Dual Hyperbolic Generalized Adrien Numbers

Abstract. This study introduces the generalized dual hyperbolic Adrien numbers, a novel extension of the classic Adrien framework, enriched by dual and hyperbolic algebraic structures. These sequences are constructed within a fourth-order linear recurrence system, offering intricate mathematical behavior and promising structural versatility. Special attention is devoted to two distinguished cases: the dual hyperbolic Adrien and dual hyperbolic Adrien–Lucas numbers, each revealing unique interrelations between dual numbers, hyperbolic units, and classical integer sequences.

For each class, explicit Binet-type expressions are derived, enabling direct computation and closed-form analysis. Ordinary and exponential generating functions are presented to encapsulate the sequences' evolution and provide analytical tools for combinatorial and algebraic exploration. Summation formulas are established to link consecutive terms and identify structural patterns across the sequences. Matrix representations are also constructed, encoding the recurrence relations and offering compact formulations suitable for computational applications.

The proposed families of numbers serve not only as theoretical constructs but also as candidates for deeper investigations into symbolic computation, algebraic identities, and discrete dynamical models. Their formulation opens doors for applications in areas such as cryptography, quantum computation, and signal processing, where dual and hyperbolic systems are gaining renewed interest.

2010 Mathematics Subject Classification. 11B37, 11B39, 11B83.

Keywords. Adrien numbers, Adrien-Lucas numbers, dual hyperbolic Adrien numbers, dual hyperbolic Adrien-Lucas numbers.

1. Introduction

Sequences defined by linear recurrence relations play a fundamental role in discrete mathematics, computer science, and theoretical physics. An *m-order linear recurrence relation* with constant coefficients takes the general form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m}, \quad \text{for } n \geq m, \tag{1.1}$$

where $c_1, c_2, \ldots, c_m \in \mathbb{R}$ (or $\mathbb{C}$), and a_n denotes the n-th term of the sequence. The initial values $a_0, a_1, \ldots, a_{m-1}$ must be specified to uniquely determine the sequence.

Characteristic Equation and General Solution. To analyze such relations, one typically considers the *characteristic polynomial* associated with Equation (1.1):

$$P(x) = x^m - c_1 x^{m-1} - c_2 x^{m-2} - \cdots - c_m. \tag{1.2}$$

Suppose the roots of $P(x)$ are $r_1, r_2, \ldots, r_k$, with respective multiplicities $m_1, m_2, \ldots, m_k$ such that $m_1 + m_2 + \cdots + m_k = m$. Then the general solution to the recurrence relation is expressed using the Binet-type formula:

$$a_n = \sum_{j=1}^{k} \left(\sum_{i=0}^{m_j - 1} \alpha_{j,i} n^i \right) r_j^n, \tag{1.3}$$

where $\alpha_{j,i}$ are constants determined by the initial conditions. This closed-form expression circumvents recursion and allows for direct computation of a_n.

Generating Functions. An alternative and powerful method for studying recurrence relations involves the use of *generating functions*. The ordinary generating function (OGF) of a sequence $\{a_n\}_{n\geq 0}$ is defined as:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{1.4}$$

Substituting the recurrence into the series leads to an algebraic equation for $G(x)$, typically of the form:

$$G(x) = \frac{P_0(x)}{Q(x)}, \tag{1.5}$$

where $P_0(x)$ encodes the initial conditions and $Q(x)$ is derived from the recurrence relation coefficients, often matching the characteristic polynomial $P(x)$. Generating functions simplify convolution operations, enable combinatorial interpretations, and allow extraction of closed-form expressions via partial fraction decomposition or coefficient extraction techniques.

Special Cases of Linear Recurrence Relations. We now examine specific instances of linear recurrence relations for orders $m = 2, 3, 4, 5$. These cases illustrate how the general theory adapts to concrete examples and provide insight into the structure and solvability of such relations.

A second-order linear recurrence relation has the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad n \geq 2. \tag{1.6}$$

Characteristic Equation:

$$x^2 - c_1 x - c_2 = 0$$

Let the roots be r_1 and r_2. Then the general solution is:

$$a_n = \alpha r_1^n + \beta r_2^n$$

where α, β are determined by initial conditions a_0, a_1.

Generating Function:

$$G(x) = \frac{a_0 + (a_1 - c_1 a_0)x}{1 - c_1 x - c_2 x^2}$$

A third-order recurrence takes the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}, \quad n \geq 3. \tag{1.7}$$

Characteristic Equation:

$$x^3 - c_1 x^2 - c_2 x - c_3 = 0$$

Let the roots be r_1, r_2, r_3. Then the general solution is:

$$a_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n$$

Generating Function:

$$G(x) = \frac{P_0(x)}{1 - c_1 x - c_2 x^2 - c_3 x^3}$$

where $P_0(x)$ encodes the initial conditions a_0, a_1, a_2.

A fourth-order recurrence is given by:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4}, \quad n \geq 4. \tag{1.8}$$

Characteristic Equation:

$$x^4 - c_1 x^3 - c_2 x^2 - c_3 x - c_4 = 0$$

Let the roots be r_1, r_2, r_3, r_4. Then the general solution is:

$$a_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n + \delta r_4^n$$

Generating Function:

$$G(x) = \frac{P_0(x)}{1 - c_1 x - c_2 x^2 - c_3 x^3 - c_4 x^4}$$

A fifth-order recurrence relation is:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4} + c_5 a_{n-5}, \quad n \geq 5. \tag{1.9}$$

Characteristic Equation:

$$x^5 - c_1 x^4 - c_2 x^3 - c_3 x^2 - c_4 x - c_5 = 0$$

Let the roots be r_1, r_2, r_3, r_4, r_5. Then the general solution is:

$$a_n = \sum_{j=1}^{5} \alpha_j r_j^n$$

Generating Function:

$$G(x) = \frac{P_0(x)}{1 - c_1x - c_2x^2 - c_3x^3 - c_4x^4 - c_5x^5}$$

Remark 1. *If the characteristic polynomial has repeated roots, the general solution includes polynomial terms multiplied by powers of the root. Generating functions provide a compact representation and are especially useful for extracting asymptotic behavior and closed-form expressions.*

1.1. Special Cases of Linear Recurrence Sequences. We now present notable examples of linear recurrence sequences, each defined by specific initial conditions and recurrence relations. For each, we provide the recurrence, initial terms, Binet-type formula (when available), and generating function.

- Fibonacci Sequence
  - Recurrence:
  $$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1$$
  - Binet's Formula: Let $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$. Then:
  $$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$
  - Generating Function:
  $$G(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$
- Lucas Sequence
  - Recurrence:
  $$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \quad L_1 = 1$$
  - Binet's Formula:
  $$L_n = \phi^n + \psi^n$$
  - Generating Function:
  $$G(x) = \sum_{n=0}^{\infty} L_n x^n = \frac{2 - x}{1 - x - x^2}$$
- Pell Sequence
  - Recurrence:
  $$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \quad P_1 = 1$$
  - Binet's Formula:Let $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$. Then:
  $$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$$
  - Generating Function:
  $$G(x) = \frac{x}{1 - 2x - x^2}$$
- Jacobsthal Sequence

- Recurrence:

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \quad J_1 = 1$$

- Binet's Formula: Let $\lambda_1 = \frac{1+\sqrt{3}}{2}$, $\lambda_2 = \frac{1-\sqrt{3}}{2}$. Then:

$$J_n = \frac{2^n - (-1)^n}{3}$$

- Generating Function:

$$G(x) = \frac{x}{1 - x - 2x^2}$$

- Tribonacci Sequence
  - Recurrence:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad T_0 = 0, \quad T_1 = 0, \quad T_2 = 1$$

  - Generating Function:

$$G(x) = \frac{x^2}{1 - x - x^2 - x^3}$$

  - Binet's Formula: Let r_1, r_2, r_3 be roots of $x^3 - x^2 - x - 1 = 0$. Then:

$$T_n = a_1 r_1^n + a_2 r_2^n + a_3 r_3^n$$

  where a_i are constants determined by initial conditions.
- Narayana Sequence
  - Recurrence:

$$N_n = N_{n-1} + N_{n-3}, \quad N_0 = 0, \quad N_1 = 1, \quad N_2 = 1$$

  - Generating Function:

$$G(x) = \frac{x}{1 - x - x^3}$$

  - Binet's Formula: Let r_1, r_2, r_3 be roots of $x^3 - x^2 - 1 = 0$. Then:

$$N_n = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n$$

- Padovan Sequence
  - Recurrence:

$$P_n = P_{n-2} + P_{n-3}, \quad P_0 = 1, \quad P_1 = 1, \quad P_2 = 1$$

  - Generating Function:

$$G(x) = \frac{x^2 + x + 1}{1 - x^2 - x^3}$$

  - Binet's Formula: Let r_1, r_2, r_3 be roots of $x^3 - x - 1 = 0$. Then:

$$P_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$$

- Tetranacci Sequence

– Recurrence:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}, \quad T_0 = 0, \quad T_1 = 0, \quad T_2 = 0, \quad T_3 = 1$$

– Generating Function:

$$G(x) = \frac{x^3}{1 - x - x^2 - x^3 - x^4}$$

– Binet's Formula: Let r_1, r_2, r_3, r_4 be roots of $x^4 - x^3 - x^2 - x - 1 = 0$. Then:

$$T_n = \sum_{i=1}^{4} d_i r_i^n$$

- Pentanacci Sequence
  – Recurrence:

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \quad P_0 = 0, \quad P_1 = 0, \quad P_2 = 0, \quad P_3 = 0, \quad P_4 = 1$$

– Generating Function:

$$G(x) = \frac{x^4}{1 - x - x^2 - x^3 - x^4 - x^5}$$

– Binet's Formula: Let r_1, r_2, r_3, r_4, r_5 be roots of $x^5 - x^4 - x^3 - x^2 - x - 1 = 0$. Then:

$$P_n = \sum_{i=1}^{5} e_i r_i^n$$

2. Background on Hypercomplex Number Systems

Hypercomplex number systems, as introduced by Kantor, [10], constitute algebraic extensions of the real number system.Among the commutative instances of such systems are the complex numbers, defined as:

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

as well as the hyperbolic numbers (also referred to as double or split-complex numbers), [8],

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

and dual numbers, [15],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Quaternions, introduced by Hamilton, extend complex numbers into a four-dimensional non-commutative algebra over the real numbers, [22],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

The octonions [11] and sedenions [24] exemplify higher-dimensional hypercomplex number systems.. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are structured as real algebras derived from the real number system $\mathbb{R}$ via a recursive doubling procedure known as the Cayley–Dickson Process.This iterative method allows for the construction of successive 2^n-dimensional algebras, extending beyond sedenions to form generalized entities referred to as 2^n-ions [5], [13], [7].

