On Gaussian Generalized Pierre Numbers

Abstract. In this article, we define and investigate Gaussian Generalized Pierre Numbers in detail, and focus on two specialized cases: Gaussian Pierre numbers, Gaussian Pierre-Lucas numbers.

In addition, we present some identities and matrices related to these sequences, as well as recurrence relations, Binet's formulas, generating functions, Simson's formulas, and summation formulas.

Keywords: Gaussian Pierre numbers, Gaussian Pierre-Lucas numbers.

1. Introduction

In this section, we highlight some preliminary results on Pierre numbers.

The generalized Pierre sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is given by the fourth-order recurrence relations as

$$W_n = 2W_{n-1} - W_{n-4}. \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 are not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ As a result, recurrence (1.1) holds for all integer n. Soykan has executed a study on this particular sequence, for more details, see [25]

Characteristic equation of $\{W_n\}$ is

$$x^4 - 2x^3 + 1 = (x^3 - x^2 + 1)(x - 1) = 0,$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}, \\
\beta &= \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3}, \\
\gamma &= \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}, \\
\delta &= 1,
\end{aligned}$$

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= 1, \\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= -2, \\
\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= -1, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Notice that

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= 2, \\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= 0, \\
\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

For $n = 1, 2, 3....$ Thus, recurrence (1.1) is true for all integer n. For the fourth-order recurrence relations has been studied by many authors, for more detail see [20, 21, 15, 16, 19, 18, 25, 14, 13, 22,].

We now present Binet's formula for the generalized Pierre numbers.

Next, we give Binet's formula of generalized Pierre numbers.

THEOREM 1.1. *[25]Binet formula for generalized Pierre numbers can be shown as follows:*

$$\begin{aligned}
W_n &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)\alpha^n}{2\alpha^2+2\alpha-2} \\
&+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)\beta^n}{2\beta^2+2\beta-2} \\
&+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)\gamma^n}{2\gamma^2+2\gamma-2} \\
&-\frac{W_3-W_2+W_1-W_0}{2}.
\end{aligned}$$

Now we define two special cases of the sequence $\{W_n\}$ as follows: The Pierre sequence $\{P_n\}_{n\geq 0}$ and the Pierre-Lucas sequence $\{C_n\}_{n\geq 0}$ are described, orderly, by the fourth-order recurrence relations as:

$$\begin{aligned} P_n &= 2P_{n-1} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2,\ P_3 = 4, \quad n \geq 4, \\ C_n &= 2C_{n-1} - C_{n-4}, \qquad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8, \quad n \geq 4. \end{aligned}$$

The sequences $\{P_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, can be extended to negative subscripts by defining,

$$P_{-n} = 2P_{-(n-3)} - P_{-(n-4)}, \tag{1.2}$$

$$C_{-n} = 2C_{-(n-3)} - C_{-(n-4)}, \tag{1.3}$$

for $n = 1, 2, 3, ...$ orderly. As a result, recurrences (1.2)-(1.3) hold for all integer n.

Pierre and Pierre-Lucas numbers can be defined using Binet's formulas as follows.

Corollary 1.2. *For all integers n, Binet's formula of Pierre and Pierre-Lucas numbers are*

$$P_n = \frac{(\alpha^2+\alpha+1)\alpha^n}{2(\alpha^2+\alpha-1)} + \frac{(\beta^2+\beta+1)\beta^n}{2(\beta^2+\beta-1)} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)} - \frac{1}{2},$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respetively.

Next, we give some information about Gaussian sequences from literature.

We provide some Gaussian numbers that satisfy second-order and third-order recurrence relations.

- Horadam [8] introduced Gaussian Fibonacci numbers and defined by

$$GF_n = F_n + iF_{n-1}$$

  where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact, he defined these numbers as $GF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.).
- Pethe and Horadam [12] studied on Gaussian generalized Fibonacci numbers.
- Halıcı and Öz [7] studied Gaussian Pell and Pell Lucas numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1} \end{aligned}$$

where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

- Aşcı and Gürel [1] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

$$GJ_n = J_n + iJ_{n-1},$$

$$Gj_n = j_n + ij_{n-1}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

- Taşçı [26] introduced and studied Gaussian Mersenne numbers defined by

$$GM_n = M_n + iM_{n-1}$$

where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

- Taşçı [28] introduced and studied Gaussian balancing and Gaussian Lucas Balancing numbers given by, respectively,

$$GB_n = B_n + iB_{n-1},$$

$$GC_n = C_n + iC_{n-1}$$

where $B_n = 6B_{n-1} - BJ_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6Cj_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

- Ertaş and Yılmaz [5] studied Gaussian Oresme numbers and defined them as

$$GS_n = S_n + iS_{n-1}$$

where Oresme numbers are given by $S_n = S_{n-1} - \frac{1}{4}S_{n-2}$, $S_0 = 0$, $S_1 = \frac{1}{2}$.

