INVESTIGATION OF ONE-POINT COMPACTIFICATION IN SEMI-NORMAL SPACES

Abstract

Let X be a non-empty set and (X, τ) be a semi-normal space. In this paper, we investigated the relationship between one-point compactification and semi-normal spaces under the fram.work of semi-open sets. In addition, we in particular proved that if (X, τ) is a semi-normal space, then its one-point compactification, X^* is also semi-normal. We also extended our work on establishing that, if (X, τ) is a semi-normal space, then its one-point compactification, X^* of X is compact if and only if (X, τ) is also Hausdorff.

Key Words and Phrases: Topological space, semi-open set, cover of a set, compact set, one-point compactification and semi-normal space

1 Introduction

A lot of research on the properties of topological spaces has been done and many results established. The properties (normality[1], regularity [1], semi regularity [8], connectedness[12] , compactification [5], etc.) have been studied and internal characterization of some spaces like Tychonoff spaces, established. Frink [2] described compactification with regard to Wallman base in a Wallman space $w(\mathscr{Z})$ under the framework of ultrafilters. Later, Piekorsz [6] characterized One-point compactification with regard to Wallman base $\mathcal{C}$ under the framework of generalized topological spaces. Marcus [11] established Schwartz functions and compactification. Recently, Alrababah [7] improved the concept of D-paracompactness under the framework of D-sets. In addition, Alrababah [7] studied the different characteristics of D-paracompactness and how they relate with topological characteristics. In this note, we have characterized one-point compactification with respect to semi-normal spaces under the framework of semi-open sets.

2 Preliminary Notes

Definition 1. Consider a space (X, τ), where X is non-empty set and τ is a topology on X if it satisfies the following properties:
(i). $\emptyset, X \in \tau$.
(ii). The arbitrary union of sets in τ belong to τ.
(iii). Any finite intersection of sets of τ belong to τ.

We made some definitions which are instrumental to this present paper.

Definition 2. ([4], Definition 3.1)
Let (X, τ) be a topological space. If every open covering in X has a countable sub-covering, then the space is called Lindelöf space.

Definition 3. ([10], Definition 7.1.7)
A subset A of a topological space (X, τ) is said to be compact if, every open covering of A has a finite sub-covering. If the compact subset A equals to X then, (X, τ) is said to be compact space.

Definition 4. , ([3], Definition1)
A topological space (X, τ) is said to be semi-normal if for each pair of disjoint semi-closed sets $A, B \subseteq X$, there exist disjoint semi-open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Definition 5. ([10], Definition 7.1.7)
Let (X, τ) be a semi-normal and discrete topological space and (Y, τ_Y) be a subspace of (X, τ). Let $f : X \longrightarrow Y$ be an embedding, that is f is a homeomorphism and $\overline{f(x)} = X$, $f(x) \subseteq Y$. Then (Y, τ_Y) is a compactification of (X, τ).

Definition 6. ([13], Definition 1.3)
A topological space (X, τ) is said to be locally compact at a point $x \in X$ if, x lies in the interior of some compact subset of X.

Definition 7. (T_1 axiom, [9])
A space (X, τ) is said to have the T_1 property if x and y are distinct points of X, there exists an open sets P and Q such that P contains x but not y, and the open set Q contains y but not x.

Definition 8. (Completely regular axiom, [9])
A space (X, τ) is said to be completely regular (or Tychonoff) if it is both $T_{3\frac{1}{2}}$ and T_1.

3 One-Point compactification

We begin our results with proposition 9 which is a basis for the main results.

Proposition 9. One-Point Compactification X^* of locally compact Semi-Normal Space is Compact Normal Space.

Proof. Let X be a locally compact semi normal space and A and B be disjoint closed subsets of X, such that there exist disjoint open sets U and V in X such that $\bar{A}x \subseteq U$ and $\bar{B}x \subseteq V$. Let X^* be one-point compactification of X with ∞ representing the point at ∞. Considering two scenarios of U and V with respect to ∞:
(i). One or both U and V contain ∞. In this case, extending them with $\{\infty\}$ does not affect their disjointness. ie. If $U \cap \{\infty\} \neq \emptyset$ or $V \cap \{\infty\} \neq \emptyset$ i.e. ∞ belong to either U or V, then extending U and V with $\{\infty\}$ does not change their relative positions. $A \cup \{\infty\} \subseteq U \cup \{\infty\}$ and $B \cup \{\infty\} \subseteq V \cup \{\infty\}$. Since U and V were disjoint $U \cup \{\infty\}$ and $V \cup \{\infty\}$ remain disjoint open sets in X^*.
(ii). Neither U nor V contain ∞: Utilize the local compactness of X. Since X is locally compact $\forall\, x \in X$ has compact neighborhood. i.e $x \in X \setminus (U \cup V)$, $\exists$ a compact neighborhood K_x such that $K_x \subseteq X \setminus (U \cup V)$ and does not contain ∞. Let $K_u = U \cap (\cup\{K_x : x \in X \setminus U\})$ and $K_v = V \cap (\cup\{K_x : x \in X \setminus V\})$. These sets are compact because they are intersections of compact sets and moreover, $K_u \subseteq U \setminus \{\infty\}$ and $K_v \subseteq V \setminus \{\infty\}$. These guarantee the existence of disjoint open sets in X^* i.e $(U \cup \{\infty\}) \cup K_u$ and $(V \cup \{\infty\}) \cup K_v$.
$\Rightarrow X^*$ is compact by definition of one point compactification and the existence of disjoint open sets $(U \cup \{\infty\}) \cup K_u$ and $(V \cup \{\infty\}) \cup K_v$ in X^*, shows that X^* is normal which completes the proof. □

