Original Research Article

Global convergence in non-relativistic limits for the non-isentropic Euler-Maxwell system near non-constant equilibrium

Abstract. This work establishes the global-in-time convergence when the light speed $c \to \infty$ ($\nu = \frac{1}{c} \to 0$), demonstrating how the non-isentropic Euler-Maxwell system reduces to the Euler-Poisson system near non-constant equilibria. The non-isentropic setting introduces new challenges due to temperature effects and energy coupling, complicating dissipation estimates for the electric field E. A div-curl decomposition is required, disrupting the systems anti-symmetric structure and L^2-estimates. By constructing a tailored strictly convex entropy functional and employing refined induction arguments, we establish global convergence. Key to our analysis is the non-singularity of E under non-relativistic scaling, alongside novel estimates for thermal-electromagnetic interactions.

Keywords. Euler-Maxwell system; global-in-time convergence; non-constant equilibrium state; non-relativistic limit.

1 Introduction

This study investigates the convergence rate in the non-relativistic limit for a compressible, non-isentropic one-fluid Euler-Maxwell system describing electron dynamics around non-uniform equilibrium states.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = -\rho(E + \nu u \times B) - \rho u, \\ \rho \partial_t \theta + \rho u \cdot \nabla \theta + \frac{2}{3} p \operatorname{div} u - \frac{2}{3} \Delta \theta = \frac{1}{3} \rho |u|^2 - \rho(\theta - 1), \\ \nu \partial_t E - \nabla \times B = \nu \rho u, \quad \operatorname{div} E = b(x) - \rho, \\ \nu \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \\ t = 0, (\rho, u, E, B) = (\rho_0, u_0, E_0, B_0). \end{cases} \tag{1.1}$$

We denote $\rho, u = (u_1, u_2, u_3)^T$ the density and velocity of the fluid, and E, B the electric and magnetic field, respectively. They are all functions of the time $t > 0$ and the position $x = (x_1, x_2, x_3)^T$. The physical parameter $c > 0$ is the speed of light. We denote its reciprocal as ν. The functions $b = b(x)$, and $p = \rho\theta$ are doping profile, and the pressure, respectively. The periodic problem of the Euler-Maxwell system then reads [1, 2, 3] due to the fact that

$$\partial_t b = \partial_t \operatorname{div} E^\nu + \partial_t \rho^\nu = \operatorname{div}(\rho^\nu u^\nu) - \operatorname{div}(\rho^\nu u^\nu) = 0.$$

Now we consider the non-constant steady state of (1.1) with zero velocity. Let $\mathcal{W}_s = (\rho_s, 0, 1, E_s, B_s)$ be the steady solution to system (1.1) satisfying

$$\begin{cases} \nabla \rho_s = \rho_s \nabla \phi_s, \quad E_s = -\nabla \phi_s, \\ \nabla \times B_s = 0, \quad \operatorname{div} E_s = b(x) - \rho_s, \\ \nabla \times E_s = 0, \quad \operatorname{div} B_s = 0. \end{cases} \tag{1.2}$$

with

$$\lim_{|x|\to\infty} \rho_s(x) = 1, \qquad \lim_{|x|\to\infty} \phi_s = 0. \tag{1.3}$$

The existence and uniqueness of solutions of (1.2)-(1.3) can be easily obtained (cf. [4, 5]). Here we learn that B_s is a constant vector.

Proposition 1.1. *Let $b(x) > 0$, $b(x) \in C^{k+1}(\mathbb{R}^3)$, $\nabla b(x) \in H^k(\mathbb{R}^3)$, $k \geq 3$ when $\rho_s - b \in H^{k+1}(\mathbb{R}^3)$, the system (1.2)-(1.3) has a unique classical solution (ρ_s, ϕ_s) which satisfies*

$$0 < \inf_{x\in\mathbb{R}^3} b(x) \leq \rho_s(x) \leq \sup_{x\in\mathbb{R}^3} b(x) < \infty,$$

and

$$\|(\nabla \rho_s, \nabla \phi_s)\|_{H^k(\mathbb{R}^3)} \leq C,$$

where C depends on $\|\nabla b\|_{H^k(\mathbb{R}^3)}$.

The well-developed theory of Euler-Maxwell systems establishes that when the density $\rho > 0$, system (1.1) constitutes a first-order symmetric hyperbolic system, guaranteeing local-in-time existence and uniqueness of smooth solutions through classical results by Lax [20] and Kato [18] (see also [19, 24]). For global-in-time solutions, existing research covers various scenarios: constant background velocities [25, 26], small perturbations from constant vectors [23], isentropic systems with generalized irrotational constraints $B + \nabla \times u$ [16, 17], and more general cases involving non-constant background velocities [14, 15, 21, 30]. It should be noted that these cited results typically assume the spatial domain is either a torus or the whole of space, and often set $\nu = 1$ for simplicity. Besides, Y, Wang and Zhao [28] study the global-in-time convergence of non-relativistic limits from Euler-Maxwell systems to Euler-Poisson systems near non-constant equilibrium states by letting the reciprocal of the speed of light $\nu := \frac{1}{c} \to 0$.

This paper primarily investigates the global convergence behavior of the non-relativistic limit ($v \to 0$) in the vicinity of a general non-constant equilibrium state $\mathcal{W}s$, where $\mathcal{W}s$ is not required to be a small perturbation of a constant vector. Our analysis begins with a formal derivation of the limiting equations. By considering $(\bar{n}, \bar{u}, \bar{\theta}, \bar{E}, \bar{B})$ as the limiting values of (ρ, u, θ, E, B) and taking the formal limit $v \to 0$ in system (1.1), we obtain

$$\begin{cases} \partial_t \bar{\rho} + \operatorname{div}(\bar{\rho}\bar{u}) = 0, \\ \partial_t(\bar{\rho}\bar{u}) + \operatorname{div}(\bar{\rho}\bar{u} \otimes \bar{u}) + \nabla p(\bar{\rho}) = -\bar{\rho}\bar{E} - \bar{\rho}\bar{u}, \\ \partial_t \bar{\theta} + \bar{u} \cdot \nabla \bar{\theta} + \frac{2}{3}\bar{\theta} \operatorname{div} \bar{u} - \frac{2}{3\bar{\rho}} \Delta \bar{\theta} = \frac{|\bar{u}|^2}{3} - (\bar{\theta} - 1), \\ \nabla \times \bar{B} = 0, \quad \operatorname{div} \bar{E} = b(x) - \bar{\rho}, \\ \nabla \times \bar{E} = 0, \quad \operatorname{div} \bar{B} = 0, \end{cases}$$

From the previous analysis, we deduce that $\bar{B}$ remains a constant vector field. Given that the curl of $\bar{E}$ vanishes ($\nabla \times \bar{E} = 0$), we can express $\bar{E}$ as the negative gradient of a scalar potential $\bar{\varphi}$, i.e., $\bar{E} = -\nabla\bar{\varphi}$. By inserting this relationship into equation (1.4), we arrive at the Euler-Poisson system of equations

$$\begin{cases} \partial_t\bar{\rho} + \operatorname{div}(\bar{\rho}\bar{u}) = 0, \\ \partial_t(\bar{\rho}\bar{u}) + \operatorname{div}(\bar{\rho}\bar{u}\otimes\bar{u}) + \nabla p(\bar{\rho}) = \bar{\rho}\nabla\bar{\phi} - \bar{\rho}\bar{u}, \\ \partial_t\bar{\theta} + \bar{u}\cdot\nabla\bar{\theta} + \dfrac{2}{3}\bar{\theta}\operatorname{div}\bar{u} - \dfrac{2}{3\bar{\rho}}\Delta\bar{\theta} = \dfrac{|\bar{u}|^2}{3} - (\bar{\theta}-1), \\ \Delta\bar{\varphi} = \bar{\rho} - b(x). \end{cases} \tag{1.5}$$