Quaternions were first formulated by the Irish mathematician W. R. Hamilton (1805–1865) [22] as a non-commutative generalization of the complex numbers. In 1848, J. Cockle introduced the notion of hyperbolic numbers with complex coefficients [12].Later, H. H. Cheng and S. Thompson [9] extended this framework by defining dual numbers with complex coefficients, which they termed complex dual numbers. More recently, Akar, Yüce, and Şahin [14] proposed the algebraic system of dual hyperbolic numbers, further enriching the landscape of generalized number systems.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and addition of dual hyperbolic numbers is defined as componentwise.

The algebra of dual hyperbolic numbers forms a commutative ring, a real vector space, and a real algebra. However, the structure denoted by $\mathbb{H}_{\mathbb{D}}$ does not constitute a field, as not every dual hyperbolic number possesses a multiplicative inverse. For further details on the algebraic properties and construction of dual hyperbolic numbers, the reader is referred to[14].

To lay the groundwork for subsequent analysis, we first recount the established definition of generalized Adrien numbers, as presented in the existing literature.

3. Background on Generalized Adrien Sequence

A generalized Adrien sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4},\ n \geq 4, \tag{3.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (3.1) holds for all integer n. Soykan has investigated this specific numerical sequence in a recent study, for more details, see [16].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 3z^3 + z^2 + 1 = (z^3 - 2z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\delta &= 1.
\end{aligned}$$

Where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 3, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Note also that

$$\begin{aligned}
\alpha + \beta + \gamma &= 2, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

Using the roots and the recurrence relation of $\{W_n\}$ the Binet's formula for the generalized Adrien numbers can be expressed for all integers n as follows

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{4\alpha^2+3\alpha-1}+\frac{p_2\beta^n}{4\beta^2+3\beta-1}+\frac{p_3\gamma^n}{4\gamma^2+3\gamma-1}+\frac{p_4\delta^n}{3} \\ &= S_1\alpha^n+S_2\beta^n+S_3\gamma^n+S_4\delta^n. \end{aligned} \tag{3.2}$$

Where p_1, p_2, p_3 and p_4 are given below

$$\begin{aligned} p_1 &= (\alpha W_3-\alpha(3-\alpha)W_2+(-\alpha^2+(3-1)\alpha+1)W_1-1W_0), \\ p_2 &= (\beta W_3-\beta(3-\beta)W_2+(-\beta^2+(3-1)\beta+1)W_1-1W_0), \\ p_3 &= (\gamma W_3-\gamma(3-\gamma)W_2+(-\gamma^2+(3-1)\gamma+1)W_1-1W_0), \\ p_4 &= -(W_3-2W_2-W_1-W_0). \end{aligned}$$

And

$$\begin{aligned} S_1 &= \frac{p_1}{4\alpha^2+3\alpha-1}, \\ S_2 &= \frac{p_2}{4\beta^2+3\beta-1}, \\ S_3 &= \frac{p_3}{4\gamma^2+3\gamma-1}, \\ S_4 &= -\frac{(W_3-2W_2-W_1-W_0)}{3}. \end{aligned} \tag{3.3}$$

Binet's formula of Adrien and Adrien-Lucas sequences are

$$A_n=\frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1}+\frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1}+\frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1}-\frac{1}{3},$$

and

$$B_n=\alpha^n+\beta^n+\gamma^n+1.$$

respectively.

If we set $W_0=0, W_1=1, W_2=3, W_3=8$ then $\{W_n\}$ is the well-known Adrien sequence and if we set $W_0=4, W_1=3, W_2=7, W_3=18$ then $\{W_n\}$ is the well-known Lucas sequence. In other words, Adrien sequence $\{A_n\}_{n\geq 0}$ and Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations as;

$$A_n = 3A_{n-1}-A_{n-2}-A_{n-4}\ ,\ A_0=0, A_1=1, A_2=3, A_3=8,\ n\geq 4, \tag{3.4}$$

$$B_n = 3B_{n-1}-B_{n-2}-B_{n-4}\ ,\ B_0=4, B_1=3, B_2=7, B_3=18,\ n\geq 4. \tag{3.5}$$

The sequences $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, can be extended to negative subscripts by defining,

$$
\begin{aligned}
A_{-n} &= -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)}, \\
B_{-n} &= -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)},
\end{aligned}
$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (3.4),(3.5) hold for all integer n. Binet's formulas as follows.

Table 1 presents the initial generalized Adrien numbers corresponding to both positive and negative subscripts.

Table 1. A few generalized Adrien numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_2 - W_1 - W_3$
2	W_2	$3W_1 - W_0 - W_2$
3	W_3	$3W_0 - 3W_2 + W_3$
4	$3W_3 - W_2 - W_0$	$10W_2 - 6W_1 - 3W_3$
5	$8W_3 - W_1 - 3W_2 - 3W_0$	$10W_1 - 6W_0 - 3W_2$
6	$21W_3 - 3W_1 - 9W_2 - 8W_0$	$10W_0 + 3W_1 - 18W_2 + 6W_3$
7	$54W_3 - 8W_1 - 24W_2 - 21W_0$	$3W_0 - 28W_1 + 36W_2 - 10W_3$
8	$138W_3 - 21W_1 - 62W_2 - 54W_0$	$33W_1 - 28W_0 - W_2 - 3W_3$

After then we can write the generating function of generalized Adrien numbers,

$$
\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + W_0)x^2 + (W_3 - 3W_2 + W_1)x^3}{1 - 3x + x^2 + x^4}.
$$

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

For more details about generalized Adrien numbers, see [16].

Lemma 2. *[4].Suppose that* $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Adrien sequence* $\{W_n\}$.

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by:

$$
\begin{aligned}
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} e^{\alpha x} \\
&+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} e^{\beta x} \\
&+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} e^{\gamma x} + (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x}.
\end{aligned}
$$

The previous Lemma 2 gives the following results as particular examples.

COROLLARY 3. *Exponential generating function of Adrien and Adrien-Lucas numbers are given by:*

a): $\sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} ((\frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3})\frac{x^n}{n!}$
$= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^x).$

b): $\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n+\beta^n+\gamma^n+1)\frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

Subsequent sections offer a comprehensive overview of the scholarly literature addressing hyperbolic and dual hyperbolic numbers.

- Cockle [12] presented the hyperbolic numbers with complex coefficients.
- Akar at al [14] introduced the dual hyperbolic numbers.
- Cheng and Thompson[9] studied dual numbers with complex coefficients.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan at al [17] introduced dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2},\ (n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Cihan at al [2] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$

$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}.$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1, L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1$.

- Soykan at al [18] introduced dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}, J_0 = a, J_1 = b$.

- Bród at al [1] studied dual hyperbolic generalized Balancing given by

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}, B_0 = 0, B_1 = 1$.

- Yılmaz and Soykan [23] introduced dual hyperbolic generalized Guglielmo numbers given by

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$.

- Dikmen [3] introduced dual hyperbolic generalised Leonardo numbers given by

$$\widehat{l}_0 = l_0 + jl_1 + \varepsilon l_2 + j\varepsilon l_3$$

$l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3$.

- Eren and Soykan [6] introduced dual hyperbolic generalized Woodall numbers given by

$$\widehat{R}_0 = R_0 + jR_1 + \varepsilon R_2 + j\varepsilon R_3$$

where $R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3},\ R_0 = -1, R_1 = 1, R_2 = 7$.

In this study, we present the dual hyperbolic generalized Adrien numbers and provide a comprehensive analysis in the subsequent section. Key structural properties are also explored, highlighting their broader mathematical significance.

4. Dual Hyperbolic Generalized Adrien Numbers and their Generating Functions and Binet's Formulas

In this section, we introduce the dual hyperbolic generalized Adrien numbers and present their corresponding generating functions and Binet-type formulas. Specifically, we define the dual hyperbolic generalized Adrien numbers over the hyperbolic dual number system $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Adrien number is given by:

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}, \tag{4.1}$$

with the initial values $\widehat{W}_0, \widehat{W}_1, \widehat{W}_2, \widehat{W}_3$, (4.1) can be written to negative subscripts by defining,

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3}, \tag{4.2}$$

so identity (4.1) holds for all integers n.

As special cases, the nth dual hyperbolic Adrien numbers and the nth dual hyperbolic Adrien-Lucas numbers are given as.

$$\widehat{A}_n = A_n + jA_{n+1} + \varepsilon A_{n+2} + j\varepsilon A_{n+3}, \tag{4.3}$$

and

$$\widehat{B}_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}, \tag{4.4}$$

respectively. It can be easily shown that

$$\widehat{A} = 3\widehat{A}_{n-1} - \widehat{A}_{n-2} - \widehat{A}_{n-4}\ ,$$

and

$$\widehat{B} = 3\widehat{B}_{n-1} - \widehat{B}_{n-2} - \widehat{B}_{n-4}.$$

The sequence $\{\widehat{W}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{A}_{-n} = -\widehat{A}_{-(n-2)} + 3\widehat{A}_{-(n-3)} - \widehat{A}_{-(n-4)}.$$

$$\widehat{B}_{-n} = -\widehat{B}_{-(n-2)} + 3\widehat{B}_{-(n-3)} - \widehat{B}_{-(n-4)}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (4.1) holds for all integer n.