Now, we present some Gaussian numbers with third order recurrence relations.

- Soykan and at al [23] presented Gaussian generalized Tribonacci numbers given by

$$GW_n = W_n + iW_{n-1}$$

where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition W_0, W_1, W_2.

- Taşcı [27] studied Gaussian Padovan and Gaussian Pell- Padovan numbers by written, respectively,

$$GP_n = P_n + iP_{n-1}$$

$$GR_n = R_n + iR_{n-1}$$

where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

- Cerda-Morales [3] defined Gaussian third-order Jacobsthal numbers as

$$GJ_n = J_n + iJ_{n-1}$$

where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_3 = 1$.

- Yılmaz and Soykan [29] presented Gaussian Guglielmo and Guglielmo-Lucas numbers by written respectively,

$$\begin{aligned} GT_n &= T_n + iT_{n-1}, \\ GH_n &= H_n + iH_{n-1} \end{aligned}$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$, and $H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, H_0 = 3, H_1 = 3, H_2 = 3$.

- Dikmen [4] presented Gaussian Leonardo and Leonardo-Lucas numbers given by respectively,

$$\begin{aligned} Gl_n &= l_n + il_{n-1}, \\ GH_n &= H_n + iH_{n-1} \end{aligned}$$

where $l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3$, and $H_n = 2H_{n-1} - H_{n-3}, H_0 = 3, H_1 = 2, H_2 = 4$.

- Ayrılma and Soykan [2] presented Gaussian Edouard and Edouard-Lucas numbers given by respectively,

$$\begin{aligned} GE_n &= E_n + iE_{n-1}, \\ GK_n &= K_n + iK_{n-1} \end{aligned}$$

where $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, E_0 = 0, E_1 = 1, E_2 = 7$, and $K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, K_0 = 3, K_1 = 7, K_2 = 35$.

- Soykan at al [24] presented Gaussian Bigollo and Bigollo-Lucas numbers given by respectively,

$$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + iC_{n-1} \end{aligned}$$

where $B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, B_0 = 0, B_1 = 1, B_2 = 4$, and $C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, C_0 = 3, C_1 = 4, C_2 = 6$.

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1.3. *Suppose that $f_{GW_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Pierre sequence $\{W_n\}$.*

Then $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{2\alpha^2+2\alpha-2}e^{\alpha x} \\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{2\beta^2+2\beta-2}e^{\beta x} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{2\gamma^2+2\gamma-2}e^{\gamma x} \\ &+(\frac{W_3-W_2+W_1-W_0}{-2})e^{x}.\end{aligned}$$

Proof: Using the Binet's formula of generating Pierre numbers we get

$$\begin{aligned}\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty}(\frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)\alpha^n}{2\alpha^2+2\alpha-2} \\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)\beta^n}{2\beta^2+2\beta-2} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)\gamma^n}{2\gamma^2+2\gamma-2}-\frac{W_3-W_2+W_1-W_0}{2})\frac{x^n}{n!}\end{aligned}$$

$$\begin{aligned}&= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{2\alpha^2+2\alpha-2}\sum_{n=0}^{\infty}\alpha^n\frac{x^n}{n!} \\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{2\beta^2+2\beta-2}\sum_{n=0}^{\infty}\beta^n\frac{x^n}{n!} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{2\gamma^2+2\gamma-2}\sum_{n=0}^{\infty}\gamma^n\frac{x^n}{n!}-\frac{W_3-W_2+W_1-W_0}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!}\end{aligned}$$

$$\begin{aligned}&= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{2\alpha^2+2\alpha-2}e^{\alpha x}+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{2\beta^2+2\beta-2}e^{\beta x} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{2\gamma^2+2\gamma-2}e^{\gamma x}-\frac{W_3-W_2+W_1-W_0}{2}e^{x}. \ \square\end{aligned}$$

The previous Lemma 1.3 gives the following results as particular examples.

Corollary 1.4. *Exponential generating function of Pierre and Pierre-Lucas numbers can be given as:*

a): $\sum_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}(\frac{(\alpha^2+\alpha+1)\alpha^n}{2(\alpha^2+\alpha-1)}+\frac{(\beta^2+\beta+1)\beta^n}{2(\beta^2+\beta-1)}+\frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)}-\frac{1}{2})\frac{x^n}{n!}$

$= .\frac{(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}.$

b): $\sum_{n=0}^{\infty} C_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}(\alpha^n+\beta^n+\gamma^n+1)\frac{x^n}{n!} = e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^{x}.$

2. Gaussian Generalized Pierre Numbers

In this section, we define Gaussian Generalized Pierre numbers and present some properties such as Binet's formula and generating function.