2 Preliminaries and main results

2.1 Notations and inequalities

For later purpose, we introduce the following notations. For multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

Throughout this paper, the positive general constants C and c can be different in different lines. The inequality $f \lesssim g$ means that there is a constant C such that $f \leq Cg$. The constant C_η denotes constants that depend on η. Here and follows the notes $\int f := \int_{\mathbb{R}^3} f dx$, $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^3)}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R}^3)}$, and $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\mathbb{R}^3)}$ will be used. For the convenience, we introduce the hybrid spaces M_k^n, T_m and M_k whose norms are denoted as

$$\|f\|_{M_k^n}^2 := \sum_{j=1}^{n} \|\nabla^{k-j}\partial_t^j f\|^2,$$

$$\|f\|_{T_m}^2 := \sum_{j=1}^{m} \|\partial_t^j f\|^2,$$

and

$$\|f\|_{M_k}^2 := \|f\|^2 + \|\nabla^k f\|^2 + \|f\|_{M_k^{k-1}}^2 + \|f\|_{T_k}^2.$$

For any given time $T > 0$, let us introduce the Banach space

$$B_{s,T} = \bigcap_{k=0}^{s} C^k([0,T]; H^{s-k}),$$

for all t in [0, T] with the norm

$$|||f(t,\cdot)|||^2 = \sum_{\ell+|\alpha|\leq k} \|\nabla^\alpha\partial_t^\ell f(t,\cdot)\|^2.$$

The Euler-Maxwell system constitutes a first-order quasilinear system of hyperbolic type that admits symmetrization. As a consequence, applying the well-established existence theory for such systems (cf. [6, 7]), we obtain the local-in-time existence of smooth solutions to the initial value problem.

Proposition 2.1. *(Local existence of classical solutions to* (1.1) *[29]) Let* $s \geq 3$ *be an integer and* $(\rho_0 - \rho_s, u_0, \theta_0 - 1, E_0 - E_s, B_0 - B_s) \in H^s$ *with* $\rho_0 \geq 2\underline{\rho}$ *for some positive constant* $\underline{\rho}$*. Then there exists* $T_\nu > 0$ *such that system* (1.1) *has a unique smooth solution* (ρ, u, θ, E, B) *satisfying*

$$(\rho - \rho_s, u, \theta - 1, E - E_s, B - B_s) \in B_{s,T_\nu}, \quad \rho \geq \underline{\rho}.$$

Next, we introduce the Moser-type calculus inequalities, which will be frequently used in later proof. For details, we refer to [8, 9].

Lemma 2.1 (Moser-type inequality [10, 11]). Let $k \geq 1$ be an integer. Suppose $u \in H^k$, $\nabla u \in L^\infty$, and $v \in H^{k-1} \cap L^\infty$. Then for every $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq k$, it holds $\partial^\alpha(uv) - u\partial^\alpha v \in L^2$, and

$$\|\partial^\alpha(uv) - u\partial^\alpha v\| \leq C_k \left(\|\nabla u\|_\infty \left\| D^{k-1} v \right\| + \left\| D^k u \right\| \|v\|_\infty \right),$$

where C_k denotes a constant only depending on k, and

$$\left\| D^k u \right\| = \sum_{|\alpha|=k} \|\partial^\alpha u\|.$$

In particular, when $k \geq 3$, the Sobolev inequality yields

$$\|\partial^\alpha(uv) - u\partial^\alpha v\| \leq C_k \|\nabla u\|_{k-1} \|v\|_{k-1}.$$

Lemma 2.2[Commutator Estimates, [12]] Let $l \geq 1$ be an integer, and define the commutator

$$[\nabla^l, g]h = \nabla^l(gh) - g\nabla^l h.$$

If $p_0, p_1, p_2, p_3, p_4 \in [1, +\infty]$ satisfy

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

then

$$\left\| [\nabla^l, g]h \right\|_{L^{p_0}} \lesssim \|\nabla g\|_{L^{p_1}} \left\| \nabla^{l-1} h \right\|_{L^{p_2}} + \left\| \nabla^l g \right\|_{L^{p_3}} \|h\|_{L^{p_4}}.$$

In addition, for $l \geq 0$,

$$\left\| \nabla^l(gh) \right\|_{L^{p_0}} \lesssim \|g\|_{L^{p_1}} \left\| \nabla^l h \right\|_{L^{p_2}} + \left\| \nabla^l g \right\|_{L^{p_3}} \|h\|_{L^{p_4}}.$$

2.2 Main results

The main results of the paper are as follows.

To make the proof more clearer, we define $\mathcal{E}(t)$ for $k \geq 3$ as follows:

$$\mathcal{E}_1(t) := \|(\rho - \rho_s, u, \theta - 1, \nabla(\phi - \phi_s), B - B_s)\|^2, \quad \mathcal{E}_2(t) := \|\nabla^k(\rho - \rho_s, u, \theta - 1, B - B_s)\|^2,$$

$$\mathcal{E}_3(t) := \|(\rho, u, \theta)\|^2_{M_k^{k-1}}, \qquad \mathcal{E}_4(t) := \|(\rho, u, \theta)\|^2_{T_k}, \qquad \mathcal{E}(t) := \sum_{i=1}^4 \mathcal{E}_i(t).$$

The main purpose is to derive a key prior estimate of (ρ, u, θ, ϕ), which is independent of time t. We will always assume $\delta < 1$ in this section.

Theorem 2.1. *For given constant $\varepsilon_0 > 0$, assume the initial data satisfy*

$$\mathcal{E}(0) \le \varepsilon_0.$$

Then there exist positive numbers $\bar{\rho}$ and $\underline{\rho}$ such that if (ρ, u, θ, ϕ) is a smooth solution of problem (1.2)-(1.3) *satisfying*

$$\mathcal{E}(t) \le 2\delta,$$

the following estimate is valid

$$\mathcal{E}(t) + \int_0^t \mathcal{E}(\tau) d\tau \le C\mathcal{E}(0). \tag{2.1}$$

Theorem 2.2. *(The non-relativistic limit) Let the conditions in Theorem 2.1 hold. Let (ρ, u, θ, E, B) be the global solution obtained in Theorem 2.1. Assume that as $\nu \to 0$,*

$$(\rho_0 - \rho_s, u_0, \theta_0 - 1) \to (\bar{\rho}_0 - \rho_s, \bar{u}_0, \bar{\theta}_0 - 1), \quad \textit{weakly in } H^s,$$

then there exist functions $(\bar{\rho}, \bar{u}, \bar{\varphi})$ satisfying

$$(\bar{\rho} - \rho_s, \bar{u}, \nabla\bar{\varphi} - \nabla\phi_s) \in L^\infty(\mathbb{R}^+; H^s),$$

and as $\nu \to 0$, up to subsequences,

$$(\rho - \rho_s, u, \theta - 1, E - E_s, B - B_s) \overset{*}{\rightharpoonup} (\bar{\rho} - \rho_s, \bar{u}, \bar{\theta} - 1, \nabla\varphi - \nabla\phi_s, 0), \ \textit{weakly-} * \textit{in} L^\infty(\mathbb{R}^+; H^s),$$

in which $(\bar{\rho}, \bar{u}, \bar{\theta}, \nabla\bar{\varphi})$ is the global smooth solution to Euler-Poisson system (1.5) *near non-constant equilibrium $(\rho_s, 0, \theta_s, \nabla\phi_s)$.*

3 Uniform energy estimates

3.1 Global existence of solutions

Here, in this subsection, we demonstrate that the non-isentropic Euler-Maxwell system (1.1) admits global solutions uniformly for ν.

