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Original Research Article

A Note on the Polynomial $\mathbf{D_t^v(G,x)}$

Abstract

In various practical scenarios, especially in communication networks, sensor grids, and surveillance systems, it becomes essential to ensure that certain critical nodes (vertices) are included in every total dominating set of a graph. A total dominating set is a subset of vertices in a graph such that every vertex is adjacent to at least one vertex in this set. However, in specific applications, some nodes may serve as vital control hubs, data aggregators or monitoring stations that must be active or functional in any dominating configuration. In this paper the total domination polynomial $D_t^v(G,x)$, in which a particular vertex v of G is present in every TD-set of G is determined for certain classes of graphs.

Keywords: total domination, vertex cover, total domination polynomial.

2010 Mathematics Subject Classification: 05C31, 05C69

1 Introduction

Graph theory is one of the most relevant and fastest growing branches of mathematics. Graph theory has granted a plethora of indispensable tools in the design and analysis of communication networks, mobile computing and social networks to mention a few. In fact, the varied applications of Graph theory in Engineering, Social science, Biological science etc. have immensely contributed to the progress and popularity of mathematics in general and Graph theory in particular.

One of the prime concerns of Graph theory today is the study of graph polynomials. For a graph G, dominating set of a given cardinality may not be unique. S. Alikhani's [Ref: Alikhani, 2009] research in this field explored the concept of domination polynomial in graphs. Subsequently, S. Sanalkumar and A. Vijayan [Ref: Kumar and Vijayan, 2012]introduced the concept of total domination polynomial in graphs. The inclusion of a particular vertex in every total dominating set of a graph is important in the study of total domination polynomials. In this paper, the polynomial $D_t^v(G,x)$ is determined for some graphs.

2 Preliminaries

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite non-empty set and $E(G)$ is a collection of unordered pairs of vertices called edges. If u and v are two vertices of a graph and if the unordered pair $\{u, v\}$ is an edge denoted by e, we say that e is an edge between u and v. We write the edge $\{u, v\}$ as uv. An edge of the form uu is known as a loop. The *open neighbourhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V|uv \in E(G)\}$. If the graph G is clear from the context, we write $N(v)$ rather than $N_G(v)$. Notations and definitions not given here can be found in [Ref: Balakrishnan and Ranganathan, 2012, Berge and Minieka, 1973 or Henning and Yeo, 2008]. A *hypergraph* $H = (V, E)$ is a finite nonempty set $V = V(H)$ of elements called *vertices*, together with a finite multi set $E = E(H)$ of subsets of V, called *hyper edges* or simply *edges*. The *order* and *size* of H are $|V|$ and $|E|$, respectively. A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. Every simple graph is a 2-uniform hypergraph. In a hypergraph, an edge E_i with $|E_i| = 2$, is drawn as a curve connecting its two vertices. An edge E_i with $|E_i| = 1$, is drawn as a loop as in a graph. A subset T of vertices in a hypergraph H is a *transversal*(also called *vertex cover*) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. For further information on hypergraphs refer [Ref: Berge and Minieka, 1973 or Voloshin, 2009]. Let $\mathcal{C}(H, i)$ be the family of vertex covering sets of H with cardinality i and let $c(H, i) = |\mathcal{C}(H, i)|$. The polynomial $\mathcal{C}(H, x) = \sum_{i=\tau(H)}^{|V(H)|} c(H, i)x^i$ is defined as *vertex cover polynomial* of H. For a graph $G = (V, E)$, the *ONH(G)* or H_G is the *open neighbourhood hypergraph* of G; $H_G = (V, C)$ is the hypergraph with vertex set $V(H_G) = V$ and with edge set $E(H_G) = C = \{N_G(x)|x \in V\}$, consisting of the open neighbourhoods of vertices of V in G. A *total dominating set,* abbreviated TD-set, of a graph $G = (V, E)$ with no isolated vertex is set S of vertices of G such that every vertex of G is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Let $\mathcal{D}_t(G, i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathcal{D}_t(G, i)|$. The polynomial $\mathcal{D}_t(G, x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G, i)x^i$ is defined as *total domination polynomial* of G [Ref: Vijayan and Kumar, 2012].