Table 2 presents the initial values of the dual hyperbolic generalized Adrien numbers for both positive and negative indices.

Table 2. A few dual hyperbolic generalized Adrien numbers

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$3\widehat{W}_2 - \widehat{W}_1 - \widehat{W}_3$
2	$\widehat{W}_2$	$3\widehat{W}_1 - \widehat{W}_0 - \widehat{W}_2$
3	$\widehat{W}_3$	$3\widehat{W}_0 - 3\widehat{W}_2 + \widehat{W}_3$
4	$3\widehat{W}_3 - \widehat{W}_2 - \widehat{W}_0$	$10\widehat{W}_2 - 6\widehat{W}_1 - 3\widehat{W}_3$
5	$8\widehat{W}_3 - \widehat{W}_1 - 3\widehat{W}_2 - 3\widehat{W}_0$	$10\widehat{W}_1 - 6\widehat{W}_0 - 3\widehat{W}_2$

Note that

$$\begin{aligned}
\widehat{W}_0 &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3 = W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3, \\
\widehat{W}_1 &= W_1 + jW_2 + \varepsilon W_3 + j\varepsilon W_4 = W_1 + jW_2 + \varepsilon W_3 + j\varepsilon(3W_3 - W_2 - W_0). \\
\widehat{W}_2 &= W_2 + jW_3 + \varepsilon(3W_3 - W_2 - W_0) + j\varepsilon(8W_3 - W_1 - 3W_2 - 3W_0).
\end{aligned}$$

We now introduce specific cases of the dual hyperbolic generalized Adrien numbers that exhibit distinct structural or computational properties.

4.1. Some dual hyperbolic generalized Adrien numbers. The nth dual hyperbolic Adrien numbers, the nth dual hyperbolic Adrien-Lucas numbers, respectively, are given as the nth dual hyperbolic Adrien numbers is given $\widehat{A}_n = A_n + jA_{n+1} + \varepsilon A_{n+2} + j\varepsilon A_{n+3}$, with the initial values

$$\begin{aligned}
\widehat{A}_0 &= A_0 + jA_1 + \varepsilon A_2 + j\varepsilon A_3, \\
\widehat{A}_1 &= A_1 + jA_2 + \varepsilon A_3 + j\varepsilon A_4, \\
\widehat{A}_2 &= A_2 + jA_3 + \varepsilon A_4 + j\varepsilon A_5,
\end{aligned}$$

the nth dual hyperbolic Adrien-Lucas numbers is given $\widehat{B}_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$ with the initial values

$$\begin{aligned}
\widehat{B}_0 &= B_0 + jB_1 + \varepsilon B_2 + j\varepsilon B_3, \\
\widehat{B}_1 &= B_1 + jB_2 + \varepsilon B_3 + j\varepsilon B_4, \\
\widehat{B}_2 &= B_2 + jB_3 + \varepsilon B_4 + j\varepsilon B_5.
\end{aligned}$$

For dual hyperbolic Adrien numbers (taking $W_n = A_n,\ A_0 = 0, A_1 = 1,\ A_2 = 3, A_3 = 8,\ \ n \geq 4$) we get

$$\begin{aligned}
\widehat{A}_0 &= j + 3\varepsilon + 8\varepsilon j \\
\widehat{A}_1 &= 1 + 3j + 8\varepsilon + 21j\varepsilon \\
\widehat{A}_2 &= 3 + 8j + 21\varepsilon + 54j\varepsilon
\end{aligned}$$

and for dual hyperbolic Adrien-Lucas numbers (taking $W_n = B_n,\ B_0 = 4, B_1 = 3, B_0 = 7, B_0 = 18,\ \ n \geq 4$) we get

$$\begin{aligned}
\widehat{B}_0 &= 4 + 3j + 7\varepsilon + 18j\varepsilon \\
\widehat{B}_1 &= 3 + 7j + 18\varepsilon + 43j\varepsilon \\
\widehat{B}_2 &= 7 + 18j + 43\varepsilon + 108j\varepsilon
\end{aligned}$$

Tables 3 and 4 present selected values of the dual hyperbolic Adrien and Adrien-Lucas numbers for positive and negative indices.

Table 3. Dual hyperbolic Adrien numbers

n	$\widehat{A}_n$	$\widehat{A}_{-n}$
0	$j + 3\varepsilon + 8\varepsilon j$	$1 + 3\varepsilon + 8\varepsilon j$
1	$1 + 3j + 8\varepsilon + 21j\varepsilon$	$\varepsilon + 3\varepsilon j$
2	$3 + 8j + 21\varepsilon + 54j\varepsilon$	$j\varepsilon$
3	$8 + 21j + 54\varepsilon + 138j\varepsilon$	-1
4	$21 + 54j + 138\varepsilon + 352j\varepsilon$	$-j$
5	$54 + 138j + 352\varepsilon + 897j\varepsilon$	$1 - \varepsilon$

Table 4. Dual hyperbolic Adrien-Lucas numbers

n	$\widehat{B}_n$	$\widehat{B}_{-n}$
0	$4 + 3j + 7\varepsilon + 18j\varepsilon$	$4 + 3j + 7\varepsilon + 18j\varepsilon$
1	$3 + 7j + 18\varepsilon + 43j\varepsilon$	$4j + 3\varepsilon + 7j\varepsilon$
2	$7 + 18j + 43\varepsilon + 108j\varepsilon$	$-2 + 4\varepsilon + 3j\varepsilon$
3	$18 + 43j + 108\varepsilon + 274j\varepsilon$	$9 - 2j + 4j\varepsilon$
4	$43 + 108j + 274\varepsilon + 696j\varepsilon$	$-2 + 9j - 2\varepsilon$
5	$108 + 274j + 696\varepsilon + 1771j\varepsilon$	$-15 - 2j + 9\varepsilon - 2j\varepsilon$

Now, we will state Binet's formula for the dual hyperbolic generalized Adrien numbers and in the rest of the paper, we fix the following notations:

$$\widehat{\alpha} = 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \tag{4.5}$$

$$\widehat{\beta} = 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3. \tag{4.6}$$

$$\widehat{\gamma} = 1 + j\gamma + \varepsilon\gamma^2 + j\varepsilon\gamma^3 \tag{4.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + j + \varepsilon + j\varepsilon, \tag{4.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha} &= 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \\
\widehat{\beta} &= 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3, \\
\widehat{\alpha}^2 &= 1 + \alpha^2 + 2\alpha j + 2\alpha^2\left(\alpha^2+1\right)\varepsilon + 4\alpha^3 j\varepsilon, \\
\widehat{\beta}^2 &= 1 + \beta^2 + 2j\beta + 2\beta^4\varepsilon + 2\beta^2\varepsilon + 4j\beta^3\varepsilon, \\
\widehat{\alpha}\widehat{\beta} &= 1 + \alpha(\beta+j) + j\beta + \varepsilon(\alpha^2+\beta^2) + j\alpha\beta^2\varepsilon + j\alpha^2\beta\varepsilon + \beta^3(\alpha\varepsilon + j\varepsilon) + \alpha^3(\beta\varepsilon + j\varepsilon), \\
\widehat{\alpha}^2\widehat{\beta} &= 1 + \alpha^2 + \beta^2 + \alpha^2\beta^2 + 2\left(\alpha\beta+1\right)\left(\alpha+\beta\right)j + 2\left(\alpha^2+\beta^2+\alpha^2\beta^2+4\alpha\beta+1\right)\left(\alpha^2+\beta^2\right)\varepsilon \\
&\quad +4\left(\alpha+\beta\right)\left(\alpha^2+\beta^2+\alpha\beta^3\right)j\varepsilon, \\
\widehat{\alpha}\widehat{\beta}^2 &= 1 + \beta^2 + 2\alpha\beta + \left(\alpha+2\beta+\alpha\beta^2\right)j + \left(\beta^2+2\alpha\beta+1\right)\left(\alpha^2+2\beta^2\right)\varepsilon + \left(\alpha+2\beta+\alpha\beta^2\right)\left(\alpha^2+2\beta^2\right)j\varepsilon, \\
\widehat{\alpha}^2\widehat{\beta}^2 &= 1 + \beta^2 + \alpha^2 + \alpha^2\beta^2 + 4\alpha\beta + 2\left(\alpha\beta+1\right)\left(\alpha+\beta\right)j + 2\left(\alpha^2+\beta^2+\alpha^2\beta^2+4\alpha\beta+1\right)\left(\alpha^2+\beta^2\right)\varepsilon \\
&\quad +4\left(\alpha\beta+1\right)\left(\alpha+\beta\right)\left(\alpha^2+\beta^2\right)j\varepsilon.
\end{aligned}$$

Theorem 4. *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Adrien number is*

$$\widehat{W}_n = \widehat{\alpha}S_1\alpha^n + \widehat{\beta}S_2\beta^n + \widehat{\gamma}S_3\gamma^n + \widehat{\delta}S_4, \tag{4.9}$$

$\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$, $\widehat{\delta}$ are given as (4.5), (4.6), (4.7) ,(4.8).

