
Edge Induced V_4 –Magic Labeling of Subdivision Graphs

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Abstract

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0 and $G = (V(G), E(G))$ be a graph. Let $f : E(G) \rightarrow V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \rightarrow V_4$ denotes the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in V(G)$. Then f^+ again induces an edge labeling $f^{++} : E(G) \rightarrow V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$, for all $uv \in E(G)$. A graph $G = (V(G), E(G))$ is said to be an edge induced V_4 -magic graph, if there exists an edge labeling f for which the function f^{++} is a constant function. The function f , so obtained is called an Edge Induced V_4 -Magic Labeling (EIML) of G . The present paper deals with basic results regarding EIML of subdivision graphs and the characterization of EIML of subdivision of certain named graphs.

Keywords: Klein 4-group, Induced V_4 -magic graphs, Subdivision graphs.

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1 Introduction

The present paper intends to deal exclusively with simple, connected, finite and undirected graphs. Also note that, the Klein 4-group is denoted by $V_4 = \{0, a, b, c\}$ which is a non cyclic abelian group of order 4 with every non identity element which has order 2. Let $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$. The reader may check [9] for the standard terminology and notations related to Graph theory. Let $f : E(G) \rightarrow V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \rightarrow V_4$ denotes the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in V(G)$. Then

f^+ again induces an edge labeling, say, $f^{++} : E(G) \rightarrow V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$, for all $uv \in E(G)$. A graph $G = (V(G), E(G))$ is said to be an edge induced V_4 -magic graph, if there exists an edge labeling f for which $f^{++}(e)$ is a constant for all $e \in E(G)$. If this constant is x , then x is said to be the induced edge sum of the graph G and the function f , so obtained is called an edge induced V_4 -magic labeling of G . This paper aims to discuss edge induced V_4 -magic labeling of some subdivision graphs which belong to the following categories:

- (i) $\sigma_a(V_4) :=$ Set of all edge induced V_4 -magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = a$ for all $u \in V$.
- (ii) $\sigma_0(V_4) :=$ Set of all edge induced V_4 -magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = 0$ for all $u \in V$.
- (iii) $\sigma(V_4) := \sigma_a(V_4) \cap \sigma_0(V_4)$.

1.1 PRELIMINARIES

The bistar $B_{m,n}$ [5] is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge. A flag graph is denoted by Fl_n [4] and it is obtained by joining one vertex of C_n to an extra vertex called the root. The sun graph [4] on $m = 2n$ vertices, denoted by Sun_n , is the graph obtained by attaching a pendant vertex to each vertex of a n -cycle. Jelly fish graph $J(m, n)$ [4] is obtained from a 4-cycle $v_1v_2v_3v_4v_1$ by joining v_1 and v_3 with an edge and appending the central vertex of $K_{1,m}$ to v_2 and appending the central vertex of $K_{1,n}$ to v_4 . A triangular snake graph TS_n [4] is obtained from a path $v_1, v_2, v_3, \dots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \dots, n-1$. The join [9] of the graphs C_n and K_1 is called a wheel graph [9] and it is denoted by W_n , that is $W_n = C_n \vee K_1$. The corona [1] of P_n and K_1 is called the comb graph CB_n [1], that is $CB_n = P_n \circ K_1$.

Theorem 1.1. [6]. Let $G = (V, E)$ be a graph with either each vertex is of odd degree or even degree then $G \in \sigma_0(V_4)$.

Theorem 1.2. [6] (Induced edge sum theorem)

For any graph G , f is an edge induced V_4 -Magic labeling of G if and only if the induced edge sum

$$x = f^{++}(uv) = \sum_{u\alpha \in E, \alpha \neq v} f(u\alpha) + \sum_{\beta v \in E, \beta \neq u} f(\beta v), \text{ for all } (u, v) \in E \quad (1.1)$$

The Equation (1.1) corresponding to an edge uv in G , is called induced edge sum equation of the edge uv .

Theorem 1.3. [6] For the path graph P_n , we have the following:

- (i) $P_2 \in \sigma_0(V_4)$ and $P_2 \notin \sigma_a(V_4)$.
- (ii) $P_3 \in \sigma_a(V_4)$ and $P_3 \notin \sigma_0(V_4)$.
- (iii) $P_4 \in \sigma_a(V_4)$ and $P_4 \notin \sigma_0(V_4)$.
- (iv) P_n is not an edge induced magic graph for any $n \geq 5$.

