widehat{S}_n - 8j - 9\varepsilon - 10j\varepsilon - 5.$

(b): $\sum_{k=0}^{n} \widehat{S}_{2k} = \frac{1}{3}(-3(n+2)\widehat{S}_{2n+2} + (3n+8)\widehat{S}_{2n+1} + 2\widehat{S}_{2n} + (3n+7)\widehat{S}_{2n-1} + -12j - 16\varepsilon - 15j\varepsilon - 7).$

(c): $\sum_{k=0}^{n} \widehat{S}_{2k+1} = \frac{1}{3}(-(3n+4)\widehat{S}_{2n+2} + (3n+8)\widehat{S}_{2n+1} + \widehat{S}_{2n} + 3(n+2)\widehat{S}_{2n-1} + -16j - 15\varepsilon - 19j\varepsilon - 12).$

Next, we give the ordinary generating functions of some special cases of dual hyperbolic generalized Pandita numbers.

THEOREM 17. *The ordinary generating functions of the sequences $\widehat{W}_{2n}$, $\widehat{W}_{2n+1}$ are given as follows:*

(a): $\sum_{n=0}^{\infty} \widehat{W}_{2n}x^n = \dfrac{\widehat{W}_2\left(x^3+3x^2-x\right)+\widehat{W}_0\left(2x^2+2x-1\right)-\widehat{W}_1\left(x^2-x^3\right)-\widehat{W}_3\left(x^3+2x^2\right)}{-x^4-x^3+x^2+2x-1}.$

(b): $\sum_{n=0}^{\infty} \widehat{W}_{2n+1}x^n = \dfrac{\widehat{W}_0\left(x^3+2x^2\right)-\widehat{W}_3\left(x^3+x^2+x\right)-\widehat{W}_1\left(x^3-2x+1\right)+\widehat{W}_2\left(2x^3+x^2\right)}{-x^4-x^3+x^2+2x-1}.$

Proof. Similary, the proof can be constructed as in [4, Theorem 4].

From the last Theorem, we have the following Corollary which gives sum formula of dual hyperbolic Pandita numbers (Take $\widehat{W}_n = \widehat{P}_n$ whit $\widehat{P}_0 = j+2\varepsilon+3j\varepsilon, \widehat{P}_1 = 2j+3\varepsilon+5j\varepsilon+1, \widehat{P}_2 = 3j+5\varepsilon+8j\varepsilon+2, \widehat{P}_3 = 5j+8\varepsilon+12j\varepsilon+3$)

COROLLARY 18. *For $n \geq 0$ dual hyperbolic Pandita numbers have the following properties.*

(a): $\sum_{n=0}^{\infty} \widehat{P}_{2n}x^n = \dfrac{(j+5\varepsilon+4j\varepsilon)+(1-\varepsilon-j\varepsilon)x+(\varepsilon+j\varepsilon)x^2+(3j\varepsilon)x^3}{1-2x+x^2-x^3+x^4},$

(b): $\sum_{n=0}^{\infty} \widehat{P}_{2n+1}x^n = \dfrac{(2j+2\varepsilon+5j\varepsilon+4)+(\varepsilon-2j-6x-4j\varepsilon)x+(3j-2\varepsilon+2)x^2+(2\varepsilon-4j+8j\varepsilon+7)x^3}{1-2x+x^2-x^3+x^4}$

.

6. Matrices related with Dual Hyperbolic Generalized Pandita Numbers

In this sectiion, using dual hyperbolic Pandita numbers, we give some matrices related to dual hyperbolic Pandita numbers.

We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

uch that $detA = 1$. Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [20].

Then we give the following lemma.