Gaussian Generalized Pierre numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2, GW_3)\}_{n\geq 0}$ are described by

$$GW_n = 2GW_{n-1} - GW_{n-4}, \tag{2.1}$$

with the initial conditions

$$\begin{aligned} GW_0 &= W_0 + i(2W_2 - W_3), \\ GW_1 &= W_1 + iW_0, \\ GW_2 &= W_2 + iW_1, \\ GW_3 &= W_3 + iW_2, \end{aligned}$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be expanded to negative subscripts by defining

$$GW_{-n} = 2GW_{-(n-3)} - GW_{-(n-4)} \tag{2.2}$$

for $n = 1, 2, 3, \ldots$. Thus, recurrence (2.1) hold for all integer n. notice that for $n \geq 0$, we obtain

$$GW_n = W_n + iW_{n-1}, \tag{2.3}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}$$

The initial few generalized Gaussian Pierre numbers with positive subscript and negative subscript are shown in the following tables.

Table 1. The initial few generalized Gaussian Pierre numbers with positive subscript

n	GW_n
0	$W_0 + i(2W_2 - W_3)$
1	$W_1 + iW_0$
2	$W_2 + iW_1$
3	$W_3 + iW_2$
4	$2W_3 - W_0 + iW_3$
5	$4W_3 - W_1 - 2W_0 + i(2W_3 - W_0)$

and with a negative subscript shown in Table 2

Table 2: The first few generalized Gaussian Pierre numbers with negative subscript

n	GW_{-n}
0	$W_0+i(2W_2-W_3)$
1	$2W_2-W_3+i(2W_1-W_2)$
2	$2W_1-W_2+i(2W_0-W_1)$
3	$2W_0-W_1+i(4W_2-W_0-2W_3)$
4	$4W_2-W_0-2W_3+i(4W_1-4W_2+4W_3)$
5	$4W_1-4W_2+4W_3+i(4W_0-4W_1+W_2)$

We can define two special cases of GW_n : $GW_n(0,1,2+i,4+2i)=GP_n$ is the sequence of Gaussian Pierre numbers , $GW_n(4,2+4i,4+2i,8+4i)=GC_n$ is the sequence of Gaussian Pierre-Lucas numbers.

Thus Gaussian Pierre numbers are defined by

$$GW_n=2W_{n-1}-W_{n-4}$$

with the initial conditions

$$GP_0=0,GP_1=1,GP_2=2+i,GP_3=4+2i.$$

Gaussian Pierre-Lucas numbers are given by

$$GC_n=2GC_{n-1}-GC_{n-4}$$

with the initial conditions

$$GC_0=4,GC_1=2+4i,GC_2=4+2i,GC_3=8+4i.$$

Note for all integer,we have

$$GP_n=P_n+iP_{n-1},$$

$$GC_n=C_n+iC_{n-1}.$$

The initial few values of Gaussian Pierre numbers, Gaussian Pierre-Lucas numbers, with positive and negative subscript are given in the Table 3.

Table 3. Gaussian Pierre numbers, Gaussian Pierre-Lucas numbers, with positive and negative subscripts, specialized cases of generalized Pierre numbers

n	0	1	2	3	4	5	6
GP_n	0	1	$2+i$	$4+2i$	$8+4i$	$15+8i$	$28+15i$
GP_{-n}	0	0	$-i$	-1	0	$-2i$	$-2+i$
GC_n	4	$2+4i$	$4+2i$	$8+4i$	$12+8i$	$22+12i$	$40+22i$
GC_{-n}	4	0	$6i$	$6-4i$	-4	$12i$	$12-14i$

Next, we show the Binet's formula for the generalized Pierre numbers.

Theorem 2.1. *The Binet's formula for the Gaussian generalized Pierre numbers are*

$$\begin{aligned}GW_n &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)\alpha^n}{2\alpha^2+2\alpha-2}\\&+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)\beta^n}{2\beta^2+2\beta-2}\\&+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)\gamma^n}{2\gamma^2+2\gamma-2}\\&+\frac{(\delta W_3-\delta(2-\delta)W_2+(-\delta^2+\delta+1)W_1-W_0)\delta^n}{2\delta^2+2\delta-2}\\&+i(\frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)\alpha^{n-1}}{2\alpha^2+2\alpha-2}\\&+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)\beta^{n-1}}{2\beta^2+2\beta-2}\\&+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)\gamma^{n-1}}{2\gamma^2+2\gamma-2}\\&+\frac{(\delta W_3-\delta(2-\delta)W_2+(-\delta^2+\delta+1)W_1-W_0)\delta^{n-1}}{2\delta^2+2\delta-2}).\end{aligned}$$

Proof. The proof follows from (1.1) and (2.3). □

The previous Theorem gives the following results.