Lemma 3.1. *(L^2-estimate)For all $t \in [0, T]$, it holds*

$$\mathcal{E}_1(t) + \int_0^t \|(u, \theta - 1, \nabla\theta)\|^2 d\tau \le C(\underline{\rho}, \bar{\rho})\mathcal{E}_1(0).$$

Proof. Dotting $(1.1)_2$ and $(1.1)_3$ with $\frac{2}{3}u$ and $(1 - \theta^{-1})$ in L^2, respectively, we obtain

$$\begin{aligned} &\frac{d}{dt}\int \left[\rho(\theta - \ln\theta - 1) + \frac{2}{3}\rho_s\left(\frac{\rho}{\rho_s}\ln\frac{\rho}{\rho_s} - \frac{\rho}{\rho_s} + 1\right) + \frac{1}{3}\rho|u|^2\right] \\ &\quad + \int \left(\frac{1}{3}(1 + \theta^{-1})\rho|u|^2 + \rho\theta^{-1}(\theta - 1)^2 + \frac{2}{3}\frac{|\nabla\theta|^2}{\theta^2}\right) = -\frac{2}{3}(\rho u E - \rho u E_s). \end{aligned} \tag{3.1}$$

By $(1.1)_4, (1.1)_5$,and (1.2) we can obtain

$$\partial_t\left(\frac{1}{2}|F|^2 + \frac{1}{2}|G|^2\right) + \frac{1}{\nu}\text{div}(F \times G) = \rho u F = \rho u E - \rho u E_s, \tag{3.2}$$

notice that

$$F=E-E_s, \qquad ,G=B-B_s.$$

Then we can obtain

$$\begin{aligned}&\frac{d}{dt}\int\left[\rho(\theta-\ln\theta-1)+\frac{2}{3}\rho_s\left(\frac{\rho}{\rho_s}\ln\frac{\rho}{\rho_s}-\frac{\rho}{\rho_s}+1\right)+\frac{1}{3}\rho|u|^2+\frac{1}{3}|F|^2+\frac{1}{3}|G|^2\right]\\&\quad+\int\left(\frac{1}{3}(1+\theta^{-1})\rho|u|^2+\rho\theta^{-1}(\theta-1)^2+\frac{2}{3}\frac{|\nabla\theta|^2}{\theta^2}\right)=0.\end{aligned}$$

We deduce

$$\rho_s\left(\frac{\rho}{\rho_s}\ln\frac{\rho}{\rho_s}-\frac{\rho}{\rho_s}+1\right)=(\frac{\rho}{\rho_s}-1)^2\int_0^1\frac{1-\xi}{\xi(\frac{\rho}{\rho_s}-1)+1}d\xi\geq C(\overline{\rho})(\rho-\rho_s)^2,$$

and

$$\theta-\ln\theta-1=(\theta-1)^2\int_0^1\frac{\xi}{\xi(\theta-1)+1}d\xi\geq C(\theta-1)^2.$$

Then, integrate (3.1) and (3.2) with respect to time over $[0,t]$, one has

$$\|(\rho-\rho_s,u,\theta-1,E-E_s,B-B_s)(t)\|^2+\int_0^t\|(u,\theta-1,\nabla\theta)\|^2d\tau\leq C(\underline{\rho},\overline{\rho})\mathcal{E}_1(0).$$

Next, we will derive the higher-order derivative estimates. In order to get them easily, we transform (1.1) into the following form:

$$\begin{cases}\partial_t\rho=-\rho\mathrm{div}u-u\cdot\nabla\rho,\\ u_t+u\cdot\nabla u+\nabla(\theta-1)+u+\nabla\rho_s(\rho_s-\rho)(\rho\rho_s)^{-1}+(\theta-1)\rho^{-1}\nabla\rho\\ \quad+\rho^{-1}\nabla(\rho-\rho_s)=\nabla(\phi-\phi_s)-\nu u\times(B-B_s)-\nu u\times B_s,\\ \theta_t+u\cdot\nabla\theta+\frac{2}{3}\theta\mathrm{div}u-\frac{2}{3}\rho^{-1}\Delta\theta=\frac{1}{3}u^2-(\theta-1),\\ \Delta(\phi-\phi_s)=\rho-\rho_s.\end{cases}\tag{3.3}$$

Lemma 3.2. *Let (ρ,u,θ,ϕ) be a smooth solution of (1.1) on $\mathbb{R}^3\times(0,T]$. Then under the conditions of Proposition 2.1, there exists a constant C depending on $\overline{\rho}$ and $\underline{\rho}$, such that*

$$\begin{aligned}&\frac{d}{dt}\int\left(\rho^{-1}\theta|\nabla^k(\rho-\rho_s)|^2+\rho|\nabla^k u|^2+\frac{3}{2}\rho\theta^{-1}|\nabla^k\theta|^2+|\nabla^{k-1}(\rho-\rho_s)|^2\right)\\&\quad+(c-\eta-C\delta)\int\left(|\nabla^k u|^2+|\nabla^k\theta|^2+|\nabla^{k+1}\theta|^2\right)\\&\quad\leqslant C_\eta\|(\nabla\theta,\nabla^k(\rho-\rho_s))\|^2+(\eta+C\delta)\|(u,\rho-\rho_s)\|^2+\delta\|\nabla^{k-1}\partial_t E\|^2,\end{aligned}$$

where the suitable small positive constants η and δ will be determined later.

Proof. Applying ∇^k to $(3.3)_1$-$(3.3)_3$ and multiplying it by $\rho^{-1}\theta\nabla^k(\rho-\rho_s)$, $\rho\nabla^k u$, and $\frac{3}{2}\rho\theta^{-1}\nabla^k(\theta-1)$ in L^2, respectively, then

$$\begin{aligned}&\frac{d}{dt}\int\left(\rho^{-1}\theta|\nabla^k(\rho-\rho_s)|^2+\rho|\nabla^k u|^2+\frac{3}{2}\rho\theta^{-1}|\nabla^k\theta|^2\right)\\&\quad+\int\left(2\rho|\nabla^k u|^2+3\rho\theta^{-1}|\nabla^k\theta|^2\right)=\sum_{i=1}^9 I_i,\end{aligned}\tag{3.4}$$

where

$$I_1 := \int \rho^{-1}\theta_t|\nabla^k(\rho-\rho_s)|^2 - \rho^{-2}\rho_t\theta|\nabla^k(\rho-\rho_s)|^2 + \rho_t|\nabla^k u|^2 + \frac{3}{2}(\rho_t\theta^{-1} - \rho\theta^{-2}\theta_t)|\nabla^k\theta|^2,$$

$$I_2 := 2\int \rho\nabla^k\nabla(\phi-\phi_s)\nabla^k u, \qquad I_3 := -2\int \rho\nabla^k[(\rho\rho_s)^{-1}\nabla\rho_s(\rho_s-\rho)]\nabla^k u,$$

$$I_4 := -2\int \rho^{-1}\theta\nabla^k(u\cdot\nabla\rho)\nabla^k(\rho-\rho_s) + \rho\nabla^k(u\cdot\nabla u)\nabla^k u,$$

$$I_5 := -3\int \rho\theta^{-1}\nabla^k(u\cdot\nabla\theta)\nabla^k\theta, \qquad I_6 := 2\int \rho\theta^{-1}\nabla^k(\rho^{-1}\Delta\theta + \frac{1}{2}|u|^2)\nabla^k\theta,$$