Definition 2.1 (Ref:Alikhani and Jafari, 2017)**.** Let G be a graph and v be a vertex of G. Let $\mathcal{D}_t^v(G, i)$ be the family of all total dominating sets of G of cardinality i containing the vertex v. If $d_t^v(G, i) = |\mathcal{D}_t^v(G, i)|$, the polynomial $D_t^v(G, x)$ is defined as $D_t^v(G, x) = \sum_{i=1}^{|V(G)|} d_t^v(G, i)x^i$.

Definition 2.2 (Ref:Latheesh kumar, 2018)**.** Let G be a graph and v be a vertex of G. Let $d_{t_v}(G, i) = |\mathcal{D}_{t_v}(G, i)|$, where $\mathcal{D}_{t_v}(G, i) = \{S \subseteq V(G)\colon v \notin S,\ N(S) = V(G),\ |\ S\ |= i\}$. Then the polynomial $D_{t_v}(G, x)$ is defined as $D_{t_v}(G, x) = \sum_{i=1}^{|V(G)|} d_{t_v}(G, i)x^i$.

Definition 2.3 (Ref:Latheesh kumar, 2018)**.** Let G be a graph and v be a vertex of G. Let $\mathcal{C}^v(G, i)$ be the family of all vertex covering sets of G of cardinality i containing the vertex v. If $c^v(G, i) = |\mathcal{C}^v(G, i)|$, the polynomial $\mathcal{C}^v(G, x)$ is defined as $\mathcal{C}^v(G, x) = \sum_{i=1}^{|V(G)|} c^v(G, i)x^i$.

Theorem 2.1 (Ref:Dong et al., 2002)**.** *Let $G = G_1 \cup G_2$ be the union of two graphs G_1 and G_2. Then $\mathcal{C}(G, x) = \mathcal{C}(G_1, x)\mathcal{C}(G_2, x)$.*

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Theorem 2.2 (Ref: Dong et al., 2002). *For the path graph P_n, where $n \geq 2$, we have*

$$\mathcal{C}(P_n, x) = \sum_{i=0}^{n} \binom{i+1}{n-i} x^i.$$

3 Main Results

Theorem 3.1. *Let G be a graph and $v \in V(G)$. Then $D_t^v(G, x) = \mathcal{C}^v(H_G, x)$.*

Proof. Let S be a subset of $V(G)$. It is clear that a total dominating set of a graph G is a vertex covering set of the open neighborhood hypergraph, H_G of G and vice versa. Therefore, a set S is a total dominating set containing v if and only if S is a vertex covering set of H_G containing v. This completes the proof. □

Theorem 3.2. *Let G be a graph and $v \in V(G)$. Then $\mathcal{C}^v(G, x) = x\mathcal{C}(G - v, x)$.*

Proof. Let $S \subseteq V(G)$ be a vertex covering set of G of cardinality i containing the vertex v. Then $S \backslash \{v\}$ is a vertex covering set of $G-v$ of cardinality $i-1$. So for $i = 1, 2, \ldots, |V(G)|,\ c(G, i) = c(G-v, i-1)$. This proves the result. □

Theorem 3.3. *If u is a vertex of the cycle graph C_{2n+1}, then*

$$D_t^u(C_{2n+1}, x) = \sum_{i=0}^{2n} \binom{i+1}{2n-i} x^{i+1}.$$

Proof. Let $H_{C_{2n+1}}$ be the open neighborhood hypergraph of the cycle C_{2n+1}. Clearly, $H_{C_{2n+1}}$ is isomorphic to C_{2n+1}. Then from Theorems 3.1 and 3.2 we have, $D_t^u(C_{2n+1}, x) = \mathcal{C}^u(H_{C_{2n+1}}, x) = \mathcal{C}^u(C_{2n+1}, x) = x\mathcal{C}(C_{2n+1} - u, x) = x\mathcal{C}(P_{2n}, x)$. Then the result follows from Theorem 2.2. □