Corollary 2.2. *For all integers n, we have following identities:*

(a):

$$\begin{aligned}GP_n = \frac{(\alpha^2+\alpha+1)\alpha^n}{2\alpha^2+2\alpha-2} &+\frac{(\beta^2+\beta+1)\beta^n}{2\beta^2+2\beta-2}+\frac{(\gamma^2+\gamma+1)\gamma^n}{2\gamma^2+2\gamma-2}-\frac{1}{2}\\&+i(\frac{(\alpha^2+\alpha+1)\alpha^{n-1}}{2\alpha^2+2\alpha-2}+\frac{(\beta^2+\beta+1)\beta^{n-1}}{2\beta^2+2\beta-2}+\frac{(\gamma^2+\gamma+1)\gamma^{n-1}}{2\gamma^2+2\gamma-2}-\frac{1}{2}).\end{aligned}$$

(b):

$$GC_n=(\alpha^n+\beta^n+\gamma^n+1)+i(\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}+1).$$

The next Theorem shows the generating function of Gaussian generalized Pierre numbers.

Theorem 2.3. *Let $f_{GW_n}(x)=\sum_{n=0}^{\infty}GW_nx^n$ give the generating function of Gaussian generalized Pierre numbers are shown as follows:*

$$\begin{aligned}f_{GW_n}(x) &= \sum_{n=0}^{\infty}GW_nx^n\\&= \frac{GW_0+(GW_1-2GW_0)x+(GW_2-2GW_1)x^2+(GW_3-2GW_2)x^3}{1-2x+x^4}. \qquad (2.4)\end{aligned}$$

Proof. Using the definition of Gaussian Pierre numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we get

$$\begin{aligned}
(1-2x+x^4)f_{GW_n}(x) &= \sum_{n=0}^{\infty} GW_n x^n - 2x\sum_{n=0}^{\infty} GW_n x^n + x^4\sum_{n=0}^{\infty} GW_n x^n, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 2\sum_{n=0}^{\infty} GW_n x^{n+1} + \sum_{n=0}^{\infty} GW_n x^{n+4}, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 2\sum_{n=1}^{\infty} GW_{(n-1)} x^n + \sum_{n=4}^{\infty} GW_{(n-4)} x^n, \\
&= (GW_0 + GW_1x + GW_2x^2 + GW_3x^3) - 2(GW_0x + GW_1x^2 + GW_2x^3) \\
&\quad + \sum_{n=4}^{\infty}(GW_n - 2GW_{n-1} + GW_{n-4})x^n, \\
&= GW_0 + (GW_1 - 2GW_0)x + (GW_2 - 2GW_1)x^2 + (GW_3 - 2GW_2)x^3.
\end{aligned}$$

And reorganizing above equation, we get (2.4). □

Theorem (2.3) gives the following results as specialized cases,

$(1-2x+x^4)f_{GP_n}(x) = GP_0 + (GP_1 - 2GP_0)x + (GP_2 - 2GP_1)x^2 + (GP_3 - 2GP_2)x^3 = x + ix^2$,

$(1-2x+x^4)f_{GC_n}(x) = GC_0 + (GC_1 - 2GC_0)x + (GC_2 - 2GC_1)x^2 + (GC_3 - 2GC_2)x^3 = 4 + (4i-6)x - 6ix^2$.

Theorem 2.4. *Binet's formula of generalized Tetranacci polynomials: Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$).*

$$\begin{aligned}
GW_n &= \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
&\quad + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
q_1 &= GW_0\alpha^3 + (GW_1 - rGW_0)\alpha^2 + (GW_2 - rGW_1 - sGW_0)\alpha + (GW_3 - rGW_2 - sGW_1 - tGW_0), \\
q_2 &= GW_0\beta^3 + (GW_1 - rGW_0)\beta^2 + (GW_2 - rGW_1 - sGW_0)\beta + (GW_3 - rGW_2 - sGW_1 - tGW_0), \\
q_3 &= GW_0\gamma^3 + (GW_1 - rGW_0)\gamma^2 + (GW_2 - rGW_1 - sGW_0)\gamma + (GW_3 - rGW_2 - sGW_1 - tGW_0), \\
q_4 &= GW_0\delta^3 + (GW_1 - rGW_0)\delta^2 + (GW_2 - rGW_1 - sGW_0)\delta + (GW_3 - rGW_2 - sGW_1 - tGW_0).
\end{aligned}$$

Corollary 2.5. *According to the above theorem, the results obtained from the generalized Gaussian Pierre numbers and Gaussian Pierre-Lucas numbers are follows*

$$\begin{aligned}
\sum_{n=0}^{\infty} GP_n x^n &= \frac{x + ix^2}{1 - 2x + x^4}, \\
\sum_{n=0}^{\infty} GC_n x^n &= \frac{4 + (4i-6)x - 6ix^2}{1 - 2x + x^4}.
\end{aligned}$$

LEMMA 2.6. *Suppose that* $f_{GW_n}(x)=\sum\limits_{n=0}^{\infty}GW_n\frac{x^n}{n!}$ *is the Exponential Gaussian Generating Function of the generalized Pierre sequence* $\{GW_n\}$.