Proof. Using Binet's formula of the generalized Adrien numbers given below

$$W_n = S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4,$$

where S_1, S_2, S_2, S_4 are given (3.3) we get

$$\begin{aligned}
\widehat{W}_n &= W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3} \\
&= S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4 \\
&\quad +j(S_1\alpha^{n+1} + S_2\beta^{n+1} + S_3\gamma^{n+1} + S_4) \\
&\quad +\varepsilon(S_1\alpha^{n+2} + S_2\beta^{n+2} + S_3\gamma^{n+2} + S_4) \\
&\quad +j\varepsilon(S_1\alpha^{n+3} + S_2\beta^{n+3} + S_3\gamma^{n+3} + S_4) \\
&= S_1\alpha^n(1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3) + S_2\beta^n(1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3) \\
&\quad +S_3\gamma^n(1 + j\gamma + \varepsilon\gamma^2 + j\varepsilon\gamma^3) + S_4(1 + j + \varepsilon + j\varepsilon) \\
&= \widehat{\alpha}S_1\alpha^n + \widehat{\beta}S_2\beta^n + \widehat{\gamma}S_3\gamma^n + \widehat{1}^n S_4.
\end{aligned}$$

This proves (4.9). □

As special cases, for any integer n, the Binet's Formula of nth dual hyperbolic Adrien number is

$$\widehat{A}_n = \frac{(2\alpha^2+\alpha+1)\alpha^n\widehat{\alpha}}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n\widehat{\beta}}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n\widehat{\gamma}}{4\gamma^2+3\gamma-1} - \frac{\widehat{1}}{3}, \tag{4.10}$$

and the Binet's Formula of nth dual hyperbolic Adrien-Lucas number is

$$\widehat{B}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + 1. \tag{4.11}$$

In the following section, we present the generating function associated with the dual hyperbolic generalized Adrien numbers.

THEOREM 5. *The generating function for the dual hyperbolic generalized Adrien numbers is*

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)x^2 + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)x^3}{1 - 3x + x^2 + x^4}. \tag{4.12}$$

Proof. Let

$$f_{\widehat{W}_n}(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n$$

be generating function of the dual hyperbolic generalized Adrien numbers. Then, using the definition of the dual hyperbolic generalized Adrien numbers, and substracting $xf_{\widehat{W}_n}(x)$ and $x^2 f_{\widehat{W}_n}(x)$ from $f_{\widehat{W}_n}(x)$, we obtain $(1 - 3x + x^2 + x^4)f_{\widehat{W}_n}(x)$

$$\begin{aligned}
(1 - 3x + x^2 + x^4)f_{\widehat{W}_n}(x) &= \sum_{n=0}^{\infty} \widehat{W}x^n - 3x\sum_{n=0}^{\infty} \widehat{W}x^n + x^2\sum_{n=0}^{\infty} \widehat{W}x^n + x^4\sum_{n=0}^{\infty} \widehat{W}x^n, \\
&= \sum_{n=0}^{\infty} \widehat{W}x^n - 3\sum_{n=0}^{\infty} \widehat{W}x^{n+1} + \sum_{n=0}^{\infty} \widehat{W}x^{n+2} + \sum_{n=0}^{\infty} \widehat{W}x^{n+4}, \\
&= \sum_{n=0}^{\infty} \widehat{W}x^n - 3\sum_{n=1}^{\infty} \widehat{W}_{(n-1)}x^n + \sum_{n=2}^{\infty} \widehat{W}_{(n-2)}x^n + \sum_{n=4}^{\infty} \widehat{W}_{(n-4)}x^n, \\
&= (\widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2 + \widehat{W}_3 x^3) - 3(\widehat{W}_0 x + \widehat{W}_1 x^2 + \widehat{W}_2 x^3) + 3(\widehat{W}_0 x^2 + \widehat{W}_1 x^3), \\
&\quad + \sum_{n=4}^{\infty} (\widehat{W}_n - \widehat{3W}_{n-1} + \widehat{W}_{n-2} + \widehat{W}_{n-4})x^n, \\
&= \widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2 + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)x^3.
\end{aligned}$$

Note that, using the recurrence relation $\widehat{A} = 3\widehat{A}_{n-1} - \widehat{A}_{n-2} - \widehat{A}_{n-4}$ and rearranging above equation the (4.12) has been obtained. □

Now we can write the generating functions of the dual hyperbolic Adrien and Adrien-Lucas numbers as:

(a): $f_{\widehat{A}_n}(x) = \sum\limits_{n=0}^{\infty} \widehat{A}_n x^n = \frac{1}{1-3x+x^2+x^4}(j + 3\varepsilon + 8\varepsilon j) + (1 - 3\varepsilon + 8\varepsilon j)x + (6j - 9\varepsilon j)x^2 + (-\varepsilon - 3\varepsilon j)x^3,$

(b): $f_{\widehat{B}_n}(x) = \sum\limits_{n=0}^{\infty} \widehat{B}_n x^n = \frac{1}{1-3x+x^2+x^4}(4 + 3j + 7\varepsilon + 18\varepsilon j) + (-9 - 2j - 3\varepsilon - 11\varepsilon j)x$
$+(2 - 4\varepsilon - 3\varepsilon j)x^2 + (-4j - 3\varepsilon - 7\varepsilon j)x^3.$

LEMMA 6. *Suppose that* $f_{\widehat{W}_n}(x) = \sum\limits_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!}$ *is the exponential generating function of the dual hyperbolic generalized Adrien sequence* $\{\widehat{W}_n\}$.

Then $\sum\limits_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}=S_1e^{\alpha x}\widehat{\alpha}+S_2e^{\beta x}\widehat{\beta}+S_3e^{\gamma x}\widehat{\gamma}+S_4e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma},\widehat{\delta}$ are given as(4.5), (4.6), (4.7) ,(4.8).Using either Binet's formula for the dual hyperbolic generalized Adrien numbers or the exponential generating function of the generalized form, we obtain the following analytical expressions. Adrien sequence we get the required identity.Let' s get the details.

Proof. Using Binet' s formula

$$W_n=S_1\alpha^n+S_2\beta^n+S_3\gamma^n+S_4,$$

where A_1,A_2,A_3,A_4 are given in (3.3) we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}W_n\frac{x^n}{n!}+j\sum_{n=0}^{\infty}W_{n+1}\frac{x^n}{n!}+\varepsilon\sum_{n=0}^{\infty}W_{n+2}\frac{x^n}{n!}+j\varepsilon\sum_{n=0}^{\infty}W_{n+3}\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}(S_1\alpha^n+S_2\beta^n+S_3\gamma^n+S_4)\frac{x^n}{n!}+j\sum_{n=0}^{\infty}(S_1\alpha^{n+1}+S_2\beta^{n+1}+S_3\gamma^{n+1}+S_4)\frac{x^n}{n!}\\
&\quad +\varepsilon\sum_{n=0}^{\infty}(S_1\alpha^{n+2}+S_2\beta^{n+2}+S_3\gamma^{n+2}+S_4)\frac{x^n}{n!}+j\varepsilon\sum_{n=0}^{\infty}(S_1\alpha^{n+3}+S_2\beta^{n+3}+S_3\gamma^{n+3}+S_4)\frac{x^n}{n!}\\
&= (S_1e^{\alpha x}+S_2e^{\beta x}+S_3e^{\gamma x}+S_4e^{x})+j(S_1\alpha e^{\alpha x}+S_2\beta e^{\beta x}+S_3\gamma e^{\gamma x}+S_4e^{x})\\
&\quad +\varepsilon(S_1\alpha^2e^{\alpha x}+S_2\beta^2e^{\beta x}+S_3\gamma^2e^{\gamma x}+S_4e^{x})+j\varepsilon(S_1\alpha^3e^{\alpha x}+S_2\beta^3e^{\beta x}+S_3\gamma^3e^{\gamma x}+S_4e^{x})\\
&= S_1e^{\alpha x}(1+j\alpha++\varepsilon\alpha^2+j\varepsilon\alpha^3)+S_2e^{\beta x}(1+j\beta++\varepsilon\beta^2+j\varepsilon\beta^3)\\
&\quad +S_3e^{\gamma x}(1+j\gamma++\varepsilon\gamma^2+j\varepsilon\gamma^3)+S_4e^{x}(1+j++\varepsilon+j\varepsilon)\\
&= S_1e^{\alpha x}\widehat{\alpha}+S_2e^{\beta x}\widehat{\beta}+S_3e^{\gamma x}\widehat{\gamma}+S_4e^{x}\widehat{1}.\ \square
\end{aligned}$$

From the previous lemma, we derive the following outcomes as particular instances.

COROLLARY 7. *Exponential generating function of dual hiperbolic Adrien and dual hiperbolic Adrien-Lucas numbers are given by:*

a):

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{A}_n\frac{x^n}{n!} &= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x}+\frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x}+\frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x}-\frac{1}{3}e^{x})\\
&\quad +j(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x}+\frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x}+\frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x}-\frac{1}{3}e^{x})\\
&\quad +\varepsilon(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x}+\frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x}+\frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x}-\frac{1}{3}e^{x})\\
&\quad +j\varepsilon(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x}+\frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x}+\frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x}-\frac{1}{3}e^{x}).\\
&= \frac{(2\alpha^2+\alpha+1)\widehat{\alpha}}{4\alpha^2+3\alpha-1}e^{\alpha x}+\frac{(2\beta^2+\beta+1)\widehat{\beta}}{4\beta^2+3\beta-1}e^{\beta x}+\frac{(2\gamma^2+\gamma+1)\widehat{\gamma}}{4\gamma^2+3\gamma-1}e^{\gamma x}-\frac{\widehat{1}}{3}e^{x}
\end{aligned}$$

b):

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{B}_n\frac{x^n}{n!} &= e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^{x}+j(\alpha e^{\alpha x}+\beta e^{\beta x}+\gamma e^{\gamma x}+e^{x})\\ &\quad +\varepsilon(\alpha^2e^{\alpha x}+\beta^2e^{\beta x}+\gamma^2e^{\gamma x}+e^{x})\\ &\quad +j\varepsilon(\alpha^3e^{\alpha x}+\beta^3e^{\beta x}+\gamma^3e^{\gamma x}+e^{x}).\\ &= e^{\alpha x}\widehat{\alpha}+e^{\beta x}\widehat{\beta}+e^{\gamma x}\widehat{\gamma}+e^{x}\widehat{1}.\end{aligned}$$

5. Obtaining Binet Formula From Generating Function

We next find Binet formula of dual hyperbolic generalized Adrien number $\{\widehat{W}_n\}$ by the use of generating function for $\widehat{W}_n$.