Lemma 19. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} \widehat{W}_{n+3} \\ \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_{n} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}.$$

Proof. The identitiy(19) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{0} \begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identitiy is true

$$\begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_{k} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_{k} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{W}_{k+4} \\ \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by mathematical induction on n, the proof completed. □

We define

$$N_{\widehat{W}} = \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix}, \tag{6.1}$$

$$E_{\widehat{W}} = \begin{pmatrix} \widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\ \widehat{W}_n & \widehat{W}_{n-1-} & \widehat{W}_{n-2} & \widehat{W}_{n-3} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with $N_{\widehat{W}}$ and $E_{\widehat{W}}$

THEOREM 20. *Using* $N_{\widehat{W}}$ *and* $E_{\widehat{W}}$ *, we get*

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{\widehat{W}} &= \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$
\begin{aligned}
a_{11} &= \widehat{W}_1\left(P_n-P_{n-1}\right)-\widehat{W}_2\left(P_n-P_{n-1}+P_{n-2}\right)-\widehat{W}_0P_n+W_3P_{n+1}=\widehat{W}_{n+3},\\
a_{12} &= \widehat{W}_0\left(P_n-P_{n-1}\right)-\widehat{W}_1\left(P_n-P_{n-1}+P_{n-2}\right)-P_n\widehat{W}_{-1}+\widehat{W}_2P_{n+1}=\widehat{W}_{n+2},\\
a_{13} &= \widehat{W}_{-1}\left(P_n-P_{n-1}\right)-\widehat{W}_0\left(P_n-P_{n-1}+P_{n-2}\right)-P_n\widehat{W}_{-2}+\widehat{W}_1P_{n+1}=\widehat{W}_{n+1},\\
a_{14} &= \widehat{W}_{-2}\left(P_n-P_{n-1}\right)-\widehat{W}_{-1}\left(P_n-P_{n-1}+P_{n-2}\right)-P_n\widehat{W}_{-3}+\widehat{W}_0P_{n+1}=\widehat{W}_n,\\
a_{21} &= \widehat{W}_3P_n-\widehat{W}_2\left(P_{n-1}-P_{n-2}+P_{n-3}\right)+\widehat{W}\left(P_{n-1}-P_{n-2}\right)-\widehat{W}_0P_{n-1}=\widehat{W}_{n+2},\\
a_{22} &= \widehat{W}_2P_n-\widehat{W}_{-1}P_{n-1}-\widehat{W}_1\left(P_{n-1}-P_{n-2}+P_{n-3}\right)+\widehat{W}\left(P_{n-1}-P_{n-2}\right)=\widehat{W}_{n+1},\\
a_{23} &= \widehat{W}_{-1}\left(P_{n-1}-P_{n-2}\right)-\widehat{W}_{-2}P_{n-1}+\widehat{W}_1P_n-\widehat{W}_0\left(P_{n-1}-P_{n-2}+P_{n-3}\right)=\widehat{W}_n,\\
a_{24} &= \widehat{W}_{-2}\left(P_{n-1}-P_{n-2}\right)-\widehat{W}_{-3}P_{n-1}+\widehat{W}_0P_n-\widehat{W}_{-1}\left(P_{n-1}-P_{n-2}+P_{n-3}\right)=\widehat{W}_{n-1},\\
a_{31} &= \widehat{W}_1\left(P_{n-2}-P_{n-3}\right)-\widehat{W}_2\left(P_{n-2}-P_{n-3}+P_{n-4}\right)-\widehat{W}_0P_{n-2}+\widehat{W}_3P_{n-1}=\widehat{W}_{n+1},\\
a_{32} &= \widehat{W}_0\left(P_{n-2}-P_{n-3}\right)-\widehat{W}_1\left(P_{n-2}-P_{n-3}+P_{n-4}\right)-\widehat{W}_{-1}P_{n-2}+\widehat{W}_2P_{n-1}=\widehat{W}_n,\\
a_{33} &= \widehat{W}_{-1}\left(P_{n-2}-P_{n-3}\right)-\widehat{W}_{-2}P_{n-2}-\widehat{W}_0\left(P_{n-2}-P_{n-3}+P_{n-4}\right)+\widehat{W}_1P_{n-1}=\widehat{W}_{n-1},\\
a_{34} &= \widehat{W}_{-2}\left(P_{n-2}-P_{n-3}\right)-\widehat{W}_{-3}P_{n-2}-\widehat{W}_{-1}\left(P_{n-2}-P_{n-3}+P_{n-4}\right)+\widehat{W}_0P_{n-1}=\widehat{W}_{n-2},\\
a_{41} &= \widehat{W}_1\left(P_{n-3}-P_{n-4}\right)-\widehat{W}_2\left(P_{n-3}-P_{n-4}+P_{n-5}\right)-\widehat{W}_0P_{n-3}+\widehat{W}_3P_{n-2}=\widehat{W}_n,\\
a_{42} &= \widehat{W}_0\left(P_{n-3}-P_{n-4}\right)-\widehat{W}_1\left(P_{n-3}-P_{n-4}+P_{n-5}\right)-\widehat{W}_{-1}P_{n-3}+\widehat{W}_2P_{n-2}=\widehat{W}_{n-1},\\
a_{43} &= \widehat{W}_{-1}\left(P_{n-3}-P_{n-4}\right)-\widehat{W}_{-2}P_{n-3}-\widehat{W}_0\left(P_{n-3}-P_{n-4}+P_{n-5}\right)+\widehat{W}_1P_{n-2}=\widehat{W}_{n-2},\\
a_{44} &= \widehat{W}_{-2}\left(P_{n-3}-P_{n-4}\right)-\widehat{W}_{-3}P_{n-3}-\widehat{W}_{-1}\left(P_{n-3}-P_{n-4}+P_{n-5}\right)+\widehat{W}_0P_{n-2}=\widehat{W}_{n-3}.
\end{aligned}
$$