Theorem 1.4. [6] For the cycle graph C_n , we have the following:

- (i) $C_n \in \sigma_0(V_4)$ for all n .
- (ii) $C_n \in \sigma_a(V_4)$ if and only if n is a multiple of 4.

Definition 1.1. The subdivision of an edge $e = uv$ in the graph G gives a new graph obtained by replacing the edge $e = uv$ by two edges $e_1 = uw$ and $e_2 = wv$. A subdivision of a graph G or simply a subdivision graph is a graph which is denoted by $S(G)$ and is obtained from the subdivision of all edges in G .

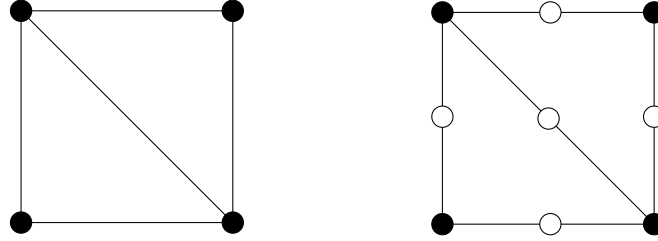


Figure 1: A Graph and its subdivision graph

2 Main Results

Theorem 2.1. *Let G be graph with every vertex is of odd degree, then $S(G) \in \sigma_a(V_4)$.*

Proof. Suppose G is a graph with every vertex is of odd degree. Then define $f : E(S(G)) \rightarrow V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(G))$.

Let $uv \in E(G)$ and α be the inserted vertex on the edge uv in $S(G)$. Then $f(u\alpha) = f(v\alpha) = a$. Therefore, $f^+(u) = f^+(v) = \deg(u)a = a$, since $\deg(u)$ is odd and $f^+(\alpha) = \deg(\alpha)a = 0$, since $\deg(\alpha) = 2$. Thus $f^{++}(u\alpha) = a$ and $f^{++}(v\alpha) = a$. Since uv is an arbitrary edge in $S(G)$, we can conclude that $f^{++}(e) = a$ for all $e \in S(G)$. Thus $S(G) \in \sigma_a(V_4)$.

Hence the proof. $\square$

Theorem 2.2. *Let G be graph with every vertex is of even degree, then $S(G) \in \sigma_0(V_4)$.*

Proof. Suppose G is a graph with every vertex is of even degree. Then define $f : E(S(G)) \rightarrow V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(G))$.

Let $uv \in E(G)$ and α be the inserted vertex on the edge uv in $S(G)$. Then $f(u\alpha) = f(v\alpha) = a$. Therefore, $f^+(u) = f^+(v) = \deg(u)a = 0$, since $\deg(u)$ is even and $f^+(\alpha) = \deg(\alpha)a = 0$, since $\deg(\alpha) = 2$. Thus $f^{++}(u\alpha) = 0$ and $f^{++}(v\alpha) = 0$. Since uv is an arbitrary edge in $S(G)$, it can be concluded that $f^{++}(e) = 0$ for all $e \in S(G)$. Thus $S(G) \in \sigma_0(V_4)$.

Hence the proof. $\square$

Theorem 2.3. *For the path graph $P_n, n \geq 2$ we have the following:*

- (i) $S(P_2) \in \sigma_a(V_4)$ and $S(P_2) \notin \sigma_0(V_4)$.
- (ii) $S(P_n) \notin \sigma_a(V_4)$ and $S(P_n) \notin \sigma_0(V_4)$ for any $n \geq 3$.

Proof. Since $S(P_2) = P_3$ proof of (i) follows directly from Theorem 1.3 (ii).

Also we have, $S(P_n) = P_{2n-1}$ and if $n \geq 3$ then $2n-1 \geq 5$, therefore, the proof of (ii) follows directly from Theorem 1.3 (iv). $\square$

Theorem 2.4. *For the cycle graph C_n , we have the following:*

- (a) $S(C_n) \in \sigma_0(V_4)$ for all n .
- (b) $S(C_n) \in \sigma_a(V_4)$ if and only if n is even.
- (c) $S(C_n) \in \sigma(V_4)$ if and only if n is even.