Theorem 10. *If (X, τ) is a semi-normal space, then its one-point compactification, X^* is also semi-normal.* .

Proof. Let X be a semi normal space with topology τ and $A, B \subseteq X$ be disjoint closed sets. By proposition [9] we consider two scenarios: (i).$A \cap \{\infty\} \neq \emptyset$ and $B \cap \{\infty\} \neq \emptyset$. (i.e. A contains ∞ and B is entirely in X), then the semi-normality of X guarantees the existence of disjoint sets U and V in X such that $\bar{A}_x \subseteq U$ and $\bar{B}_x \subseteq V$. Extending U and V with $\{\infty\}$ in X^* creates disjoint sets used in $U \cup \{\infty\}$ and V in X^* that separates A and B.
$\Rightarrow A \subseteq A \cup \{\infty\}$ and $B \subseteq V$.
(ii). Both sets contain ∞: Analyzing the open sets used in X, consider the case when the original open sets are used to separate A and B, already contain ∞, extending them with $\{\infty\} \in X^*$ would maintain their disjoint property i.e, Let U' and V' be two original open sets in X. Then $U' \cup \{\infty\}$ and $V' \cup \{\infty\}$ are disjoint open sets separating A and B.

(iii). Both sets A and B are entirely contained in X i.e Neither A nor B contain ∞: Since X is semi-normal, $\exists$ disjoint open sets U and V in X such that $A \subseteq U$, $B \subseteq V$, then U and V are also disjoint open in X^*. Therefore, for a semi-normal space X, its One-point compactification X^* is semi-normal. □

Theorem 11. *If (X, τ) is a semi-normal space, then its one-point compactification, X^* of X is compact if and only if (X, τ) is also Hausdorff.*

Proof. Let X be a Hausdorff and semi-normal space. We want to show that its one point compactification X^* is compact. From definition [5], to show that $X^*$is compact, we consider any open cover $\{U_i\}_{i \in I}$ of X^*. We split this in to 2 cases:
Case 1: Some open set U_j in the cover contains ∞. If one of the open set contain ∞, the remaining set covers X. Moreover, since X is locally compact and Hausdorff, and X is covered by open sets, $\exists$a finite subcover that covers all points in X. Hence, combining the subcover with the set U_j that contain ∞, we have a finite subcover for X^*, proving that X^* is compact in this case.
case 2: No open set U_i contains ∞.
Here the open cover $\{U_i\}$ only covers points $X^* \setminus \{\infty\}$, which is homeomorphic to X. Since X is locally compact and Haudorff, every point in X has a compact neighborhood. Because $X^* \setminus \{\infty\}$ is homeomorphic to X, $\exists$ a finite subcover that covers all of X. Since ∞ is not covered by any U_i, this leads to a contradiction (as all open cover include ∞).
$\Rightarrow$ We can always find a finite subcover for any open cover of X^*. This shows that X^* is compact.
Conversely:
Assume that the one-point compactification X^* of X is compact. We need to show that X is Hausdorff. Suppose X is not Hausdorff. Then, $\exists$ distinct points $x, y \in X$ that cannot be separated by disjoint open sets.
$\Rightarrow$ no open neighborhoods of x and y are disjoint.
$\Rightarrow$ the space X does not satisfy the separation axioms [7 and 8]. However, X^* is compact and in a compact space, distinct points can always be separated by disjoint open sets because compact spaces are normal (by Tychonoff theorem).
Since X^* is compact and normal, the points x and $y \in X$ must be

separable by disjoint open sets, contradicting our assumption that X is not Hausdorff.
Thus, if X^* is compact, X must be Hausdorff, completing the proof. $\square$

Example 12. Let $X = \mathbf{Q}$, the space of rational numbers, with the subspace topology from $\mathbf{R}$. X is locally compact, non compact and Hausdorff. $X^* = \{\mathbf{Q} \cup \infty\}$, which is compact. The topology $\tau^* = \{X^* \setminus G : G \subseteq \mathbf{Q}\}$. X^* is compactified and semi-normal because X is locally compact and Hausdorff and we can separate compact sets from closed sets.

4 Conclusion

Finally, the results we discussed, outline clearly the relationship between semi normal space and one-point compactification.

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