Then $\sum\limits_{n=0}^{\infty}GW_n\frac{x^n}{n!}$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty}GW_n\frac{x^n}{n!} &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{2\alpha^2+2\alpha-2}e^{\alpha x}\\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{2\beta^2+2\beta-2}e^{\beta x}\\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{2\gamma^2+2\gamma-2}e^{\gamma x}+(\frac{W_3-W_2+W_1-W_0}{-2})e^{x}.\\ &+i(\frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{\alpha(2\alpha^2+2\alpha-2)}e^{\alpha x}\\ &+\frac{\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{\beta(2\beta^2+2\beta-2)}e^{\beta x}\\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{\gamma(2\gamma^2+2\gamma-2)}e^{\gamma x}+(\frac{W_3-W_2+W_1-W_0}{-2})e^{x}).\end{aligned}$$

Proof. The proof follows from the Binet's formula of GW_n and $GW_n=W_n+iW_{n-1}$ Lemma1.3.

The previous Lemma 2.6 gives the following results as particular examples.

COROLLARY 2.7. *Exponential Gaussian Generating Function of Pierre and Pierre-Lucas numbers*

(a): $\sum\limits_{n=0}^{\infty}P_n\frac{x^n}{n!}=\sum\limits_{n=0}^{\infty}((\frac{(\alpha^2+\alpha+1)\alpha^n}{2(\alpha^2+\alpha-1)}+\frac{(\beta^2+\beta+1)\beta^n}{2(\beta^2+\beta-1)}+\frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)}-\frac{1}{2})+i(\frac{(\alpha^2+\alpha+1)\alpha^{n-1}}{2(\alpha^2+\alpha-1)}+\frac{(\beta^2+\beta+1)\beta^{n-1}}{2(\beta^2+\beta-1)}+\frac{(\gamma^2+\gamma+1)\gamma^{n-1}}{2(\gamma^2+\gamma-1)}-\frac{1}{2}))\frac{x^n}{n!}$

$=.\frac{(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}+i(\frac{(\alpha^2+\alpha+1)}{2\alpha(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{(\beta^2+\beta+1)}{2\beta(\beta^2+\beta-1)}e^{\beta x}+\frac{(\gamma^2+\gamma+1)\gamma^n}{2\gamma(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}).$

(b): $\sum\limits_{n=0}^{\infty}C_n\frac{x^n}{n!}=\sum\limits_{n=0}^{\infty}(\alpha^n+\beta^n+\gamma^n+1+i(\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}+1))\frac{x^n}{n!}$

$=e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^{x}+i(\frac{1}{\alpha}e^{\alpha x}+\frac{1}{\beta}e^{\beta x}+\frac{1}{\gamma}e^{\gamma x}+e^{x}).$

3. Obtaining Binet Formula From Generating Function

We next find Binet formula generalized Gaussian Pierre number $\{GW_n\}$ by the use of generating function for GW_n.

THEOREM 3.1. *(Binet formula of generalized Gaussian Pierre numbers)*

$$G_n=\frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

where

$$\begin{aligned} q_1 &= GW_0\alpha^3+(GW_1-2GW_0)\alpha^2+(GW_2+2GW_1)\alpha+(GW_3+2GW_2), \\ q_2 &= GW_0\beta^3+(GW_1-2GW_0)\beta^2+(GW_2+2GW_1\beta+(GW_3+2GW_2), \\ q_3 &= GW_0\gamma^3+(GW_1-2GW_0)\gamma^2+(GW_2+2GW_1)\gamma+(GW_3+2GW_2), \\ q_4 &= GW_0\delta^3+(GW_1-2GW_0)\delta^2+(GW_2+2GW_1)\delta+(GW_3+2GW_2). \end{aligned}$$

Proof. Let

$$h(x)=1-2x+x^4.$$

Then for some α,β,γ and δ we write

$$h(x)=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)$$

i.e.,

$$1-2x+x^4=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).$$

Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α,β,γ and δ as the roots of

$$h(\frac{1}{x})=1-\frac{2}{x}+\frac{1}{x^4}=0.$$

This implies$1-2x+x^4=0$. Now, by it follows that

$$\sum_{n=0}^{\infty}GW_nx^n=\frac{GW_0+(GW_1-2GW_0)x+(GW_2-2GW_1)x^2+(GW_3-2GW_2)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\begin{aligned} & \frac{GW_0+(GW_1-2GW_0)x+(GW_2-2GW_1)x^2+(GW_3-2GW_2)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} \\ = & \frac{B_1}{(1-\alpha x)}+\frac{B_2}{(1-\beta x)}+\frac{B_3}{(1-\gamma x)}+\frac{B_4}{(1-\delta x)}. \end{aligned}$$