Theorem 3.4. *If u is a vertex of the cycle graph C_{2n}, then*

$$D_t^u(C_{2n}, x) = x\mathcal{C}(C_n, x)\mathcal{C}(P_{n-1}, x).$$

Proof. Let (X, Y) be the bipartition of C_{2n}. Assume that $u \in X$. Note that the components H_X, H_Y of $ONH(C_{2n})$ are cycles of length n. Then, from Theorems 2.1, 3.1 and 3.2 we have, $D_t^u(C_{2n}, x) = \mathcal{C}^u(H_{C_{2n}}, x) = \mathcal{C}^u(H_X, x)\mathcal{C}(H_Y, x) = x\mathcal{C}(H_X - u, x)\mathcal{C}(H_Y, x) = x\mathcal{C}(P_{n-1}, x)\mathcal{C}(C_n, x)$. This completes the proof. □

For Theorems 3.5 and 3.6 we take the path graph as $P_n = (1, 2, \ldots, n)$.

Theorem 3.5. *For the path $P_{2n} = (1, 2, \ldots, 2n)$, we have*

(i) $D_t^1(P_{2n}, x) = x^3\mathcal{C}(P_{n-1}, x)\mathcal{C}(P_{n-2}, x)$,

(ii) *For* $1 \leq r \leq n-2,\ D_t^{2r+1}(P_{2n}, x) = x^3\mathcal{C}(P_{n-1}, x)\mathcal{C}(P_r, x)\mathcal{C}(P_{n-r-2}, x)$.

Proof. Let $X = \{1, 3, \ldots, 2n-1\}$ and $Y = \{2, 4, \ldots, 2n\}$ be the bipartition of P_{2n}. Let H_X and H_Y (shown in figure 1) be the components of $ONH(P_{2n})$ corresponding to X and Y respectively.