Using the theorem (9) the proof is done. □

By taking $\widehat{W}_n = \widehat{P}_n$ with $\widehat{P}_0, \widehat{P}_1, \widehat{P}_2, \widehat{P}_3$ in (6.1) and (6.2)

$\widehat{W}_n = S_n$ with $\widehat{S}_0, \widehat{S}_1, \widehat{S}_2, \widehat{S}_3$ in (6.1) and (6.2) respectively, we get:

$$N_{\widehat{P}} = \begin{pmatrix} 5j+8\varepsilon+12j\varepsilon+3 & 3j+5\varepsilon+8j\varepsilon+2 & 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon \\ 3j+5\varepsilon+8j\varepsilon+2 & 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon \\ 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon & -j\varepsilon \\ j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon & -j\varepsilon & -1 \end{pmatrix},$$

$$E_{\widehat{P}} = \begin{pmatrix} \widehat{P}_{n+3} & \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_{n} \\ \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_{n} & \widehat{P}_{n-1} \\ \widehat{P}_{n+1} & \widehat{P}_{n} & \widehat{P}_{n-1} & \widehat{P}_{n-2} \\ \widehat{P}_{n} & \widehat{P}_{n-1} & \widehat{P}_{n-2} & \widehat{P}_{n-3} \end{pmatrix},$$

$$N_{\widehat{S}} = \begin{pmatrix} 6j+7\varepsilon+11j\varepsilon+5 & 5j+6\varepsilon+7j\varepsilon+2 & 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 \\ 5j+6\varepsilon+7j\varepsilon+2 & 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 & 4j+2\varepsilon+2j\varepsilon+1 \\ 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 & -4j+2\varepsilon+2j\varepsilon+1 & j+4\varepsilon+2j\varepsilon-1 \\ 2j+2\varepsilon+5j\varepsilon+4 & -4j+2\varepsilon+2j\varepsilon+1 & j+4\varepsilon+2j\varepsilon-1 & \varepsilon-j+4j\varepsilon+4 \end{pmatrix},$$

$$E_{\widehat{S}} = \begin{pmatrix} \widehat{S}_{n+3} & \widehat{S}_{n+2} & \widehat{S}_{n+1} & \widehat{S}_{n} \\ \widehat{S}_{n+2} & \widehat{S}_{n+1} & \widehat{S}_{n} & \widehat{S}_{n-1} \\ \widehat{S}_{n+1} & S_{n} & \widehat{S}_{n-1} & \widehat{S}_{n-2} \\ \widehat{S}_{n} & \widehat{S}_{n-1} & \widehat{S}_{n-2} & \widehat{S}_{n-3} \end{pmatrix}.$$

From Theorem [20], we can write the following corollary.

Corollary 21. *The following identities are hold:*

- **a):** $A^n N_{\widehat{P}} = E_{\widehat{P}}$.
- **b):** $A^n N_{\widehat{S}} = E_{\widehat{S}}$.

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