Proof. Since $S(C_n) = C_{2n}$, proof of (a) follows from Theorem 1.4 (i). Similarly, we have, $S(C_n) = C_{2n}$, therefore, proof of (b) follows directly from Theorem 1.4 (ii). Note that, the proof of (c) follows from (a) and (b). $\square$

Theorem 2.5. *For the complete graph K_n with n vertices, we have the following:*

- (i) $S(K_n) \in \sigma_0(V_4)$ for n odd.
- (ii) $S(K_n) \in \sigma_a(V_4)$ for n even.

Proof. Consider the subdivision graph of the complete graph $S(K_n)$. Let vu be an edge in $S(K_n)$, where $v \in V(K_n)$ and u be an inserted vertex in $S(K_n)$. Define $f : E(S(K_n)) \rightarrow V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(K_n))$. Then $f^+(v) = (n-1)a$ and $f^+(u) = a + a = 0$. Therefore, $f^{++}(vu) = (n-1)a$. Since the vertices u and v are arbitrary, we have $f^{++}(vu)$ is a constant.

Case (i) n is an odd integer.

In this case, $f^{++}(vu) = (n-1)a = 0$. Therefore, $S(K_n) \in \sigma_0(V_4)$.

Case (ii) n is an even integer.

In this case, $f^{++}(vu) = (n-1)a = a$. Therefore, $S(K_n) \in \sigma_a(V_4)$.

Hence the proof. $\square$

Theorem 2.6. *For the star graph $K_{1,n}$, we have the following:*

- (i) $S(K_{1,n}) \in \sigma_a(V_4)$ if and only if n is odd.
- (ii) $S(K_{1,n}) \notin \sigma_0(V_4)$ for any n .

Proof. Consider $K_{1,n}$ with vertex set $\{v, v_1, v_2, v_3, \dots, v_n\}$, where $vv_i \in E(K_{1,n})$ for $i = 1, 2, 3, \dots, n$. Let u_i be the inserted vertices on the edge vv_i for $i = 1, 2, 3, \dots, n$ in $S(K_{1,n})$.

Let $f : E(S(K_{1,n})) \rightarrow V_4 \setminus \{0\}$ with $f(vu_i) = x_{i1}$, and $f(u_i v_i) = x_{i2}$ for $i = 1, 2, 3, \dots, n$. Then from the induced edge sum equation of each edge we have the following equation.

$$\begin{aligned}
 x_{11} = x_{21} = x_{31} = \dots = x_{n1} &= x_{21} + x_{31} + x_{41} + \dots + x_{n1} + x_{12} \\
 &= x_{11} + x_{31} + x_{41} + \dots + x_{n1} + x_{22} \\
 &= x_{11} + x_{21} + x_{31} + \dots + x_{n1} + x_{32} \\
 &\vdots \\
 &= x_{11} + x_{31} + x_{41} + \dots + x_{(n-1)1} + x_{n2}.
 \end{aligned}$$

Let $x = x_{11} = x_{21} = x_{31} = \dots = x_{n1}$ then above equations become

$$\begin{aligned}
 x &= (n-1)x + x_{12} \\
 &= (n-1)x + x_{22} \\
 &= (n-1)x + x_{32} \\
 &\vdots \\
 &= (n-1)x + x_{n2}.
 \end{aligned}$$

Note that the above system implies that $x_{12} = x_{22} = x_{32} = \dots = x_{n2} = y$ (say).

Then the above system of equations reduces to $x = (n-1)x + y$.

That is $(n-2)x + y = 0$.

Case (i) n is an odd integer.

In this case, the equation $(n-2)x + y = 0$ reduces to $x + y = 0$, that is $x = y$. Thus by taking $x = y = a$ that is, by defining $f : E(S(K_{1,n})) \rightarrow V_4 \setminus \{0\}$ as $f(e_i) = a$, for all $e_i \in E(S(K_{1,n}))$ we can prove that $S(K_{1,n}) \in \sigma_a(V_4)$.

Case (ii) n is an even integer.

In this case, the equation $(n-2)x + y = 0$ reduces to $y = 0$. That is $f(u_i v_i) = x_{i2} = 0$, which is a contradiction to the choice for f . Therefore, in this case, $S(K_{1,n})$ is not an edge induced magic graph.

Note that $S(K_{1,n}) \in \sigma_0(V_4)$ only when $x = 0$. But $x = 0$ is not possible. Therefore, $S(K_{1,n}) \notin \sigma_0(V_4)$ for any n .