$$I_7 := -2\int \rho^{-1}\theta\nabla^k(\rho\mathrm{div}u)\nabla^k(\rho-\rho_s) + \rho\nabla^k[\rho^{-1}\nabla(\rho-\rho_s)]\nabla^k u + \rho\nabla^k[\rho^{-1}\nabla\rho(\theta-1)]\nabla^k u,$$

$$I_8 := -2\int \rho\theta^{-1}\nabla^k\theta\nabla^k(\theta\mathrm{div}u) + \rho\nabla^k\nabla\theta\nabla^k u$$

$$I_9 := 2\int \nu\rho\nabla^k[u\times(B-B_s)]\nabla^k u, \qquad .$$

By the similar way as that in Lemma 2.3 of [28], lemma 2.1, and lemma2.2, we can obtain

$$I_1 \lesssim \delta\|\nabla^k(u,\theta,\rho-\rho_s)\|^2. \tag{3.5}$$

$$I_2 \le -\frac{d}{dt}\int |\nabla^k\nabla(\phi-\phi_s)|^2 + C_\eta\|\nabla^k(\rho-\rho_s)\|^2 + \eta\|(u,\nabla^k u,\rho-\rho_s)\|^2. \tag{3.6}$$

$$I_3 \lesssim \|\nabla^k u\|\|\nabla^k(\rho-\rho_s)\| + \|\nabla^k u\|\|\rho-\rho_s\|_{L^\infty} \le C_\eta\|\nabla^k(\rho-\rho_s)\|^2 + \eta\|(\nabla^k u,\rho-\rho_s)\|^2. \tag{3.7}$$

$$I_4 \le C_\eta\|\nabla^k(\rho-\rho_s)\|^2 + \eta\|u\|^2 + (\eta + C\delta)\|\nabla^k u\|^2. \tag{3.8}$$

$$I_5 \lesssim \delta\|\nabla^k(u,\theta,\nabla\theta)\|^2. \tag{3.9}$$

$$I_6 \le -2\int \theta^{-1}|\nabla^k\nabla\theta|^2 + (\eta + C\delta)\|\nabla^k(u,\nabla\theta)\|^2 + C_\eta\|\nabla\theta\|^2. \tag{3.10}$$

$$I_7 \le (\eta + C\delta)\|(u,\nabla^k u,\nabla^{k+1}\theta,\rho-\rho_s)\|^2 + C_\eta\|(\nabla^k(\rho-\rho_s),\nabla\theta)\|^2. \tag{3.11}$$

$$I_8 \le C_\eta\|\nabla\theta\|^2 + \eta\|\nabla^k(u,\nabla\theta)\|^2.$$

Applying ∇^{k-1} to $(1.1)_4$ and then multiplying the resulting identities by $\nabla^k B$ then integrating in $\mathbb{R}^3$, we obtain

$$\begin{aligned}\|\nabla\times(B-B_s)\|_{k-1}^2 &\le \|(\nu\nabla^{k-1}\partial_t E - \nu\nabla^{k-1}(\rho u))\|\|\nabla\times(B-B_s)\|_{k-1}\\ &\le \delta\|\nabla^{k-1}(\partial_t E,u)\|^2.\end{aligned} \tag{3.12}$$

By Hölder's and Cauchy's inequalities, we obtain

$$\begin{aligned}I_9 &\le \|\nabla^k u\|(\|B-B_s\|_{L^\infty}\|\nabla^k u\| + \|u\|_{L^\infty}\|\nabla^k(B-B_s)\|)\\ &\le \delta\|\nabla^k(u,B-B_s)\|^2 \le \delta\|\nabla^{k-1}(\partial_t E,u,\nabla u)\|^2.\end{aligned} \tag{3.13}$$

Plugging (3.5), (3.6)-(3.7), (3.8)-(3.9), (3.10), and (3.11)-(3.13) into (3.4), the proof the Lemma 3.2 can be completed.

Next, we will derive the higher-order estimate of density.

Lemma 3.3. *Let* (ρ, u, θ, ϕ) *be a solution of* (1.1) *on* $\mathbb{R}^3 \times (0, T]$, *it hold*

$$\begin{aligned}
&(c-\eta-C\delta)\|\nabla^k(\rho-\rho_s)\|^2+\|\nabla^{k-1}(\rho-\rho_s)\|^2\\
&\leqslant C_\eta\|\nabla^{k-1}u_t\|^2+C_\eta\|(u,\nabla\theta,\rho-\rho_s)\|^2\\
&\quad+(\eta+C\delta)\|\nabla^k(u,\nabla\theta)\|^2+\delta\|\nabla^{k-1}\partial_t E\|^2+\delta\|B-B_s\|^2,
\end{aligned} \tag{3.14}$$

where the suitable small positive constants η *and* δ *will be determined later.*

Proof. Applying ∇^{k-1} to $(3.3)_2$ and then multiplying the resulting identities by $\nabla^{k-1}\nabla(\rho-\rho_s)$ then integrating in $\mathbb{R}^3$, we obtain

$$\int \rho^{-1}|\nabla^{k-1}\nabla(\rho-\rho_s)|^2+\|\nabla^{k-1}(\rho-\rho_s)\|^2=\sum_{i=1}^{7} II_i, \tag{3.15}$$

where

$$\begin{aligned}
&II_1:=\int[\nabla^{k-1},\rho^{-1}]\nabla(\rho-\rho_s)\nabla^{k-1}\nabla(\rho-\rho_s), \qquad II_2:=\int\nabla^{k-1}(u\cdot\nabla u)\nabla^{k-1}\nabla(\rho-\rho_s),\\
&II_3:=\int\nabla^{k-1}u\nabla^{k-1}\nabla(\rho-\rho_s), \qquad II_4:=\int\nabla^{k-1}[\nabla\rho_s(\rho_s-\rho)(\rho\rho_s)^{-1}]\nabla^{k-1}\nabla(\rho-\rho_s),\\
&II_5:=\int\nabla^{k-1}u_t\nabla^{k-1}\nabla(\rho-\rho_s), \qquad II_6:=\int\nabla^{k-1}\nabla(\theta-1)\nabla^{k-1}\nabla(\rho-\rho_s),\\
&II_7:=\int\nabla^{k-1}[(\theta-1)\rho^{-1}\nabla\rho]\nabla^{k-1}\nabla(\rho-\rho_s),\\
&II_8=\int\nabla^{k-1}[-\nu u\times(B-B_s)-\nu u\times B_s]\nabla^{k-1}\nabla(\rho-\rho_s).
\end{aligned}$$

By the similar way as that in Lemma 2.4 of [28], we can obtain

$$\begin{aligned}
&II_1\le C_\eta\|\rho-\rho_s\|^2+\eta\|\nabla^k(\rho-\rho_s)\|^2, \qquad II_2\lesssim\delta\|\nabla^k(u,\rho-\rho_s)\|^2+\delta\|u\|^2,\\
&II_3\le C_\eta\|u\|^2+\eta\|\nabla^k(\rho-\rho_s,u)\|^2, \qquad II_4\le\eta\|\nabla^k(\rho-\rho_s)\|^2+C_\eta\|\rho-\rho_s\|^2,\\
&II_5\le\eta\|\nabla^k(\rho-\rho_s)\|^2+C_\eta\|\nabla^{k-1}u_t\|^2, \qquad II_6\le\eta\|\nabla^k(\rho-\rho_s,\nabla\theta)\|^2+C_\eta\|\nabla\theta\|^2,\\
&II_7\le\eta\|\nabla^k(\rho-\rho_s,\nabla\theta)\|^2+C_\eta\|\nabla\theta\|^2.
\end{aligned} \tag{3.16}$$