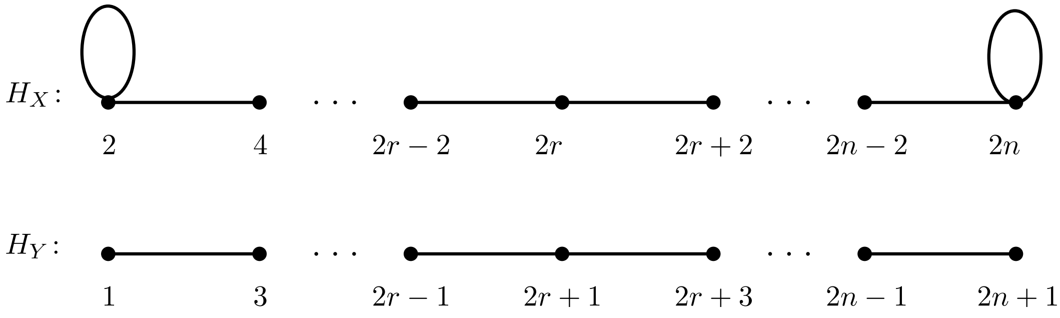

Figure 1: H_X and H_Y

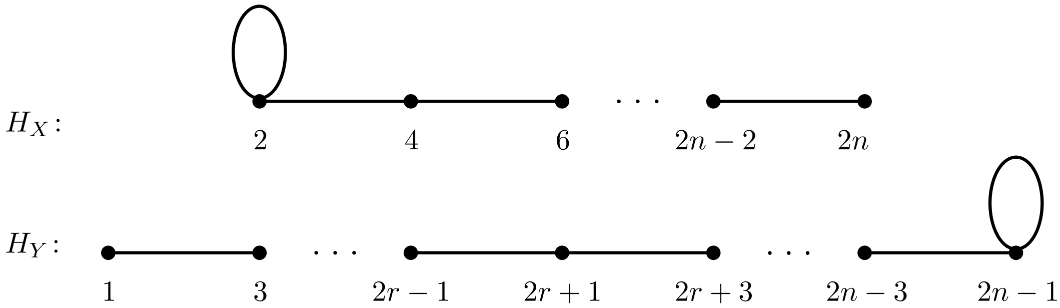

Figure 2: H_X and H_Y

(i) Using Theorems 2.1, 3.1 and 3.2 we have, $D_t^1(P_{2n},x) = \mathcal{C}^1(H_{P_{2n}},x)$
$= \mathcal{C}(H_X,x)\mathcal{C}^1(H_Y,x) = \mathcal{C}(H_X,x)\ x\ \mathcal{C}(H_Y-1,x) = x^3\mathcal{C}(P_{n-1},x)\mathcal{C}(P_{n-2},x)$.

(ii) From Theorems 2.1 and 3.2 we have, $\mathcal{C}(H_X,x) = x\mathcal{C}(P_{n-1},x)$ and $\mathcal{C}^{2r+1}(H_Y,x) = x\mathcal{C}(H_Y - (2r+1),x) = x^2\mathcal{C}(P_r,x)\mathcal{C}(P_{n-r-2},x)$. Applying Theorem 3.1, $D_t^{2r+1}(P_{2n},x) = \mathcal{C}^{2r+1}(H_{P_{2n}},x) = \mathcal{C}(H_X,x)\mathcal{C}^{2r+1}(H_Y,x)$.

Thus the result follows. □

Remark 3.1. Since $f\colon V(P_{2n}) \to V(P_{2n})$ defined by $f(k) = 2n-(k-1)$ is an isomorphism, we have $D_t^{2n}(P_{2n},x) = D_t^1(P_{2n},x)$ and $D_t^{2r}(P_{2n},x) = D_t^{2r+1}(P_{2n},x)$.

Theorem 3.6. *For the path $P_{2n+1} = (1,2,\ldots,2n+1)$, we have*

(i) $D_t^1(P_{2n+1},x) = x^3\mathcal{C}(P_n,x)\mathcal{C}(P_{n-2},x)$,

(ii) *For* $1 \le r \le n-1,\ D_t^{2r}(P_{2n+1},x) = x^3\mathcal{C}(P_{r-2},x)\mathcal{C}(P_{n-r-1},x)\mathcal{C}(P_{n+1},x)$,

(iii) *For* $1 \le r \le n-2,\ D_t^{2r+1}(P_{2n+1},x) = x^3\mathcal{C}(P_{n-2},x)\mathcal{C}(P_r,x)\mathcal{C}(P_{n-r},x)$.

Proof. Let $X = \{1,3,\ldots,2n+1\}$ and $Y = \{2,4,\ldots,2n\}$ be the bipartition of P_{2n+1}. Let H_X and H_Y (shown in figure 2) be the components of $ONH(P_{2n+1})$ corresponding to X and Y respectively. Then from Theorems 2.1, 3.1 and 3.2 we have,

(i) $\mathcal{C}(H_X,x) = x^2\mathcal{C}(P_{n-2},x)$ and $\mathcal{C}^1(H_Y,x) = x\mathcal{C}(H_Y-1,x) = x\mathcal{C}(P_n,x)$. Since $D_t^1(P_{2n+1},x) = \mathcal{C}^1(H_{P_{2n+1}},x) = \mathcal{C}(H_X,x)\mathcal{C}^1(H_Y,x)$, the proof follows.