Theorem 8. *(Binet formula of dual hyperbolic generalized Adrien numbers)*

$$\widehat{W}_n=\frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{5.1}$$

where

$$\begin{aligned}p_1 &= \widehat{W}_0\alpha^3+(\widehat{W}_1-3\widehat{W}_0)\alpha^2+(\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\alpha+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1),\\ p_2 &= \widehat{W}_0\beta^3+(\widehat{W}_1-3\widehat{W}_0)\beta^2+(\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\beta+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1),\\ p_3 &= \widehat{W}_0\gamma^3+(\widehat{W}_1-3\widehat{W}_0)\gamma^2+(\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\gamma+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1),\\ p_4 &= \widehat{W}_0\delta^3+(\widehat{W}_1-3\widehat{W}_0)\delta^2+(\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\delta+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1).\end{aligned}$$

Proof. Let

$$h(x)=1-3x+x^2+x^4.$$

Then for some α,β,γ and δ we write

$$h(x)=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x),$$

i.e.,

$$1-3x+x^2=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x), \tag{5.2}$$

Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α,β,γ and δ as the roots of

$$h(\frac{1}{x})=1-\frac{3}{x}+\frac{1}{x^2}+\frac{1}{x^4}=0.$$

This implies $x^4-3x^3+x^2+u=0$. Now, by it follows that

$$\sum_{n=0}^{\infty}\widehat{W}x^{n}=\frac{\widehat{W}_0+(\widehat{W}_1-3\widehat{W}_0)x+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)x^2+(\widehat{W}_3-3\widehat{W}_2+\widehat{W}_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\begin{aligned}&\frac{\widehat{W}_0+(\widehat{W}_1-3\widehat{W}_0)x+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)x^2+(\widehat{W}_3-3\widehat{W}_2+\widehat{W}_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}\\ =\;&\frac{B_1}{(1-\alpha x)}+\frac{B_2}{(1-\beta x)}+\frac{B_3}{(1-\gamma x)}+\frac{B_4}{(1-\delta x)}.\end{aligned}\tag{5.3}$$

So

$$\begin{aligned}&\widehat{W}_0+(\widehat{W}_1-3\widehat{W}_0)x+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)x^2+(\widehat{W}_3-3\widehat{W}_2+\widehat{W}_1)x^3\\ =\;&B_1(1-\beta x)(1-\gamma x)(1-\delta x)+B_2(1-\alpha x)(1-\gamma x)(1-\delta x)\\ &+B_3(1-\alpha x)(1-\beta x)(1-\delta x)+B_3(1-\alpha x)(1-\beta x)(1-\gamma x).\end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get $\widehat{W}_0+(\widehat{W}_1-3\widehat{W}_0)\frac{1}{\alpha}+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)\frac{1}{\alpha^2}+(\widehat{W}_3-3\widehat{W}_2+\widehat{W}_1)\frac{1}{\alpha^3}$ $=B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$.

This gives

$$\begin{aligned}B_1&=\frac{\alpha^3(\widehat{W}_0+(\widehat{W}_1-3\widehat{W}_0)\frac{1}{\alpha}+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)\frac{1}{\alpha^2}+(\widehat{W}_3-6\widehat{W}_2+\widehat{W}_1)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\\ &=\frac{\widehat{W}_0\alpha^3+(\widehat{W}_1-\widehat{W}_0)\alpha^2+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)\alpha+(\widehat{W}_3-3\widehat{W}_2+\widehat{W}_1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}B_2&=\frac{\widehat{W}_0\beta^3+(\widehat{W}_1-3\widehat{W}_0)\beta^2+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)\beta+(\widehat{W}_3-3\widehat{W}_2+\widehat{W}_1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},\\ B_3&=\frac{\widehat{W}_0\gamma^3+(\widehat{W}_1-3\widehat{W}_0)\gamma^2+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)\gamma+(\widehat{W}_3-3\widehat{W}_2+\widehat{W}_1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},\\ B_4&=\frac{\widehat{W}_0\delta^3+(\widehat{W}_1-3\widehat{W}_0)\delta^2+(\widehat{W}_2-3\widehat{W}_1+\widehat{W}_0)\delta+(\widehat{W}_3-3\widehat{W}_2+\widehat{W}_1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.\end{aligned}$$

Thus (5.3) can be written as

$$\sum_{n=0}^{\infty}\widehat{W}x^{n}=B_1(1-\alpha x)^{-1}+B_2(1-\beta x)^{-1}+B_3(1-\gamma x)^{-1}+B_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{W}x^{n}&=B_1\sum_{n=0}^{\infty}\alpha^n x^n+B_2\sum_{n=0}^{\infty}\beta^n x^n+B_3\sum_{n=0}^{\infty}\gamma^n x^n+B_4\sum_{n=0}^{\infty}\delta^n x^n\\ &=\sum_{n=0}^{\infty}(B_1\alpha^n+B_2\beta^n+B_3\gamma^n+B_4\delta^n)x^n.\end{aligned}$$

Therefore, equating coefficients from both sides of the above expression yields the following formulation.

$$\widehat{W} = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

and then we get (5.1). □

We can get an identitiy related to dual hyperbolic Adrien numbers given below.

THEOREM 9. *For all integers m, n the following identities hold:*

$\widehat{W}_{m+n} = A_{m-2}\widehat{W}_{n+3} + (-A_{m-3} - A_{m-5})\widehat{W}_{n+2} + (-A_{m-4})\widehat{W}_{n+1} - A_{m-3}\widehat{W}_n.$

Proof. First we assume that $m, n \geq 0$ then (9) can be proved by mathematical induction on m. If $m = 0$ we get

$$\widehat{W}_n = A_{-2}\widehat{W}_{n+3} + (-A_{-3} - A_{-5})\widehat{W}_{n+2} + (-A_{-4})\widehat{W}_{n+1} - A_{-3}\widehat{W}_n.$$

which is true since $A_{-2} = 0$, $A_{-3} = -1$, $A_{-4} = 0$, $A_{-5} = 1$. Assume that the equality holds for $m \leq k$. For $m = k + 1$, we get

$$\begin{aligned}
\widehat{W}_{k+1+n} &= 3\widehat{W}_{n+k} - \widehat{W}_{n+k-1} - \widehat{W}_{n+k-3}, \\
&= 3A_{k-2}\widehat{W}_{n+3} + (-A_{k-3} - A_{k-5})\widehat{W}_{n+2} \\
&\quad +3(-A_{k-4})\widehat{W}_{n+1} - A_{k-3}\widehat{W}_n \\
&\quad -(A_{k-3}\widehat{W}_{n+3} + (-A_{k-4} - A_{k-6})\widehat{W}_{n+2} + (-A_{-5})\widehat{W}_{n+1} - A_{k-4}\widehat{W}_n) \\
&\quad -A_{k-5}\widehat{W}_{n+3} + (-A_{k-6} - A_{k-8})\widehat{W}_{n+2} + (-A_{k-6})\widehat{W}_{n+1} - A_{k-6}\widehat{W}_n.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem (9).

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $\widehat{W}_n = A_n$ or $\widehat{W}_n = B_n$ in above Theorem, respectively, we get:

COROLLARY 10.

$$\begin{aligned}
\widehat{A}_{m+n} &= A_{m-2}\widehat{A}_{n+3} + (-A_{m-3} - A_{m-5})\widehat{A}_{n+2} + (-A_{m-4})\widehat{A}_{n+1} - A_{m-3}\widehat{A}_n, \\
\widehat{B}_{m+n} &= A_{m-2}\widehat{B}_{n+3} + (-A_{m-3} - A_{m-5})\widehat{B}_{n+2} + (-A_{m-4})\widehat{B}_{n+1} - A_{m-3}\widehat{B}_n.
\end{aligned}$$

6. Simson's Formulas

This section introduces Simson's formula as applied to the dual hyperbolic generalized Adrien numbers. This is a special case of [21,Theorem 4.1].

THEOREM 11. *For all integers n, we have*

$$\begin{vmatrix}
\widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\
\widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\
\widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\
\widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} & \widehat{W}_{n-3}
\end{vmatrix} = (\widehat{W}_0+\widehat{W}_1+2\widehat{W}_2-\widehat{W}_3)(-\widehat{W}_3^3+5\widehat{W}_2^3+\widehat{W}_1^3+\widehat{W}_0^3-(\widehat{W}_0+3\widehat{W}_1-7\widehat{W}_2)\widehat{W}_3^2$$

$+(3\widehat{W}_0 - 4\widehat{W}_1 - 14\widehat{W}_3)\widehat{W}_2^2 + (2\widehat{W}_0 + \widehat{W}_2 - 6\widehat{W}_3)\widehat{W}_1^2 - (\widehat{W}_1 + 2\widehat{W}_3)\widehat{W}_0^2 + 13\widehat{W}_1\widehat{W}_2\widehat{W}_3 + \widehat{W}_0\widehat{W}_2\widehat{W}_3 + 5\widehat{W}_0\widehat{W}_1\widehat{W}_3 - 7\widehat{W}_0\widehat{W}_1\widehat{W}_2)$.