So

$$\begin{aligned} & GW_0+(GW_1-2GW_0)x+(GW_2-2GW_1)x^2+(GW_3-2GW_2)x^3 \\ = & B_1(1-\beta x)(1-\gamma x)(1-\delta x)+B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\ & +B_3(1-\alpha x)(1-\beta x)(1-\delta x)+B_3(1-\alpha x)(1-\beta x)(1-\gamma x). \end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get $GW_0+(GW_1-2GW_0)\frac{1}{\alpha}+(GW_2-2GW_1)\frac{1}{\alpha^2}+(GW_3-2GW_2)\frac{1}{\alpha^3}=B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$.

This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3(GW_0+(GW_1-2GW_0)\frac{1}{\alpha}+(GW_2-2GW_1)\frac{1}{\alpha^2}+(GW_3-4GW_2)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{GW_0\alpha^3+(GW_1-2GW_0)\alpha^2+(GW_2-2GW_1)\alpha+(GW_3-2GW_2)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{GW_0\beta^3 + (GW_1 - 2GW_0)\beta^2 + (GW_2 - 2GW_1)\beta + (GW_3 - 2GW_2)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{GW_0\gamma^3 + (GW_1 - 2GW_0)\gamma^2 + (GW_2 - 2GW_1)\gamma + (GW_3 - 2GW_2)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{GW_0\delta^3 + (GW_1 - 2GW_0)\delta^2 + (GW_2 - 2GW_1)\delta + (GW_3 - 2GW_2)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus it can be written as

$$\sum_{n=}^{\infty} GW_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} GW_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$GW = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n$$

In this section,we present Simson's formula of generalized Gaussian Pierre numbers. For all integers n we have

4. Some Identities About Recurrence Relations of Gaussian Generalized Pierre Numbers

In this section, we show some identities on Gaussian Pierre, Gaussian Pierre-Lucas.

THEOREM 4.1. *The following equations hold for all integer n*

$$\begin{aligned} GP_n &= \frac{4}{11}GC_{n+5} - \frac{2}{11}GC_{n+4} - \frac{3}{11}GC_{n+3} - \frac{9}{22}GC_{n+2}, \qquad (4.1) \\ GC_n &= 4GP_{n+3} - 6GP_{n+2}. \end{aligned}$$

Proof. To proof identity (4.1), we can write

$$GP_n = aGC_{n+3} + bGC_{n+2} + cGC_{n+1} + dGC_n$$

and solving the system of equations

$$\begin{aligned} GP_0 &= aGC_3 + bGC_2 + cGC_1 + dGC_0 \\ GP_1 &= aGC_4 + bGC_3 + cGC_2 + dGC_1 \\ GP_2 &= aGC_5 + bGC_4 + cGC_3 + dGC_2 \\ GP_3 &= aGC_6 + bGC_5 + cGC_4 + dGC_3 \end{aligned}$$

we get $a = \frac{4}{11}, b = -\frac{2}{11}, c = -\frac{3}{11}, d = -\frac{9}{22}$.

The other identities can be found similarly.

$$GC_n = aGP_{n+3} + bGP_{n+2} + cGP_{n+1} + dGP_n$$

From the above equation, the following can be obtained.

$$\begin{aligned} GC_0 &= aGP_3 + bGP_2 + cGP_1 + dGP_0 \\ GC_1 &= aGP_4 + bGP_3 + cGP_2 + dGP_1 \\ GC_2 &= aGP_5 + bGP_4 + cGP_3 + dGP_2 \\ GC_3 &= aGP_6 + bGP_5 + cGP_4 + dGP_3 \end{aligned}$$

we have $a = 0, b = 0, c = 4, d = -6$.

LEMMA 4.2. *6, Let's consider that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2},$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}},$$

respectively.

The generating functions of the even and odd-indexed generalized Pierre sequences are ensured by the following theorem.