(ii) $D_t^{2r}(P_{2n+1},x) = \mathcal{C}^{2r}(H_{P_{2n+1}},x) = \mathcal{C}^{2r}(H_X,x)\mathcal{C}(H_Y,x)$. Since $\mathcal{C}^{2r}(H_X,x)$ $= x\mathcal{C}(H_X-(2r),x) = x^3\mathcal{C}(P_{r-2},x)\mathcal{C}(P_{n-r-1},x)$ and $\mathcal{C}(H_Y,x) = \mathcal{C}(P_{n+1},x)$, the proof follows.

(iii) Proceeding as above, we get $D_t^{2r+1}(P_{2n+1},x) = \mathcal{C}(H_X,x)\mathcal{C}^{2r+1}(H_Y,x) = x^2\mathcal{C}(P_{n-2},x)x\mathcal{C}(H_Y-(2r+1),x) = x^3\mathcal{C}(P_{n-2},x)\mathcal{C}(P_r,x)\mathcal{C}(P_{n-r},x)$.

□

The following results can be easily derived from the definition of $D_t^v(G,x)$.

Theorem 3.7. *For any vertex* v *of* K_n, $D_t^v(K_n,x) = x\left[(1+x)^{n-1}-1\right]$.

Theorem 3.8. *For* $v \in V(K_{n,n})$, $D_t^v(K_{n,n},x) = x(1+x)^{n-1}\left[(1+x)^n-1\right]$.

Theorem 3.9. *If* $K_{n+1}^{(k)}$ *denotes the one point union of* k *copies of the complete graph* K_{n+1}*, then* $D_t(K_{n+1}^{(k)},x) = x\left[(1+x)^{nk}-1\right]\left[(1+x)^n-1-nx\right]^k$.

Proof. Let u be the vertex common to the k copies of K_{n+1}. Let S be a total dominating set of $K_{n+1}^{(k)}$. Then we have two possibilities. Either $u \in S$ or $u \notin S$.

Case i: If $u \in S$, then for any vertex $v \neq u$ of $K_{n+1}^{(k)}$, the set $\{u,v\}$ is a total dominating set. Since there are nk vertices in $K_{n+1}^{(k)}-u$, the number of total dominating sets of $K_{n+1}^{(k)}$ containing the vertex u of cardinality i is $\binom{nk}{i-1}$. Therefore,

$$\begin{aligned} D_t^u(K_{n+1}^{(k)},x) &= x\left[\binom{nk}{1}x + \binom{nk}{2}x^2 + \ldots + \binom{nk}{nk}x^{nk}\right] \\ &= x\left[(1+x)^{nk}-1\right]. \end{aligned}$$

Case ii: Let $u \notin S$. Let $V_1,V_2,\ldots,V_k$ be the sets of vertices of the components of $K_{n+1}^{(k)}-u$. Then $\mid S \cap V_i \mid \geq 2$. In other words, a set containing at least two vertices from each and every component of $K_{n+1}^{(k)}-u$ forms a total dominating set. Since we can select i vertices from the set V_j in $\binom{n}{i}$ ways,

$$\begin{aligned} D_{t_u}(K_{n+1}^{(k)},x) &= \left[\binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n\right]^k \\ &= \left[(1+x)^n-1-nx\right]^k. \end{aligned}$$

The proof then follows from $D_t(K_{n+1}^{(k)},x) = D_t^u(K_{n+1}^{(k)},x) + D_{t_u}(K_{n+1}^{(k)},x)$. □

4 Conclusion

In wireless sensor networks, central base stations or power-rich gateway nodes often serve as indispensable components of the monitoring infrastructure and must be included in every active configuration. Similarly, in security systems, high-value checkpoints or control hubs require uninterrupted surveillance, necessitating their inclusion in every total dominating set of the corresponding network graph. Recognizing such application-driven constraints, this paper presents a systematic approach to compute the total domination polynomial $D_t^v(G,x)$, under the condition that a specified vertex is included in every total dominating set of the graph G. This formulation not only captures essential structural properties but also enables precise modeling of real-world networks with critical node dependencies.

Disclaimer (Artificial Intelligence)

Author hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

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