Proof. Take $r = 3, s = -1, t = 0, u = -1$. □

Corollary 12. *For all integers n, the Simson's formulas of dual hyperbolic generalized Adrien number and dual hyperbolic generalized Adrien-Lucas numbers are given as:*

$$\begin{vmatrix} \widehat{A}_{n+3} & \widehat{A}_{n+2} & \widehat{A}_{n+1} & \widehat{A}_{n} \\ \widehat{A}_{n+2} & \widehat{A}_{n+1} & \widehat{A}_{n} & \widehat{A}_{n-1} \\ \widehat{A}_{n+1} & \widehat{A}_{n} & \widehat{A}_{n-1} & \widehat{A}_{n-2} \\ \widehat{A}_{n} & \widehat{A}_{n-1} & \widehat{A}_{n-2} & \widehat{A}_{n-3} \end{vmatrix} = 18j - 97\varepsilon + 139j\varepsilon - 12,$$

$$\begin{vmatrix} \widehat{B}_{n+3} & \widehat{B}_{n+2} & \widehat{B}_{n+1} & \widehat{B}_{n} \\ \widehat{B}_{n+2} & \widehat{B}_{n+1} & \widehat{B}_{n} & \widehat{B}_{n-1} \\ \widehat{B}_{n+1} & \widehat{B}_{n} & \widehat{B}_{n-1} & \widehat{B}_{n-2} \\ \widehat{B}_{n} & \widehat{B}_{n-1} & \widehat{B}_{n-2} & \widehat{B}_{n-3} \end{vmatrix} = -2349j - 16\,443\varepsilon - 16\,443j\varepsilon - 2349,$$

respectively.

7. Linear Sums

This section presents the summation formulas for the dual hyperbolic generalized Adrien numbers, encompassing both positive and negative subscripts. We then proceed to introduce the summation formulas for the generalized Adrien numbers.

Theorem 13. *For the generalized Adrien numbers with positive and negative subscript, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} W_k = \frac{1}{3}(-(n+3)W_{n+3} + (2n+7)W_{n+2} + (n+2)W_{n+1} + (n+4)W_n + 3W_3 - 7W_2 - 2W_1 - W_0)$.
- **(b):** $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(-(n+2)W_{2n+2} + (2n+5)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 4W_2 - 3W_1)$.
- **(c):** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(n+1)W_{2n+2} + (2n+5)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 5W_2 - 2W_0)$.
- **(d):** $\sum_{k=1}^{n} W_{-k} = \frac{1}{3}(-(n+1)W_{-n+3} + (2n+1)W_{-n+2} + (n+2)W_{-n+1} + (n+3)W_{-n} + W_3 - W_2 - 2W_1 - 3W_0)$.
- **(e):** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(-(n+2)W_{-2n+2} + (2n+3)W_{-2n+1} + (n+4)W_{-2n} + (n+2)W_{-2n-1} + 2W_3 - 4W_2 - W_1 - 4W_0)$.
- **(f):** $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(n+3)W_{-2n+2} + 2(n+3)W_{-2n+1} + (n+2)W_{-2n} + (n+2)W_{-2n-1} + 2W_3 - 3W_2 - 4W_1 - 2W_0)$.

Proof. For the proof, see Soykan [19]. □

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic numbers.

Theorem 14. *For the dual hyperbolic numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n}\widehat{W}_k = \frac{1}{3}(-(n+3)\widehat{W}_{n+3}+(2n+7)\widehat{W}_{n+2}+(n+2)\widehat{W}_{n+1}+(n+4)\widehat{W}_n+3\widehat{W}_3-7\widehat{W}_2-2\widehat{W}_1-\widehat{W}_0)$.

(b): $\sum\limits_{k=0}^{n}\widehat{W}_{2k} = \frac{1}{3}(-(n+2)\widehat{W}_{2n+2}+(2n+5)\widehat{W}_{2n+1}+(n+3)\widehat{W}_{2n}+(n+2)\widehat{W}_{2n-1}+2\widehat{W}_3-4\widehat{W}_2-3\widehat{W}_1)$.

(c): $\sum\limits_{k=0}^{n}\widehat{W}_{2k+1} = \frac{1}{3}(-(n+1)\widehat{W}_{2n+2}+(2n+5)\widehat{W}_{2n+1}+(n+2)\widehat{W}_{2n}+(n+2)\widehat{W}_{2n-1}+2\widehat{W}_3-5\widehat{W}_2-2\widehat{W}_0)$.

(d): $\sum\limits_{k=1}^{n}\widehat{W}_{-k} = \frac{1}{3}(-(n+1)\widehat{W}_{-n+3}+(2n+1)\widehat{W}_{-n+2}+(n+2)\widehat{W}_{-n+1}+(n+3)\widehat{W}_{-n}+\widehat{W}_3-\widehat{W}_2-2\widehat{W}_1-3\widehat{W}_0)$.

(e): $\sum\limits_{k=1}^{n}\widehat{W}_{-2k} = \frac{1}{3}(-(n+2)\widehat{W}_{-2n+2}+(2n+3)\widehat{W}_{-2n+1}+(n+4)\widehat{W}_{-2n}+(n+2)\widehat{W}_{-2n-1}+2\widehat{W}_3-4\widehat{W}_2-\widehat{W}_1-4\widehat{W}_0)$.

(f): $\sum\limits_{k=1}^{n}\widehat{W}_{-2k+1} = \frac{1}{3}(-(n+3)\widehat{W}_{-2n+2}+2(n+3)\widehat{W}_{-2n+1}+(n+2)\widehat{W}_{-2n}+(n+2)\widehat{W}_{-2n-1}+2\widehat{W}_3-3\widehat{W}_2-4\widehat{W}_1-2\widehat{W}_0)$.

Proof.

(a): Note that using (4.1), we get

$$\sum_{k=0}^{n}\widehat{W}_k = \sum_{k=0}^{n}W_k + j\sum_{k=0}^{n}W_{k+1} + \varepsilon\sum_{k=0}^{n}W_{k+2} + j\varepsilon\sum_{k=0}^{n}W_{k+3}$$

and using Theorem (13) then proof is straightforward.

(b): Note that using (4.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{2k} = \sum_{k=0}^{n}W_{2k} + j\sum_{k=0}^{n}W_{2k+1} + \varepsilon\sum_{k=0}^{n}W_{2k+2} + j\varepsilon\sum_{k=0}^{n}W_{2k+3}$$

and using Theorem (13) the proof is easily attainable.

(c): Note that using (4.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{2k+1} = \sum_{k=0}^{n}W_{2k+1} + j\sum_{k=0}^{n}W_{2k+2} + \varepsilon\sum_{k=0}^{n}W_{2k+3} + j\varepsilon\sum_{k=0}^{n}W_{2k+4}$$

and using Theorem (13) the proof is straightforward.

Proof.

(d): Note that using (4.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-k} = \sum_{k=0}^{n}W_{-k} + j\sum_{k=0}^{n}W_{-k+1} + \varepsilon\sum_{k=0}^{n}W_{-k+2} + j\varepsilon\sum_{k=0}^{n}W_{-k+3}$$

and using Theorem (13) the proof is easily attainable.

(e): Note that using (4.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-2k} = \sum_{k=0}^{n}W_{-2k} + j\sum_{k=0}^{n}W_{-2k+1} + \varepsilon\sum_{k=0}^{n}W_{-2k+2} + j\varepsilon\sum_{k=0}^{n}W_{-2k+3}$$

and using Theorem (13) then proof is straightforward.

(f): Note that using (4.1), we get using Theorem (13), we get

$$\sum_{k=0}^{n}\widehat{W}_{-2k+1}=\sum_{k=0}^{n}W_{-2k+1}+j\sum_{k=0}^{n}W_{-2k+2}+\varepsilon\sum_{k=0}^{n}W_{2k+3}+j\varepsilon\sum_{k=0}^{n}W_{-2k+4}$$

and using Theorem (14) the proof is easily completed. □

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic Adrien numbers.

Theorem 15. *For $n\geq 0$, dual hyperbolic generalized Adrien numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n}\widehat{A}_k=\frac{1}{3}(-(n+3)\widehat{A}_{n+3}+(2n+7)\widehat{A}_{n+2}+(n+2)\widehat{A}_{n+1}+(n+4)\widehat{A}_n+j-4\varepsilon-14j\varepsilon)$.

(b): $\sum\limits_{k=0}^{n}\widehat{A}_{2k}=\frac{1}{3}(-(n+2)\widehat{A}_{2n+2}+(2n+5)\widehat{A}_{2n+1}+(n+3)\widehat{A}_{2n}+(n+2)\widehat{A}_{2n-1}+1+j-3j\varepsilon)$.

(c): $\sum\limits_{k=0}^{n}\widehat{A}_{2k+1}=\frac{1}{3}(-(n+1)\widehat{A}_{2n+2}+(2n+5)\widehat{A}_{2n+1}+(n+2)\widehat{A}_{2n}+(n+2)\widehat{A}_{2n-1}-1+2j-3\varepsilon-10j\varepsilon)$.

(d): $\sum\limits_{k=1}^{n}\widehat{A}_{-k}=\frac{1}{3}(-(n+1)\widehat{A}_{-n+3}+(2n+1)\widehat{A}_{-n+2}+(n+2)\widehat{A}_{-n+1}+(n+3)\widehat{A}_{-n}+7j+8\varepsilon+18j\varepsilon)$.

(e): $\sum\limits_{k=1}^{n}\widehat{A}_{-2k}=\frac{1}{3}(-(n+2)\widehat{A}_{-2n+2}+(2n+3)\widehat{A}_{-2n+1}+(n+4)\widehat{A}_{-2n}+(n+2)\widehat{A}_{-2n-1}-1+7j+4\varepsilon+7j\varepsilon)$.