THEOREM 4.3. *The generating functions of the sequence* GW_{2n} *and* GW_{2n+1} *are ensured by*

$$f_{GW_{2n}}(x) = \frac{-2x^2GW_3 + (-x^3 + 4x^2 - x)GW_2 + 2x^3GW_1 + (-x^2 + 4x - 1)GW_0}{x^4 - 2x^2 + 4x - 1}, \tag{4.2}$$

$$f_{GW_{2n+1}}(x) = \frac{(-x^3 - x)GW_3 + 2x^3GW_2 + (-x^2 + 4x - 1)GW_1 + 2x^2GW_0}{x^4 - 2x^2 + 4x - 1}. \tag{4.3}$$

Proof. We only proof (4.2). From Theorem 2.3 we can obtain following identities.

$$f_{GW_n}(\sqrt{x}) = \frac{GW_0 + (GW_1 - 2GW_0)\sqrt{x} + (GW_2 - 2GW_1)x + (GW_3 - 2GW_2)x\sqrt{x}}{1 - 2\sqrt{x} + x^2},$$

$$\begin{aligned} f_{GW_n}(-\sqrt{x}) &= -\frac{(GW_0 + (GW_1 - GW_0)(-\sqrt{x}) + (GW_2 - GW_1 - 2GW_0)x}{1 + 2\sqrt{x} + x^2}. \\ &\quad \frac{+(GW_3 - GW_2 - 2GW_1 + GW_0)(-x\sqrt{x})}{1 + 2\sqrt{x} + x^2} \end{aligned}$$

Thereby, Theorem 4.3 can be proved by Lemma 4.2. The other identity can be found similarly. □

From Theorem 4.3, we get the following corollary.

Corollary 4.4.

$$
\begin{aligned}
f_{GP_{2n}}(x) &= \frac{(2+i)\,x+ix^3}{x^4+2x^2-4x+1},\\
f_{GP_{2n+1}}(x) &= \frac{1+2ix+x^2}{x^4+2x^2-4x+1}\\
f_{GC_{2n}}(x) &= \frac{4+x(-12+2i)+4x^2-6ix^3}{x^4+2x^2-4x+1},\\
f_{GK-C_{2n+1}}(x) &= \frac{2+4i-12ix+x^2(4i-6)}{x^4+2x^2-4x+1}.
\end{aligned}
$$

From Corollary 4.4 we can get the following corollary which shows the identities on Gaussian Pierre sequences.

Corollary 4.5. *For all integer n*

a): $(2+i)\,GC_{2n-2}+(i)GC_{2n-6}=4GP_{2n}+(2i-12)GP_{2n-2}+4GP_{2n-4}-6iGP_{2n-6}$,

b): $GC_{2n+1}+2iGC_{2n-1}+GC_{2n-3}=(4i-6)GP_{2n-3}-12iGP_{2n-1}+(2+4i)GP_{2n+1}$,

c): $GC_{2n}+2iGC_{2n-2}+GC_{2n-4}=4GP_{2n+1}+(-12+2i)GP_{2n-1}+4GP_{2n-3}-6iGP_{2n-5}$,

d): $(2+i)GC_{2n-1}+iGC_{2n-5}=(2+4i)\,GP_{2n}-12iGP_{2n-2}+(4i-6)GP_{2n-4}$,

e): $(2+i)\,GP_{2n-1}+iGP_{2n-5}=GP_{2n}+2iGP_{2n-2}+GP_{2n-4}$,

f): $4GC_{2n+1}+(2i-12)\,GC_{2n-1}+4GC_{2n-3}-6iGC_{2n-5}=(2+4i)\,GC_{2n}-12iGC_{2n-2}+(4i-6)GC_{2n-4}$.

Proof. From corollary 4.4 we have

$$((2+i)\,x+ix^3)f_{GC_{2n}}(x)=(4+x(-12+2i)+4x^2-6ix^3)f_{GP_{2n}}(x).$$

LHS is equal to

$$
\begin{aligned}
LHS &= ((2+i)\,x+ix^3)\sum_{n=0}^{\infty}GC_{2n}x^n,\\
&= (2+i)\,x\sum_{n=0}^{\infty}GC_{2n}x^n+ix^3\sum_{n=0}^{\infty}GC_{2n}x^n,\\
&= (2+i)\sum_{n=0}^{\infty}GC_{2n}x^{n+1}-i\sum_{n=0}^{\infty}GC_{2n}x^{n+3},\\
&= (2+i)\sum_{n=1}^{\infty}GC_{2n-2}x^n-i\sum_{n=3}^{\infty}GC_{2n-6}x^n,\\
&= (2+i)\,4x+(2+i)\,(4+2i)x^2+\sum_{n=2}^{\infty}((2+i)GC_{2n-2}-iGC_{2n-6})x^n.
\end{aligned}
$$