Hence the proof. $\square$

Theorem 2.7. For the bistar graph $B_{m,n}$, $S(B_{m,n})$ is not an edge induced magic graph for any m and n .

Proof. Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \dots, v_m, u_1, u_2, u_3, \dots, u_n\}$, where $uv, vv_i, uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$. Also let w_i, t_j and w be the inserted vertices on the edge vv_i, uu_j and uv respectively for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$ in the graph $S(B_{m,n})$.

Let $f : E(S(B_{m,n})) \rightarrow V_4 \setminus \{0\}$ with $f(vw) = \gamma, f(wu) = \delta, f(vw_i) = x_i, f(w_i v_i) = \alpha_i, f(ut_j) = y_j$ and $f(t_j u_j) = \beta_j$, then by considering the induced edge sum equation of each edge we have the following equations.

The induced edge sum equation of the edges $w_i v_i$ gives: $x_1 = x_2 = x_3 = \dots = x_m = x$ (say). Similarly the induced edge sum equation of the edges $t_j u_j$ gives: $y_1 = y_2 = y_3 = \dots = y_n = y$ (say). The induced edge sum equation of the edges vw_i gives:

$$\begin{aligned} \alpha_1 + \gamma + (m-1)x &= \alpha_2 + \gamma + (m-1)x \\ &= \alpha_3 + \gamma + (m-1)x \\ &\vdots \\ &= \alpha_m + \gamma + (m-1)x. \end{aligned}$$

It should be noted that the above system of equations imply that $\alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_m = \alpha$ (say). Thus, each induced edge sum in above system reduces to $\alpha + \gamma + (m-1)x$.

Similarly by considering the induced edge sum equation of the edges ut_j , we get the induced edge sum is $\beta + \delta + (n-1)y$, where $\beta = \beta_1 = \beta_2 = \beta_3 = \dots = \beta_n$.

Also we get, the induced edge sum of the edges vw is $mx + \delta$ and the edge sum of the edge wu is $ny + \gamma$.

Thus the edge sum equation of the graph $S(B_{m,n})$ is given by:

$$x = y = \alpha + \gamma + (m-1)x = \beta + \delta + (n-1)y = mx + \delta = ny + \gamma. \quad (2.1)$$

Case 1: m and n are even integers.

In this case, Equation (2.1) becomes

$$x = y = \alpha + \gamma + x = \beta + \delta + y = \delta = \gamma.$$

Therefore, $x = \gamma$, which implies that $\alpha = 0$, which is not possible. Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Case 2: m and n are odd integers.

In this case, Equation (2.1) becomes

$$x = y = \alpha + \gamma = \beta + \delta = x + \delta = y + \gamma.$$

Therefore, $x = x + \delta$ which implies that $\delta = 0$, which is not possible. Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Case 3: m is even and n is odd.

In this case, Equation (2.1) becomes

$$x = y = \alpha + \gamma + x = \beta + \delta = \delta = y + \gamma.$$

Therefore, $\beta + \delta = \delta$ which implies that $\beta = 0$, which is not possible. Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Case 4: m is odd and n is even.

In this case, Equation (2.1) becomes

$$x = y = \alpha + \gamma = \beta + \delta + y = x + \delta = \gamma.$$

Therefore, $\alpha + \gamma = \gamma$ which implies that $\alpha = 0$, which is not possible. Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Thus in all cases, we get $S(B_{m,n})$ is not an edge induced magic graph.

Hence the proof. $\square$

Theorem 2.8. For the flag graph Fl_n , we have the following.

Case (i) $S(Fl_n) \notin \sigma_0(V_4)$ for any n .

Case (ii) $S(Fl_n) \in \sigma_a(V_4)$ if and only if n is odd.

Proof. Let $V(Fl_n) = \{v, v_1, v_2, v_3, \dots, v_n\}$, where $v_1, v_2, v_3, \dots, v_n$ are the vertices of corresponding cycle graph C_n and v is the root vertex adjacent to the vertex v_1 . Also let u be the inserted vertex on the edge v_1v and $u_1, u_2, u_3, \dots, u_n$ be the inserted vertices on the edges $v_1v_2, v_2v_3, v_3v_4, \dots, v_nv_1$ respectively in the graph $S(Fl_n)$.

If possible, let $g : E(S(Fl_n)) \rightarrow V_4 \setminus \{0\}$ be an edge label with $g^{++}(e) = 0$ for all edge in $S(Fl_n)$. Then consider the induced edge sum of the edge uv . Note that $g^{++}(uv) = g(uv_1)$. Therefore, $g(uv_1) = 0$, which is a contradiction and it proves (i).