(f): $\sum\limits_{k=1}^{n}\widehat{A}_{-2k+1}=\frac{1}{3}(-(n+3)\widehat{A}_{-2n+2}+2(n+3)\widehat{A}_{-2n+1}+(n+2)\widehat{A}_{-2n}+(n+2)\widehat{A}_{-2n-1}+1+6j+7\varepsilon+14j\varepsilon)$.

In the following, we derive the ordinary generating functions corresponding to certain special cases of the dual hyperbolic generalized Adrien numbers.

Theorem 16. *The ordinary generating functions of the sequences $\widehat{W}_{2n}$, $\widehat{W}_{2n+1}$ are given as follows:*

(a): $\sum_{n=0}^{\infty}\widehat{W}_{2n}x^n=\dfrac{3x^2\widehat{W}_3+(x^3-8x^2+x)\widehat{W}_2-3x^3\widehat{W}_1+(x^3+2x^2-7x+1)\widehat{W}_0}{x^4+2x^3+3x^2-7x+1}$

(b): $\sum_{n=0}^{\infty}\widehat{W}_{2n+1}x^n=\dfrac{(x^3+x^2+x)\widehat{W}_3-(3x^3+3x^2)\widehat{W}_2+(x^3+2x^2-7x+1)\widehat{W}_1-3x^2\widehat{W}_0}{x^4+2x^3+3x^2-7x+1}$

From the last Theorem, we have the following Corollary which gives sum formula of dual hyperbolic Adrien numbers (Take $\widehat{W}_n=\widehat{A}_n$ with

$\widehat{A}_0=j+3\varepsilon+8\varepsilon j,\ \widehat{A}_1=1+3j+8\varepsilon+21j\varepsilon,\ \widehat{A}_2=3+8j+21\varepsilon+54j\varepsilon,\ \widehat{A}_3=8+21j+54\varepsilon+138j\varepsilon$)

Corollary 17. *For $n\geq 0$ dual hyperbolic Adrien numbers have the following properties.*

(a): $\sum_{n=0}^{\infty}\widehat{A}_{2n}x^n=\frac{1}{x^4+2x^3+3x^2-7x+1}3x^2\left(21j+54\varepsilon+138j\varepsilon+8\right)-3x^3\left(3j+8\varepsilon+21j\varepsilon+1\right)+\left(3\varepsilon+8j\varepsilon+1\right)\left(x^3+2x^2\right.$ $\left.\left(x^3-8x^2+x\right)\left(8j+21\varepsilon+54j\varepsilon+3\right)\right.$.

(b): $\sum_{n=0}^{\infty}\widehat{A}_{2n+1}x^n=\frac{1}{-(x^4+2x^3+3x^2-7x+1)}3x^2\left(3\varepsilon+8j\varepsilon+1\right)+\left(3x^3+3x^2\right)\left(8j+21\varepsilon+54j\varepsilon+3\right)-\left(3j+8\varepsilon+21j\varepsilon+1\right)\left(x^3+2x^2-7x+1\right)-\left(x^3+x^2+x\right)\left(21j+54\varepsilon+138j\varepsilon+8\right)$.

8. Matrices related with Dual Hyperbolic Generalized Adrien Numbers

This part of the study introduces matrix identities that arise in connection with the dual hyperbolic Adrien numbers.

By using the $\{A_n\}$ which is defined by the fourth-order recurrence relation as follows:

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,$$

with the initial conditions

$$A_0 = 0,\ A_1 = 1,\ A_2 = 3,\ A_3 = 8. \tag{8.1}$$

We define the square matrix M of order 4 as

$$M = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detM = 1$. Then, we give the following Lemma.

Lemma 18. *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} \widehat{W}_{n+3} \\ \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_{n} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} \tag{8.2}$$

Proof. First, we prove the assertion for the case $n \geq 0$. Lemma 18 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}$$

which is true. We assume that (8.2) is true for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_{k} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widehat{W}_{3} \\ \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix}$$
$$= \begin{pmatrix} \widehat{W}_{k+4} \\ \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}.$$

Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^n = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}$$

For the proof see [20].

We define

$$N_{H_{\widehat{W}}} = \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix}, \tag{8.3}$$

$$E_{H_{\widehat{W}}} = \begin{pmatrix} \widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} & \widehat{W}_{n-3} \end{pmatrix}. \tag{8.4}$$

Now, we have the following theorem with $N_{\widehat{W}}$ and $E_{\widehat{W}}$.

THEOREM 19. *Using $N_{\widehat{W}}$ and $E_{\widehat{W}}$, we get*

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. Note that we get

$$
\begin{aligned}
A^{n}N_{\widehat{W}} &= \begin{pmatrix} A_{n+1} & -A_{n}-A_{n-2} & -A_{n-1} & -A_{n} \\ A_{n} & -A_{n-1}-A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2}-A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3}-A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix} \begin{pmatrix} \widehat{W}_{3} & \widehat{W}_{2} & \widehat{W}_{1} & \widehat{W}_{0} \\ \widehat{W}_{2} & \widehat{W}_{1} & \widehat{W}_{0} & \widehat{W}_{-1} \\ \widehat{W}_{1} & \widehat{W}_{0} & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_{0} & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
a_{11} &= A_{n+1}\widehat{W}_{3}+(-A_{n}-A_{n-2})\widehat{W}_{2}+(-A_{n-1})\widehat{W}_{1}+(-A_{n})\widehat{W}_{0}, \\
a_{12} &= A_{n+1}\widehat{W}_{2}+(-A_{n}-A_{n-2})\widehat{W}_{1}+(-A_{n-1})\widehat{W}_{0}+(-A_{n})\widehat{W}_{-1}, \\
a_{13} &= A_{n+1}\widehat{W}_{1}+(-A_{n}-A_{n-2})\widehat{W}_{0}+(-A_{n-1})\widehat{W}_{-1}+(-A_{n})\widehat{W}_{-2}, \\
a_{14} &= A_{n+1}\widehat{W}_{0}+(-A_{n}-A_{n-2})\widehat{W}_{-1}+(-A_{n-1})\widehat{W}_{-2}+(-A_{n})\widehat{W}_{-3}, \\
a_{21} &= A_{n}\widehat{W}_{3}+(-A_{n-1}-A_{n-3})\widehat{W}_{2}+(-A_{n-2})\widehat{W}_{1}+(-A_{n-1})\widehat{W}_{0}, \\
a_{22} &= A_{n}\widehat{W}_{2}+(-A_{n-1}-A_{n-3})\widehat{W}_{1}+(-A_{n-2})\widehat{W}_{0}+(-A_{n-1})\widehat{W}_{-1}, \\
a_{23} &= A_{n}\widehat{W}_{1}+(-A_{n-1}-A_{n-3})\widehat{W}_{0}+(-A_{n-2})\widehat{W}_{-1}+(-A_{n-1})\widehat{W}_{-2}, \\
a_{24} &= A_{n}\widehat{W}_{0}+(-A_{n-1}-A_{n-3})\widehat{W}_{-1}+(-A_{n-2})\widehat{W}_{-2}+(-A_{n-1})\widehat{W}_{-3}, \\
a_{31} &= A_{n-1}\widehat{W}_{3}+(-A_{n-2}-A_{n-4})\widehat{W}_{2}+(-A_{n-3})\widehat{W}_{1}+(-A_{n-2})\widehat{W}_{0}, \\
a_{32} &= A_{n-1}\widehat{W}_{2}+(-A_{n-2}-A_{n-4})\widehat{W}_{1}+(-A_{n-3})\widehat{W}_{0}+(-A_{n-2})\widehat{W}_{-1}, \\
a_{33} &= A_{n-1}\widehat{W}_{1}+(-A_{n-2}-A_{n-4})\widehat{W}_{0}+(-A_{n-3})\widehat{W}_{-1}+(-A_{n-2})\widehat{W}_{-2}, \\
a_{34} &= A_{n-1}\widehat{W}_{0}+(-A_{n-2}-A_{n-4})\widehat{W}_{-1}+(-A_{n-3})\widehat{W}_{-2}+(-A_{n-2})\widehat{W}_{-3}, \\
a_{41} &= A_{n-2}\widehat{W}_{3}+(-A_{n-3}-A_{n-5})\widehat{W}_{2}+(-A_{n-4})\widehat{W}_{1}+(-A_{n-3})\widehat{W}_{0}, \\
a_{42} &= A_{n-2}\widehat{W}_{2}+(-A_{n-3}-A_{n-5})\widehat{W}_{1}+(-A_{n-4})\widehat{W}_{0}+(-A_{n-3})\widehat{W}_{-1}, \\
a_{43} &= A_{n-2}\widehat{W}_{1}+(-A_{n-3}-A_{n-5})\widehat{W}_{0}+(-A_{n-4})\widehat{W}_{-1}+(-A_{n-3})\widehat{W}_{-2}, \\
a_{44} &= A_{n-2}\widehat{W}_{0}+(-A_{n-3}-A_{n-5})\widehat{W}_{-1}+(-A_{n-4})\widehat{W}_{-2}+(-A_{n-3})\widehat{W}_{-3}.
\end{aligned}
$$

Using the theorem (9) the proof is done. □

by taking $W_n = A_n$ with A_0, A_1, A_2, A_3 in (8.3) and (8.4)