It follows from Hölder's, Gagliardo-Nirenberg's, and Cauchy's inequalities that

$$\begin{aligned}
II_8&\le\delta\|\nabla^{k-1}(u,B-B_s)\|^2+\delta\|\nabla^k(\rho-\rho_s)\|^2\\
&\le\delta\|\nabla^{k-1}(u,\nabla(\rho-\rho_s),\nabla(B-B_s))\|^2+\delta\|B-B_s\|^2.
\end{aligned} \tag{3.17}$$

Plugging (3.16)-(3.17) into (3.15), we can prove (3.14). □

Lemma 3.4. *Let* (ρ, u, θ, ϕ) *be a solution of* (1.1) *on* $\mathbb{R}^3 \times (0, T]$. *Then under the conditions of Proposition 2.1, there exists constants* C *and* c *depending on* $\overline{\rho}$ *and* $\underline{\rho}$, *such that for* $1 \le r \le k-1$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\Big(\rho^{-1}\theta|\nabla^{k-r}\partial_t^r\rho|^2+\rho|\nabla^{k-r}\partial_t^r u|^2+\frac{3}{2}\rho\theta^{-1}|\nabla^{k-r}\partial_t^r\theta|^2\Big)\\
&\quad+(c-C\delta)\int\Big(|\nabla^{k-r}\partial_t^r u|^2+|\nabla^{k-r}\partial_t^r\theta|^2+|\nabla^{k-r}\partial_t^r\nabla\theta|^2\Big)\\
&\le C\|\nabla^{k-r}\partial_t^r\rho\|^2+C\|(\rho,u,\theta,\nabla\theta)\|^2_{T_{k-1}}+C\delta\|(\rho-\rho_s,\theta-1)\|^2\\
&\quad+C\delta\|\rho,u,\theta,\nabla\theta\|^2_{M_k^{k-2}}+C\delta\|\nabla^k(\rho-\rho_s,\nabla\theta)\|^2,
\end{aligned} \tag{3.18}$$

where the suitable small positive constant δ *will be determined later.*

Proof. Applying $\nabla^{k-r}\partial_t^r$ to $(3.3)_1$-$(3.3)_3$, multiplying it by $\rho^{-1}\theta\nabla^{k-r}\partial_t^r\rho$, $\rho\nabla^{k-r}\partial_t^r u$, and $\frac{3}{2}\rho\theta^{-1}\nabla^{k-r}\partial_t^r\theta$ in L^2, respectively, we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\Big[\rho^{-1}\theta|\nabla^{k-r}\partial_t^r\rho|^2+\rho|\nabla^{k-r}\partial_t^r u|^2+\frac{3}{2}\rho\theta^{-1}|\nabla^{k-r}\partial_t^r\theta|^2\Big]\\&+\int\Big[\rho|\nabla^{k-r}\partial_t^r u|^2+\frac{3}{2}\rho\theta^{-1}|\nabla^{k-r}\partial_t^r\theta|^2\Big]=\sum_{i=1}^{8}J_i,\end{aligned}\tag{3.19}$$

where J_i, $i=1,\cdots,8$ are defined as

$$\begin{aligned}J_1&:=\frac{1}{2}\int\Big[(\rho^{-1}\theta)_t|\nabla^{k-r}\partial_t^r\rho|^2+\rho_t|\nabla^{k-r}\partial_t^r u|^2+(\rho\theta^{-1})_t|\nabla^{k-r}\partial_t^r\theta|^2\Big],\\J_2&:=\int\Big[\rho\nabla^{k-r}\partial_t^r\nabla(\phi-\phi_s)\nabla^{k-r}\partial_t^r u\Big],\\J_3&:=-\int\big[\rho\nabla^{k-r}\partial_t^r[\nabla\rho_s(\rho_s-\rho)(\rho\rho_s)^{-1}]\nabla^{k-r}\partial_t^r u\big],\\J_4&:=-\int\Big[\rho^{-1}\theta\nabla^{k-r}\partial_t^r(u\cdot\nabla\rho)\nabla^{k-r}\partial_t^r\rho+\rho\nabla^{k-r}\partial_t^r(u\cdot\nabla u)\nabla^{k-r}\partial_t^r u\Big],\\J_5&:=-\int\Big[\frac{3}{2}\rho\theta^{-1}\nabla^{k-r}\partial_t^r(u\cdot\nabla\theta)\nabla^{k-r}\partial_t^r\theta\Big],\\J_6&:=\int\Big[\rho\theta^{-1}\nabla^{k-r}\partial_t^r(\rho^{-1}\Delta\theta)\nabla^{k-r}\partial_t^r\theta+\frac{1}{2}\rho\theta^{-1}\nabla^{k-r}\partial_t^r|u|^2\nabla^{k-r}\partial_t^r\theta\Big],\\J_7&:=-\int\Big[\rho\nabla^{k-r}\partial_t^r\nabla(\theta-1)\nabla^{k-r}\partial_t^r u+\rho\theta^{-1}\nabla^{k-r}\partial_t^r(\theta\mathrm{div}u)\nabla^{k-r}\partial_t^r\theta\Big],\\J_8&:=-\int\big[\rho\nabla^{k-r}\partial_t^r\big(\rho^{-1}\nabla(\rho-\rho_s)+(\theta-1)\rho^{-1}\nabla\rho\big)\nabla^{k-r}\partial_t^r u\\&\qquad+\rho^{-1}\theta\nabla^{k-r}\partial_t^r(\rho\mathrm{div}u)\nabla^{k-r}\partial_t^r\rho\big]\\J_9&:=-\int\rho\nabla^{k-r}\partial_t^r\rho\nabla^{k-r}\partial_t^r[\nu u\times(B-B_s)+\nu u\times B_s].\end{aligned}$$

By the similar way as that in Lemma 2.5 of [28], we can obtain

$$\begin{aligned}&J_1\lesssim\delta\|\nabla^{k-r}\partial_t^r(\rho,u,\theta)\|^2,\qquad J_2\le C_\eta\|\rho\|^2_{T_{k-1}}+\eta\|\nabla^{k-r}\partial_t^r(u,\rho)\|^2,\\&J_3\le(\eta+C\delta)\|\nabla^{k-r}\partial_t^r u\|^2+C_\eta\|\nabla^{k-r}\partial_t^r\rho\|^2+C_\eta\|\rho\|^2_{T_{k-1}}+C\delta\Big(\|\rho\|^2_{M_k^{k-2}}+\|\rho-\rho_s\|_k^2\Big),\\&J_4\le C_\eta\|\nabla^{k-r}\partial_t^r\rho\|^2+(C\delta+\eta)\|(u,\rho)\|^2_{M_k^{k-1}}+C_\eta\|u\|^2_{T_{k-1}},\\&J_5\lesssim\delta\|\nabla^{k-r}\partial_t^r(\theta,\nabla\theta,u)\|^2+\delta\|(u,\nabla\theta)\|^2_{M_k^{k-1}}+\delta\|(u,\nabla\theta)\|^2_{T_{k-1}}.\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}J_6\le&-\int\theta^{-1}|\nabla^{k-r}\partial_t^r\nabla\theta|^2+(\eta+C\delta)\|\nabla^{k-r}\partial_t^r\nabla\theta\|^2+(\eta+C\delta)\|(\rho,u,\nabla\theta)\|^2_{M_k^{k-1}}\\&+C\delta\|(\nabla^k\nabla\theta,\nabla\theta)\|^2+C_\eta\|(\rho,u,\theta,\nabla\theta)\|^2_{T_{k-1}}.\end{aligned}\tag{3.21}$$

and

$$J_7\le(\eta+C\delta)\|\nabla^{k-r}\partial_t^r(u,\theta,\nabla\theta)\|^2+C_\eta\|(u,\theta)\|^2_{T_{k-1}}+C\delta\|(u,\theta-1)\|_k^2+C\delta\|u\|_{M_k^{k-1}}.\tag{3.22}$$

and

$$J_8 \leq (\eta + C\delta)\|\nabla^{k-r}\partial_t^r u\|^2 + C_\eta\|\nabla^{k-r}\partial_t^r \rho\|^2 + C_\eta\|(\rho, u, \theta, \nabla\theta)\|_{T_{k-1}}^2 + (C\delta + \eta)\|(\rho, \theta, \nabla\theta)\|_{M_k^{k-1}}^2 + C\delta\|\rho - \rho_s\|_2^2.$$