Suppose n is an odd integer. In this case, define $f : E(S(Fl_n)) \rightarrow V_4 \setminus \{0\}$ as follows.

$$f(e) = \begin{cases} a & \text{if } e = uv, uv_1 \\ b & \text{if } e = u_1v_1, u_3v_3, u_5v_5, \dots, u_{n-2}v_{n-2}, u_nv_n \\ c & \text{if } e = u_2v_2, u_4v_4, u_6v_6, \dots, u_{n-3}v_{n-3}, u_{n-1}v_{n-1} \\ b & \text{if } e = u_1v_2, u_3v_4, u_5v_6, \dots, u_{n-2}v_{n-1}, u_nv_1 \\ c & \text{if } e = u_2v_3, u_4v_5, u_6v_7, \dots, u_{n-3}v_{n-2}, u_{n-1}v_n. \end{cases}$$

Then $f^{++}(e) = a$ for all $e \in E(S(Fl_n))$. Thus $S(Fl_n) \in \sigma_a(V_4)$.

To prove the converse part, suppose n is an even integer. If possible, let $h : E(S(Fl_n)) \rightarrow V_4 \setminus \{0\}$ be an edge label with $h^{++}(e) = a$ for all edge in $S(Fl_n)$. Consider the induced edge sum of the edge uv . We have $h^{++}(uv) = h(uv_1)$. Similarly if we let $h(u_iv_{i+1}) = y_i$, for $i = 1, 2, 3, \dots, n$ with $i + 1$ is taken modulo n . Then the induced edge sum of the edges v_iv_i for $i = 1, 2, 3, \dots, n$ gives

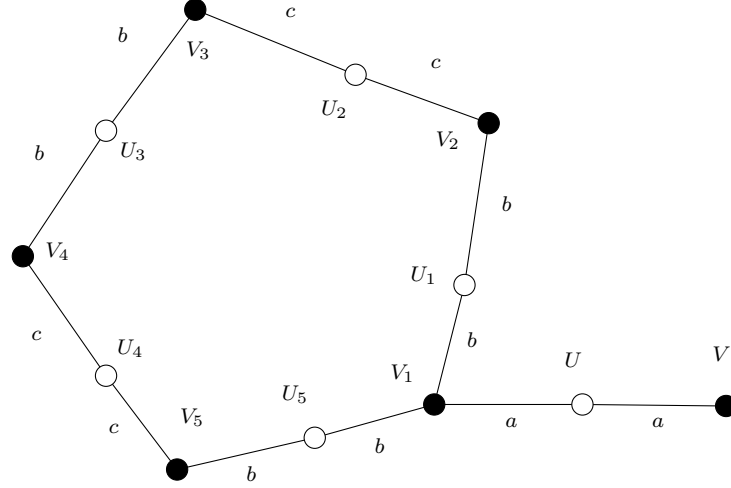
$$y_n + y_1 + h(uv_1) = y_1 + y_2 = y_2 + y_3 = \dots = y_{n-1} + y_n. \quad (2.2)$$

Since n is an even integer the above equation implies that $y_1 = y_3 = y_5 = \dots = y_{n-1} = x$ (say) and $y_2 = y_4 = y_6 = \dots = y_n = y$ (say). Thus the Equation (2.2) reduces to $x + y + h(uv_1) = x + y$, which implies that $h(uv_1) = 0$, which is not admissible. Hence there exists no such edge label h . Hence if n is an even integer, then $S(Fl_n) \notin \sigma_a(V_4)$.

Hence the proof. $\square$

Corollary 2.9. $S(Fl_n) \in \sigma(V_4)$ if and only if n is odd.

Proof. Proof follows from the above Theorem 2.8. $\square$


 Figure 2: EIML of $S(Fl_5)$

Theorem 2.10. For the sun graph Sun_n , we have $S(Sun_n) \in \sigma_a(V_4)$ for all n .

Proof. Let $\{u_i, v_i : i = 1, 2, 3, \dots, n\}$ be the vertex set of CB_n , where v_i are the pendant vertices adjacent to u_i . Also let t_i and w_i , be the inserted vertices on the edge $u_i u_{i+1}$, $u_i v_i$, for $i = 1, 2, 3, \dots, n$ and $i + 1$ is taken modulo n .