$\widehat{W}_n = \widehat{W}B_n$ with $\widehat{W}B_0$, $\widehat{W}B_1$, $\widehat{W}B_2$, $\widehat{W}B_3$ in (8.3) and (8.4)

respectively, we get:

$$N_{\widehat{A}} = \begin{pmatrix} 8+21j+54\varepsilon+138j\varepsilon & 3+8j+21\varepsilon+54j\varepsilon & 1+3j+8\varepsilon+21j\varepsilon & j+3\varepsilon+8\varepsilon j \\ 3+8j+21\varepsilon+54j\varepsilon & 1+3j+8\varepsilon+21j\varepsilon & j+3\varepsilon+8\varepsilon j & \varepsilon+3\varepsilon j \\ 1+3j+8\varepsilon+21j\varepsilon & j+3\varepsilon+8\varepsilon j & \varepsilon+3\varepsilon j & j\varepsilon \\ j+3\varepsilon+8\varepsilon j & \varepsilon+3\varepsilon j & j\varepsilon & -1 \end{pmatrix},$$

$$E_{\widehat{A}} = \begin{pmatrix} \widehat{A}_{n+3} & \widehat{A}_{n+2} & \widehat{A}_{n+1} & \widehat{A}_{n} \\ \widehat{A}_{n+2} & \widehat{A}_{n+1} & \widehat{A}_{n} & \widehat{A}_{n-1} \\ \widehat{A}_{n+1} & \widehat{A}_{n} & \widehat{A}_{n-1} & \widehat{A}_{n-2} \\ \widehat{A}_{n} & \widehat{A}_{n-1} & \widehat{A}_{n-2} & \widehat{A}_{n-3} \end{pmatrix},$$

$$N_{\widehat{B}} = \begin{pmatrix} 18+43j+108\varepsilon+274j\varepsilon & 7+18j+43\varepsilon+108j\varepsilon & 3+7j+18\varepsilon+43j\varepsilon & 4+3j+7\varepsilon+18j\varepsilon \\ 7+18j+43\varepsilon+108j\varepsilon & 3+7j+18\varepsilon+43j\varepsilon & 4+3j+7\varepsilon+18j\varepsilon & 4j+3\varepsilon+7j\varepsilon \\ 3+7j+18\varepsilon+43j\varepsilon & 4+3j+7\varepsilon+18j\varepsilon & 4j+3\varepsilon+7j\varepsilon & -2+4\varepsilon+3j\varepsilon \\ 4+3j+7\varepsilon+18j\varepsilon & 4j+3\varepsilon+7j\varepsilon & -2+4\varepsilon+3j\varepsilon & 9-2j+4j\varepsilon \end{pmatrix},$$

$$E_{\widehat{B}} = \begin{pmatrix} \widehat{B}_{n+3} & \widehat{B}_{n+2} & \widehat{B}_{n+1} & \widehat{B}_{n} \\ \widehat{B}_{n+2} & \widehat{B}_{n+1} & \widehat{B}_{n} & \widehat{B}_{n-1} \\ \widehat{B}_{n+1} & \widehat{B}_{n} & \widehat{B}_{n-1} & \widehat{B}_{n-2} \\ \widehat{B}_{n} & \widehat{B}_{n-1} & \widehat{B}_{n-2} & \widehat{B}_{n-3} \end{pmatrix}.$$

From Theorem [19], we can write the following corollary.

Corollary 20. *The following identities are hold:*

a): $A^n N_{\widehat{W}A} = E_{\widehat{W}A}$.

b): $A^n N_{\widehat{W}B} = E_{\widehat{W}B}$.

9. Conclusions

In this study, we introduce a new class of fourth-order recurrence relations, referred to as the dual hyperbolic Generalized Adrien numbers, accompanied by two notable special cases. We investigate a comprehensive set of structural properties associated with these sequences, including closed-form (Binet-type) expressions, ordinary and exponential generating functions, Simson-type identities, summation formulas, recurrence dynamics, and matrix-based representations.

Recurrence relations have long attracted significant scholarly attention due to their versatility and wide-ranging applicability across disciplines such as physics, engineering, architecture, the natural sciences, and the arts. Among these, sequences governed by second-order recurrence relations most notably the Fibonacci, Lucas, Pell, and Jacobsthal sequences occupy a central position in mathematical literature. The Fibonacci sequence, for example, gained historical prominence through its role in modeling rabbit population growth in Leonardo de Pisa's 1202 treatise Liber Abaci. Both Fibonacci and Lucas sequences have inspired extensive research, owing to their elegant algebraic structure and numerous remarkable identities.

Recurrence relations play a foundational role in both theoretical and applied mathematics, offering a powerful framework for modeling sequential processes and dynamic systems. Their significance extends across diverse domains including algorithm analysis in computer science, population modeling in biology, signal processing in engineering, and even aesthetic patterns in architecture and the arts. By expressing each term of a sequence in relation to its predecessors, recurrence relations enable the formulation of efficient computational strategies, particularly in recursive algorithms and divide-and-conquer paradigms. Classical examples such as the Fibonacci sequence illustrate their utility in modeling natural growth phenomena, while more complex relations underpin the analysis of sorting algorithms, dynamic programming, and combinatorial structures. In this study, the exploration of fourth-order recurrence relations specifically the dual hyperbolic Generalized Adrien numbers demonstrates how such constructs can yield rich algebraic properties, closed-form expressions, and generating functions, thereby contributing to the broader understanding of discrete mathematical systems.

The structural properties derived from the dual hyperbolic Generalized Adrien numbers open promising avenues for interdisciplinary applications. In quantum physics, recurrence relations underpin the analysis of quantum recurrences, such as those observed in the quantum kicked top and Poincaré-type phenomena, where unitary evolution exhibits periodic behavior in finite-dimensional Hilbert spaces. These sequences may contribute to modeling quantum coherence, entanglement dynamics, and state evolution in chaotic systems. In biology, recurrence-based models are instrumental in describing population dynamics, gene regulatory networks, and disease spread. For instance, nonlinear recurrence relations have been used to simulate genetic feedback loops and epidemiological transitions, offering insights into temporal biological computation and memory encoding in cellular systems. In statistics, recurrence relations support time series analysis, autoregressive modeling, and recurrence quantification techniques, which are essential for detecting non-stationarity, hidden periodicities, and complex dependencies in stochastic data.

Looking ahead, several future research directions emerge from this study. These include exploring the algebraic and spectral properties of higher-order Adrien-type sequences, investigating their role in reservoir computing architectures, and applying them to nonlinear dynamical systems and chaotic attractors. Additionally, the integration of recurrence-based models with machine learning frameworks such as recurrent neural networks and symbolic regression may yield novel computational tools for pattern recognition and predictive modeling. The matrix representations and generating functions derived here also invite further exploration in operator theory, combinatorial identities, and quantum information geometry, potentially bridging discrete mathematics with continuous physical systems.

References

[1] Bród D, Liana A, Wloch Two generalizations of dual-hyperbolic balancing numbers. Symmetry. (2020);12(11):1866.

[2] Cihan A, Azak AZ, Güngör MA, Tosun M.A study on dual hyperbolic Fibonacci and Lucas numbers. An Şt.Univ.Ovidius.Constanta. (2019);27(1):35-48.

[3] Dikmen, C. M. A study on dual hyperbolic generalised leonardo numbers. Asian Research Journal of mathematics, 21(6), 143-165, (2025).

[4] Demirci F, Soykan Y, Gaussian generalized Adrien numbers. Archives of Current Research International 25(7),469-491,(2025).

[5] D.K. Biss, D. Dugger, D.C. Isaksen, Large annihilators in Cayley-Dickson algebras, Communication in Algebra, 36 (2), 632-664, (2008).

[6] Eren ,O. On dual hyperbolic generalized Woodall numbers, Archives of Current Research International, 24(11), 398-423 ,(2024). Science & Computational.

[7] G. Moreno, The zero divisors of the Cayley-Dickson algebras over the real numbers, Bol. Soc. Mat. Mexicana 3(4) (1998), 13-28.

[8] G. Sobczyk, The Hyperbolic Number Plane, The College Mathematics Journal, 26(4) (1995), 268-280.

[9] H. H. Cheng, S. Thompson, D.

[10] I. Kantor, A. Solodovnikov, Hypercomplex Numbers, Springer-Verlag, New York (1989).ual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanisms, Proc. of ASME 24th Biennial Mechanisms Conference, Irvine, CA, August, (1996), 19-22.

[11] J. Baez, The octonions, Bull. Amer. Math. Soc. 39(2) (2002), 145-205.

[12] J. Cockle, On a New Imaginary in Algebra, Philosophical magazine, London-Dublin-Edinburgh, 3(34) (1849), 37-47.

[13] K. Imaeda, M. Imaeda, Sedenions: algebra and analysis, Applied Mathematics and Computation, 115 (2000), 77-88.

[14] M. Akar, S. Yüce, Ş. Şahin, On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers, Journal of Computer Mathematics, 8(1) (2018), 1-6.

[15] P. Fjelstad, S.G. Gal, n-dimensional Hyperbolic Complex Numbers, Advances in Applied Clifford Algebras, 8(1) (1998), 47-68.

[16] Soykan, Y., Generalized Adrien Numbers, Applied Mathematic and Computer Science,7(1),37-51.

[17] Soykan Y, Gümüş M, Göcen M. A study on dual hyperbolic generalized Pell numbers. Malaya Journal of Matematik. (2021);09(03):99-116.

[18] Soykan Y, Taşdemir E, Okumuş İ. On dual hyperbolic numbers with generalized Jacobsthal numbers Components. Indian J Pure Appl Math.(2023);27(1):35-48.

[19] Soykan, Y., Sums and Generating Functions of Special Cases of Generalized Tetranacci Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 12(4), 34-10, 2025.

[20] Soykan, Y., Properties of Generalized (r,s,t,u)-Numbers, Earthline Journal of Mathematical Sciences, 5(2), 297-327, (2021). https://doi.org/10.34198/ejms.5221.297327.

[21] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56, (2019).

[22] W.R. Hamilton, Elements of Quaternions, Chelsea Publishing Company, New York (1969).

[23] Yılmaz B, Soykan Y. On Dual Hyperbolic Guglielmo Numbers Journal of Advances in Mathematics and Computer Science. (2024);39(4):37-61.

[24] Y. Soykan, Tribonacci and Tribonacci-Lucas Sedenions. Mathematics 7(1) (2019), 74.