With the help of $(1.1)_5$,we can obtain

$$\nabla \times (B - B_s) = \nu\partial_t E - \nu\rho u.$$

Then we can obtain (see[29])

$$|||\nabla \times G|||_{k-1} = |||\nu\nabla F - \nu\rho u|||_{k-1} \leq C_1|||F|||_k + C_2|||u|||_{k-1}. \tag{3.23}$$

So we have

$$J_9 \leq C_\eta\|\nabla^{k-r}\partial_t^r \rho\|^2 + (C\delta + \eta)\|(u, \rho)\|_{M_k^{k-1}}^2 + C_\eta\|u\|_{T_{k-1}}^2, \tag{3.24}$$

Finally, putting (3.20), (3.21), (3.22), and (3.24) into (3.19), we can complete the proof of (3.18). □

From the result of Lemma 3.4, we still need to estimate the mixed derivative of density as follows.

Lemma 3.5. *Let (ρ, u, θ, ϕ) be a solution of (1.1) on $\mathbb{R}^3 \times (0, T]$, then under the conditions of Proposition 2.1, there exists constants C and c depending on $\overline{\rho}$ and $\underline{\rho}$, such that*

$$\begin{aligned}&\int |\nabla^{k-r-1}\partial_t^r \rho|^2 + (c - \eta - C\delta)\int |\nabla^{k-r}\partial_t^r \rho|^2 \\ &\leq C_\eta\|\nabla^{k-r-1}\partial_t^{r+1} u\|^2 + C_\eta\|(\rho, u, \theta)\|_{T_{k-1}}^2 + (\eta + C\delta)\|(\rho, u)\|_{M_k^{k-1}}^2 \\ &\quad + C\delta\|\nabla^k(\rho - \rho_s, u)\|^2 + C\delta\|(\rho - \rho_s, u)\|^2 + \eta\|\nabla^{k-r}\partial_t^r(u, \theta, \nabla\theta)\|^2,\end{aligned}$$

where the suitable small positive constants η and δ will be determined later.

Proof. Applying $\nabla^{k-r-1}\partial_t^r$ to the $(3.3)_2$ and multiplying it by $\nabla^{k-r}\partial_t^r \rho$ in L^2, we arrive at

$$\begin{aligned}\int |\nabla^{k-r-1}\partial_t^r \rho|^2 &= -\int \nabla^{k-r-1}\partial_t^r u_t \nabla^{k-r}\partial_t^r \rho - \int \nabla^{k-r-1}\partial_t^r (u \cdot \nabla u)\nabla^{k-r}\partial_t^r \rho \\ &- \int \nabla^{k-r-1}\partial_t^r u \nabla^{k-r}\partial_t^r \rho - \int \nabla^{k-r-1}\partial_t^r[\rho^{-1}\nabla(\rho - \rho_s)]\nabla^{k-r}\partial_t^r \rho \\ &- \int \nabla^{k-r-1}\partial_t^r \nabla\theta\nabla^{k-r}\partial_t^r \rho - \int \nabla^{k-r-1}\partial_t^r[\nabla\rho_s(\rho_s - \rho)(\rho\rho_s)^{-1}]\nabla^{k-r}\partial_t^r \rho \\ &- \int \nabla^{k-r-1}\partial_t^r[(\theta - 1)\rho^{-1}\nabla\rho]\nabla^{k-r}\partial_t^r \rho - \int \nabla^{k-r}\partial_t^r \rho\nabla^{k-r-1}\partial_t^r[\nu u \times (B - B_s) + \nu u \times B_s] \\ &:= \sum_{i=1}^{8} P_i.\end{aligned} \tag{3.25}$$

It is easily to deduce by [28] that

$$\begin{aligned}&P_1 \leq C_\eta\|\nabla^{k-r-1}\partial_t^{r+1} u\|^2 + \eta\|\nabla^{k-r}\partial_t^r \rho\|^2, \quad P_2 \leq C\delta\|(\rho, u)\|_{M_k^{k-1}}^2 + C\delta\|u\|_k^2, \\ &P_3 \leq C_\eta\|\nabla^{k-r-1}\partial_t^r u\|^2 + \eta\|\nabla^{k-r}\partial_t^r \rho\|^2 \leq \eta\|\nabla^{k-r}\partial_t^r(\rho, u)\|^2 + C_\eta\|u\|_{T_{k-1}}, \\ &P_4 \leq -\int \rho^{-1}|\nabla^{k-r}\partial_t^r \rho|^2 + (\eta + C\delta)\|\rho\|_{M_k^{k-1}}^2 + C_\eta\|\rho\|_{T_{k-1}}^2, \\ &P_5 + P_6 + P_7 \leq \eta\|\nabla^{k-r}\partial_t^r(\rho, \theta, \nabla\theta)\|^2 + C\delta\|\rho\|_{M_k^{k-1}}^2 + C_\eta\|(\rho, \theta)\|_{T_{k-1}}^2 + C\delta\|(\rho - \rho_s)\|_{k-2}^2.\end{aligned} \tag{3.26}$$

By means of Cauchy's inequality and (3.23), one can deduce that

$$P_8 \le \delta \|\nabla^{k-r-1}\partial_t^r(\nabla\rho, u)\|^2. \tag{3.27}$$

Inserting (3.26) and (3.27) into (3.25), the proof of Lemma 3.5 is complete. □

Lemma 3.6. *Let (ρ, u, θ, ϕ) be a solution of (1.1) on $\mathbb{R}^3 \times (0, T]$, $1 \le r \le k$. Then under the conditions of Proposition 2.1, there exists constants C and c depending on $\overline{\rho}$ and $\underline{\rho}$, such that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left(\rho^{-1}\theta|\partial_t^r\rho|^2 + \rho|\partial_t^r u|^2 + \frac{3}{2}\rho\theta^{-1}|\partial_t^r\theta|^2\right) + (c - \eta - C\delta)\|(\partial_t^r u, \partial_t^r\theta, \partial_t^r\nabla\theta)\|^2 \\
&\quad \le C_\eta\|(\partial_t^r\rho, \partial_t^r\theta, \partial_t^{r-1}u, u)\|^2 + C\delta\|(\rho, u, \theta)\|_{T_k}^2 + C\delta\|(\rho, u, \theta)\|_{M_k^{k-1}}^2 + \eta\|\nabla^k u\|^2.
\end{aligned}$$

where the suitable small positive constants η and δ will be determined later.