Suppose $f : E(S(Sun_n)) \rightarrow V_4 \setminus \{0\}$ is an edge induced magic label of Sun_n with $f(u_i t_i) = e_i$, $f(t_i u_{i+1}) = \alpha_i$, $f(u_i w_i) = \beta_i$ and $f(w_i v_i) = \gamma_i$.

Then using the induced edge sum equation of the edges $w_i v_i$, we get

$$\beta_1 = \beta_2 = \beta_3 = \dots = \beta_n = \beta \text{ (say)}. \quad (2.3)$$

By the induced edge sum equation of the edges $u_i t_i$, we get

$$\alpha_n + \alpha_1 + \beta = \alpha_1 + \alpha_2 + \beta = \alpha_2 + \alpha_3 + \beta = \dots = \alpha_{n-1} + \alpha_n + \beta. \quad (2.4)$$

By the induced edge sum equation of the edges $t_i u_{i+1}$, we get

$$e_1 + e_2 + \beta = e_2 + e_3 + \beta = e_3 + e_4 + \beta = \dots = e_n + e_1 + \beta. \quad (2.5)$$

By the induced edge sum equation of the edges $u_i w_i$, we get

$$\alpha_n + e_1 + \gamma_1 = \alpha_1 + e_2 + \gamma_2 = \alpha_2 + e_3 + \gamma_3 = \dots = \alpha_{n-1} + e_n + \gamma_n. \quad (2.6)$$

Case (i) n is an odd integer.

In this case, Equation (2.4) and Equation(2.5) imply that

$$\alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_n = \alpha \text{ (say)}.$$

$$e_1 = e_2 = e_3 = \dots = e_n = e \text{ (say)}.$$

Therefore, equation (2.6) implies that

$$\gamma_1 = \gamma_2 = \gamma_3 = \dots = \gamma_n = \gamma \text{ (say)}.$$

Therefore, in this case, the induced edge sum equation of the graph $S(Sun_n)$ is given by:

$$\beta = 2\alpha + \beta = 2e + \beta = \alpha + e + \gamma.$$

Since $\alpha \in V_4$, the above equation reduces to $\beta = \alpha + e + \gamma$.

Therefore, in this case, if we choose $\alpha = e = b$ and $\beta = \gamma = a$, then we can easily prove that $S(Sun_n) \in \sigma_a(V_4)$.

Case (ii) n is an even integer.

In this case, Equation (2.4) implies

$$\begin{aligned}\alpha_1 &= \alpha_3 = \alpha_5 = \cdots = \alpha_{n-1} = x_1 \text{ (say).} \\ \alpha_2 &= \alpha_4 = \alpha_6 = \cdots = \alpha_n = x_2 \text{ (say).}\end{aligned}$$

Also in this case Equation (2.5) implies

$$\begin{aligned}e_1 &= e_3 = e_5 = \cdots = e_{n-1} = y_1 \text{ (say).} \\ e_2 &= e_4 = e_6 = \cdots = e_n = y_2 \text{ (say).}\end{aligned}$$

Therefore, Equation (2.6) reduces to

$$x_2 + y_1 + \gamma_1 = x_1 + y_2 + \gamma_2 = x_2 + y_1 + \gamma_3 = x_1 + y_2 + \gamma_4 = \cdots = x_1 + y_2 + \gamma_n. \quad (2.7)$$

Note that Equation (2.7) implies that

$$\begin{aligned}\gamma_1 &= \gamma_3 = \gamma_5 = \cdots = \gamma_{n-1} = z_1 \text{ (say).} \\ \gamma_2 &= \gamma_4 = \gamma_6 = \cdots = \gamma_n = z_2 \text{ (say).}\end{aligned}$$

Therefore, in this case, the induced edge sum equation of the graph $S(Sun_n)$ is given by:

$$\beta = x_1 + x_2 + \beta = y_1 + y_2 + \beta = x_2 + y_1 + z_1 = x_1 + y_2 + z_2.$$

Therefore, in this case, if we choose $x_1 = x_2 = y_1 = y_2 = b$ and $\beta = z_1 = z_2 = a$ then we can easily prove that $f^{++}(e) = a$ for all $e \in E(S(Sun_n))$. Thus $S(Sun_n) \in \sigma_a(V_4)$.

Hence the proof. $\square$

Theorem 2.11. Let $J(m, n)$ be the jelly fish graph. Then $S(J(m, n)) \in \sigma_a(V_4)$ for m and n are of same parity.

Proof. Consider the jelly fish graph with $V(J(m, n)) = \{v_k : k = 1, 2, 3, 4\} \cup \{u_i : i = 1, 2, 3, \dots, m\} \cup \{w_j : j = 1, 2, 3, \dots, n\}$, where v'_k s are the vertices of the corresponding C_4 , u_i, w_j are the vertices of the corresponding $K_{1,m}$ and $K_{1,n}$ respectively and α_i ($1 \leq i \leq m$), β_j ($1 \leq j \leq n$) are the inserted vertices at the edges v_2u_i, v_4w_j respectively and $\alpha, \beta, \gamma, \delta, \mu$ are the vertices inserted on the edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3$ respectively.

Case (i) m and n are even integers.

In this case, define $f : E(S(J(m, n))) \rightarrow V_4 \setminus \{0\}$ by

$$f(e) = \begin{cases} a & \text{if } e = \alpha_i u_i, v_2 \alpha_i, v_4 \beta_j, \beta_j w_j, v_1 \mu, v_3 \mu \\ b & \text{if } e = v_2 \alpha, \alpha v_1, v_1 \delta, \delta v_4 \\ c & \text{if } e = v_4 \gamma, \gamma v_3, v_3 \beta, \beta v_2. \end{cases}$$

Case (ii) m and n are n odd integers.

In this case, define $f : E(S(J(m, n))) \rightarrow V_4 \setminus \{0\}$ by

$$f(e) = \begin{cases} a & \text{if } e = v_2 \alpha, \alpha v_1, v_3 \beta, \beta v_2, v_2 \alpha_i, v_4 \beta_j, \alpha_i u_i, \beta_j w_j \\ b & \text{if } e = v_1 \delta, \delta v_4, v_4 \gamma, \gamma v_3, v_1 \mu, v_3 \mu. \end{cases}$$

Then in both cases we can verify that $f^{++}(e) = a$ for all $e \in E(S(J(m, n)))$. That is f is an EIML of $S(J(m, n))$. Thus in both cases $S(J(m, n)) \in \sigma_a(V_4)$.

Hence the proof. $\square$

Theorem 2.12. *For the triangular snake graph TS_n , we have $S(TS_n) \in \sigma_0(V_4)$ for all n .*

Proof. Since every vertex is of even degree, the proof follows from Theorem 2.2. $\square$

Theorem 2.13. *For the wheel graph W_n , we have $S(W_n) \in \sigma_a(V_4)$ for n is odd.*

Proof. Suppose n is odd. Since every vertex is of odd degree, the proof follows from Theorem 2.1. $\square$

Theorem 2.14. *For the comb graph CB_n , we have $S(CB_n)$ is not an edge induced magic graph, for any n .*

Proof. Let $\{u_i, v_i : 1, 2, 3, \dots, n\}$ be the vertex set of CB_n , where v_i is the pendant vertex adjacent to u_i . Let w_i and t_j be the inserted vertices in the edges $u_i v_i$ and $u_j u_{j+1}$ for $i = 1, 2, 3, \dots, n$ and $j = 1, 2, 3, \dots, n-1$ respectively. If possible, suppose $f : E(S(CB_n)) \rightarrow V_4 \setminus \{0\}$ is an edge induced magic label of $S(CB_n)$. Then using the induced edge sum equation of the edges $v_1 w_1$ and $u_1 t_1$, we get $f(u_1 w_1) = f(t_1 u_2) + f(u_1 w_1)$. That is $f(t_1 u_2) = 0$, which is a contradiction. Hence $S(CB_n)$ is not an edge induced magic graph, for any n .
Hence the proof. $\square$

3 CONCLUSIONS

This paper has attempted to investigate the key results pertaining to edge-induced V_4 - magic labeling of graphs with same parity. Subsequently it establishes edge-induced V_4 -magic labeling characteristics for a selection of graphs, such as $P_n, C_n, K_n, K_{1,n}$ as well as the Bi-star graph, Flag graph, Sun graph, Jelly graph, Triangular Snake graph, Wheel graph and Comb graph.

4 Disclaimer (Artificial Intelligence)

Author hereby declares that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

5 Competing Interests

Author has declared that no competing interests exist.

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