Proof. Applying ∂_t^r to the first three equations in system (3.3) and then multiplying the resulting identities by $\rho^{-1}\theta\partial_t^r(\rho - \rho_s)$, $\rho\partial_t^r u$, and $\frac{3}{2}\rho\theta^{-1}\partial_t^r(\theta - 1)$, respectively, summing them up and then integrating in $\mathbb{R}^3$, we derive

$$\frac{1}{2}\frac{d}{dt}\int \rho^{-1}\theta|\partial_t^r\rho|^2 + \rho|\partial_t^r u|^2 + \frac{3}{2}\rho\theta^{-1}|\partial_t^r\theta|^2 dx + \int \rho|\partial_t^r u|^2 + \frac{3}{2}\rho\theta^{-1}|\partial_t^r\theta|^2 = \sum_{i=1}^{8} V_i, \tag{3.28}$$

It is easily to deduce by the lemma 2.7 of [28]

$$\begin{aligned}
&V_1 = \frac{1}{2}\int (\rho^{-1}\theta)_t|\partial_t^r\rho|^2 + \rho_t|\partial_t^r u|^2 + (\rho\theta^{-1})_t|\partial_t^r\theta|^2, \\
&V_2 = \int \rho\partial_t^r\nabla(\phi - \phi_s)\partial_t^r u, \qquad V_3 = -\int \rho\partial_t^r\left[\nabla\rho_s(\rho_s - \rho)(\rho\rho_s)^{-1}\right]\partial_t^r u, \\
&V_4 = -\int \rho\partial_t^r(u\cdot\nabla u)\partial_t^r u + \rho^{-1}\theta\partial_t^r(u\cdot\nabla\rho)\partial_t^r\rho, \\
&V_5 = -\frac{3}{2}\int \rho\theta^{-1}\partial_t^r(u\cdot\nabla\theta)\partial_t^r\theta, \quad V_6 = -\int \rho\theta^{-1}\partial_t^r(\mathrm{div}u\theta)\partial_t^r\theta + \rho\partial_t^r\nabla\theta\partial_t^r u, \\
&V_7 = \int \rho\theta^{-1}\partial_t^r(\rho^{-1}\Delta\theta)\partial_t^r\theta + \frac{1}{2}\rho\theta^{-1}\partial_t^r|u|^2\partial_t^r\theta, \\
&V_8 = -\int \rho^{-1}\theta\partial_t^r(\rho\mathrm{div}u)\partial_t^r\rho + \rho\partial_t^r[\rho^{-1}\nabla(\rho - \rho_s)]\partial_t^r u + \rho\partial_t^r[(\theta - 1)\rho^{-1}\nabla\rho]\partial_t^r u \\
&V_9 = -\int \rho\partial_t^r u\partial_t^r[\nu u \times (B - B_s) + \nu u \times B_s].
\end{aligned}$$

Notice that

$$-\int \rho\partial_t^r u\partial_t^r(\nu u \times B_s) = 0.$$

and

$$\nu\partial_t(B - B_s) = -\nabla\times(E - E_s),$$

then we can obtain

$$\|\nu\partial_t^r(B - B_s)\| \le C\|\rho\|_{T_r}^2. \tag{3.29}$$

So we have

$$V_9 \le \delta\|(u.\rho)\|_{T_r}^2. \tag{3.30}$$

By the similar way as that in Lemma 2.7 of [28] and (3.30), we can prove the Lemma 4.10. We omit it for the sake of simplicity.

Lemma 3.7. *Let (ρ, u, θ, ϕ) be a solution of (1.1) on $\mathbb{R}^3 \times (0, T]$, $1 \le r \le k$. Then under the conditions of Proposition 2.1, there exists constants C and c depending on $\overline{\rho}$ and $\underline{\rho}$, such that*

$$\begin{aligned}
&-\frac{d}{dt}\int \left[\partial_t^{r-1}(\rho u_t)\partial_t^{r-1}(\rho u) + \partial_t^{r-1}(\rho u \cdot \nabla u)\partial_t^{r-1}(\rho u) + \frac{1}{2}|\partial_t^{r-1}(\rho u)|^2\right] \\
&+ (c - \eta - C\delta)\int \left(|\partial_t^r \rho|^2 + |\partial_t^{r-1}(\rho u)|^2 + |\partial_t^{r-1}(\rho u_t)|^2\right) \\
&\le C_\eta \|(\partial_t^{r-1} u, u)\|^2 + C\delta\|(\rho, u, \theta, \nabla\theta)\|^2_{T_{k-1}} + C\delta\|(\rho, u, \nabla\theta)\|_{M_k^{k-1}} \\
&\quad + (\eta + C\delta)\|\partial_t^r(u, \theta, \nabla\theta)\|^2 + \eta\|\nabla^k u\|^2 + C\delta\|\rho - \rho_s\|_2^2,
\end{aligned} \tag{3.31}$$

where the suitable small positive constants η and δ will be determined later.

Proof. Multiplying $(3.3)_2$ by ρ and applying ∂_t^r to it, then dotting it by $\partial_t^{r-1}(\rho u)$ in L^2, we arrive at

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int |\partial_t^{r-1}(\rho u)|^2 + \int |\partial_t^k \rho|^2 = \int \partial_t^k(\rho u_t)\partial_t^{k-1}(\rho u) + \int \partial_t^k(\rho u \cdot \nabla u)\partial_t^{k-1}(\rho u) \\
&- \int \partial_t^k(\rho \nabla \phi)\partial_t^{k-1}(\rho u) + \int \partial_t^k[\rho \nabla(\theta - 1)]\partial_t^{k-1}(\rho u) \\
&+ \int \partial_t^k[\rho^{-1}\nabla\rho_s(\rho_s - \rho)]\partial_t^{k-1}(\rho u) + \int \partial_t^k[\nabla\rho(\theta - 1)]\partial_t^{k-1}(\rho u) \\
&- \int \partial_t^k[\nu\rho u \times (B - B_s) + \nu\rho u \times B_s]\partial_t^{k-1}(\rho u) := \sum_{i=1}^{7} X_i.
\end{aligned}$$

Based on integration by parts and (3.29), we can obtain

$$X_7 \le \delta\|(u.\rho)\|^2_{T_k}. \tag{3.32}$$

By the similar way as that in Lemma 2.8 of [28] and (3.32), we can prove the (3.31). We omit it for the sake of simplicity. Time derivatives cannot be bounded by Gagliardo-Nirenberg's inequality under L^2-norms in space. Therefore, we derive the time integral estimate of the higher-order derivative of temperature through the relaxation effect of (3.3).

Lemma 3.8. *Let (ρ, u, θ, ϕ) be a solution of (1.1) on $\mathbb{R}^3 \times (0, T]$, $1 \le r \le k$. Then under the conditions of Proposition (2.1), there exists constants C and c depending on $\overline{\rho}$ and $\underline{\rho}$, such that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left(|\partial_t^{r-1}(\theta - 1)|^2 + \rho^{-1}|\partial_t^{r-1}\nabla\theta|^2\right) + (c - C\delta)\int |\partial_t^r \theta|^2 \\
&\quad \le C\delta\|(u, \theta, \nabla\theta)\|^2_{T_k} + C\delta\|\nabla\theta\|_2^2 + C\|\partial_t^{r-1}(u, \nabla\theta)\|^2,
\end{aligned}$$

where the suitable small positive constant δ will be determined later.

By the similar way as that in Lemma 2.8 of [28] and (3.30), we can prove the Lemma 4.10. We omit it for the sake of simplicity.

Lemma 3.9. *Let $G = B - B_e$, it holds:*

$$\frac{d}{dt}\left(\|\partial^\alpha E\|^2 + \|\partial^\alpha G\|^2\right) \le \delta\|\partial^\alpha(u, E - E_s)\|^2.$$

Proof. Start with the fifth equation in system (1.3):

$$\partial_t(E - E_s) - \frac{1}{\varepsilon}\,\mathrm{curl}(G) = \rho u.$$

Apply ∂^α and take the inner product with $\partial^\alpha(E - E_s)$:

$$\frac{d}{dt}\|\partial^\alpha(E - E_s)\|^2 - \frac{2}{\varepsilon}\left(\mathrm{curl}(\partial^\alpha G), \partial^\alpha(E - E_s)\right) = 2\left(\partial^\alpha(\rho u), \partial^\alpha(E - E_s)\right).$$

Next, take the fourth equation in (1.3):

$$\partial_t G + \frac{1}{\varepsilon}\,\mathrm{curl}(E - E_s) = 0.$$

Apply ∂^α and take the inner product with $\partial^\alpha G$:

$$\frac{d}{dt}\|\partial^\alpha G\|^2 + \frac{2}{\varepsilon}\left(\mathrm{curl}(\partial^\alpha(E - E_s)), \partial^\alpha G\right) = 0.$$

By using the vector identity $\mathbf{a}\cdot\mathrm{curl}(\mathbf{b}) + \mathbf{b}\cdot\mathrm{curl}(\mathbf{a}) = \mathrm{div}(\mathbf{a}\times\mathbf{b})$,

$$\begin{aligned}\frac{d}{dt}\left(\|\partial^\alpha E\|^2 + \|\partial^\alpha G\|^2\right) &= 2\left(\partial^\alpha(\rho u), \partial^\alpha(E - E_s)\right)\\ &\le \delta\|\partial^\alpha(u, E - E_s)\|^2.\end{aligned}$$

□

Proof of Theorem 1.1. Theorem 2.1. follows by combining Lemma 3.1 - Lemma 3.9.
Proof of Theorem 2.2. Recall that

$$N = \rho - \rho_s, \quad \Theta = \theta - 1, \quad F = E - E_s, \quad G = B - B_s.$$

We initially observe that the tuple (N, u, Θ, F, G) satisfies the following coupled system of equations:

$$\begin{cases} \partial_t N + u\cdot\nabla N + (N + \rho_s)\mathrm{div}u + u\cdot\nabla\rho_s = 0,\\ u_t + u\cdot\nabla u + \nabla\Theta + u + \nabla\rho_s N(\rho\rho_s)^{-1} + \Theta\rho^{-1}\nabla\rho\\ \quad +\rho^{-1}\nabla N = -F - \nu u\times G - \nu u\times B_s,\\ \theta_t + u\cdot\nabla\theta + \frac{2}{3}\theta\mathrm{div}u - \frac{2}{3}\rho^{-1}\Delta\theta = \frac{1}{3}u^2 - (\theta - 1),\\ \nu\partial_t F - \nabla\times G = \nu n u, \quad \mathrm{div}F = -N,\\ \nu\partial_t G + \nabla\times F = 0, \quad \mathrm{div}G = 0. \end{cases} \tag{3.33}$$

The uniform bound given in (2.1) shows that the family $(N_\nu, u_\nu, F_\nu, G_\nu)\nu > 0$ is uniformly bounded in the space $L^\infty(\mathbb{R}^+; H^s)$. Consequently, we can extract subsequences (still denoted by the same symbols) and find limit functions $(\bar{N}, \bar{u}, \bar{F}, \bar{G})$ belonging to $L^\infty(\mathbb{R}^+; H^s)$ such that $(N\nu, u_\nu, F_\nu, G_\nu) \rightharpoonup (\bar{N}, \bar{u}, \bar{F}, \bar{G})$ in the weak-* topology as $\nu \to 0^+$.

$$(N, u, F, G) \xrightarrow{*} (\bar{N}, \bar{u}, \bar{F}, \bar{G}), \quad \text{weakly-}* \text{ in } L^\infty(\mathbb{R}^+; H^s).$$

Furthermore, in the limit as ν approaches zero, the convergence holds in the distributional sense,

$$\nu(u\times B) \to 0, \quad \nu\partial_t G \to 0,$$

$$\nu\rho u \to 0, \quad \nu\partial_t F \to 0.$$

These results suffice for taking the limit in the Maxwell equations given in (3.33), allowing us to conclude that

$$\begin{cases} \nabla \times \bar{G} = 0, \quad \mathrm{div}\bar{F} = -\bar{N}, \\ \nabla \times \bar{F} = 0, \quad \mathrm{div}\bar{G} = 0. \end{cases} \tag{3.34}$$

Notice that ρ_s and $b(x)$ is independent of ν. Let

$$\bar{n} := \bar{N} + \rho_s, \quad \bar{E} := \bar{F} + E_s, \quad \bar{B} := \bar{G} + B_s.$$

Then one obtains from (1.2) that $(\bar{E}, \bar{B})$ satisfies

$$\begin{cases} \nabla \times \bar{B} = 0, \quad \mathrm{div}\bar{E} = b(x) - \bar{n}, \\ \nabla \times \bar{E} = 0, \quad \mathrm{div}\bar{B} = 0, \end{cases} \tag{3.35}$$

The analysis reveals that $\bar{B}$ is time-independent and spatially uniform. The curl-free condition $\nabla \times \bar{E} = 0$ implies the existence of an electric potential ϕ satisfying $\bar{E} = -\nabla\phi$. Under these conditions, equation (3.35) reduces to

$$\Delta\phi = b(x) - \bar{n}, \quad \bar{E} = -\nabla\phi. \tag{3.36}$$

Furthermore, the uniform estimate (2.1) also implies that sequence $\{(\partial_t N, \partial_t u)\}_{\nu>0}$ is bounded in $L^2([0,T]; H^{s-1})$ for any $T > 0$. By the classical compactness theories (See for instance [13]), sequences $(N)_{\nu>0}$ and $(u)_{\nu>0}$ are bounded in $C([0,T]; H^{s'})$ for any $s' \in [0, s)$, which yields that up to subsequence, (N, u) converges strongly to $(\bar{N}, \bar{u})$ by the uniqueness of the limit. These are sufficient for us to pass to the limit $\nu \to 0$ in the first two equations in (3.33) to obtain

$$\begin{cases} \partial_t \bar{N} + \bar{u} \cdot \nabla \bar{N} + (\bar{N} + n_e)\mathrm{div}\bar{u} + \bar{u} \cdot \nabla n_e = 0, \\ \partial_t(\bar{\rho}\bar{u}) + \mathrm{div}(\bar{\rho}\bar{u} \otimes \bar{u}) + \nabla p(\bar{\rho}) = -\bar{\rho}\bar{E} - \bar{\rho}\bar{u}, \\ \partial_t\bar{\theta} + \bar{u} \cdot \nabla\bar{\theta} + \dfrac{2}{3}\bar{\theta}\mathrm{div}\,\bar{u} - \dfrac{2}{3\bar{\rho}}\Delta\bar{\theta} = \dfrac{|\bar{u}|^2}{3} - (\bar{\theta} - 1). \end{cases} \tag{3.37}$$

Similarly, noticing (1.2) and (3.36), one obtains that (3.37) together with (3.36) can be rewritten into

$$\begin{cases} \partial_t \bar{n} + \bar{u} \cdot \nabla \bar{n} + \bar{n}\mathrm{div}\bar{u} = 0, \\ \partial_t \bar{u} + (\bar{u} \cdot \nabla)\bar{u} + \nabla(\bar{\rho}\bar{\theta}) = \nabla\phi - \bar{u}, \\ \partial_t\bar{\theta} + \bar{u} \cdot \nabla\bar{\theta} + \dfrac{2}{3}\bar{\theta}\mathrm{div}\,\bar{u} - \dfrac{2}{3\bar{\rho}}\Delta\bar{\theta} = \dfrac{|\bar{u}|^2}{3} - (\bar{\theta} - 1) \\ \Delta\phi = \bar{\rho} - b(x), \end{cases}$$

which is exactly the compressible Euler-Poisson equations (1.5). Then Theorem 2.2 is proofed.

4 conclusion

In the next step, we aim to calculate the concrete decay speed.

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