Whereas the RHS is equal to

$$\begin{aligned}
RHS &= (4+x(-12+2i)+4x^2-6ix^3)\sum_{n=0}^{\infty} GP_{2n}x^n, \\
&= 4\sum_{n=0}^{\infty} GP_{2n}x^n+(-12+2i)\,x\sum_{n=0}^{\infty} GP_{2n}x^n+4x^2\sum_{n=0}^{\infty} GP_{2n}x^n-6ix^3\sum_{n=0}^{\infty} GP_{2n}x^n \\
&= 4\sum_{n=0}^{\infty} GP_{2n}x^n+(-12+2i)\sum_{n=0}^{\infty} GP_{2n}x^{n+1}+4\sum_{n=0}^{\infty} GP_{2n}x^{n+2}-6i\sum_{n=0}^{\infty} GP_{2n}x^{n+3} \\
&= 4\sum_{n=0}^{\infty} GP_{2n}x^n+(-12+2i)\sum_{n=1}^{\infty} GP_{2n-2}x^n+4\sum_{n=2}^{\infty} GP_{2n-4}x^n-6i\sum_{n=3}^{\infty} GP_{2n-6}x^n \\
&= (4+2i)x+4(8+4i)x^2(2i-12)(2+i)x^2+\sum_{n=2}^{\infty}(4GP_{2n}+(-12+2i)\,GP_{2n-2} \\
&\quad +4GP_{2n-4}x^n-6iGP_{2n-6})
\end{aligned}$$

by comparing with the coefficients the proof of the first identity (a) is done. We can prove other identities similarly. □

We can get an identity consisted of Gaussian Pierre numbers and Pierre-Lucas numbers given below.

Corollary 4.6. *For all integers m,n the following identities holds:*

$$GW_{m+n}=P_{m-2}GW_{n+3}-P_{m-5}GW_{n+2}-P_{m-4}GW_{n+1}-P_{m-3}GW_n.$$

Proof. First we assume that $m,n\geq 0$ theorem 4.6 can be proved by mathematical induction on m. If $m=0$ we get

$$GW_n=P_{-2}GW_{n+3}-P_{-5}GW_{n+2}-P_{-4}GW_{n+1}-P_{-3}GW_n.$$

which is true since $P_{-2}=0, P_{-3}=-1, P_{-4}=0, P_{-5}=0$. Suppose that the equality holds for $m\leq k$. For $m=k+1$, we obtain

$$\begin{aligned}
GW_{k+1+n} &= 2GW_{n+k}+-GW_{n+k-3}, \\
&= 2\left(P_{k-2}GW_{n+3}-P_{k-5}GW_{n+2}-P_{k-4}GW_{n+1}-P_{k-3}GW_n\right) \\
&\quad -\left(2P_{k-5}GW_{n+3}-P_{k-8}GW_{n+2}-P_{k-6}GW_{n+1}-P_{k-6}GW_n\right)
\end{aligned}$$

as a result, by mathematical induction on m, this proves Theorem 4.6.

The other cases of m,n can be proved smilarly for all integers m,n. □

Taking $GW_n=GP_n$ or $GW_n=GC_n$ in above Theorem, respectively, we obtain:

Corollary 4.7.

$$\begin{aligned}
GP_{m+n} &= P_{m-2}GP_{n+3}-P_{m-5}GP_{n+2}-P_{m-4}GP_{n+1}-P_{m-3}GP_n, \\
GC_{m+n} &= P_{m-2}GC_{n+3}-P_{m-5}GC_{n+2}-P_{m-4}GC_{n+1}-P_{m-3}GC_n.
\end{aligned}$$

5. SIMSON'S FORMULA

In this section, we show Simson's formula of generalized Gaussian Pierre numbers. This is a specialized case of [17,Theorem 4.1]. We give the proof by computing determinant and using Binet's formula of Gaussian generalized Pierre numbers.

THEOREM 5.1. *For all integers n, we can write the following equality:*

$$\begin{vmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1} & GW_{n-2} & GW_{n-3} \end{vmatrix} = \begin{vmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{vmatrix}$$

$$= (GW_3 - 2GW_2 + GW_0)(GW_3 - 2GW_1 + GW_0)(GW_3^2 - GW_2^2$$

$$+GW_1^2 - GW_0^2 - GW_2GW_3 - 2GW_1GW_3 + GW_1GW_2 + GW_0GW_3 + 2GW_0GW_2 - GW_0GW_1).$$

Proof. Using Theorem 2.1 it can be proved by using induction use [17,Theorem 4.1]

From Theorem 5.1 we obtain the following Corollary.

COROLLARY 5.2. *For all integers n, the Simson's formulas of Pierre and Pierre- Lucas numbers are deduced as respectively*

a): $$\begin{vmatrix} GP_{n+3} & GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+2} & GP_{n+1} & GP_n & GP_{n-1} \\ GP_{n+1} & GP_n & GP_{n-1} & GP_{n-2} \\ GP_n & GP_{n-1} & GP_{n-2} & GP_{n-3} \end{vmatrix} = 2 - 2i.$$

b): $$\begin{vmatrix} GC_{n+3} & GC_{n+2} & GC_{n+1} & GC_n \\ GC_{n+2} & GC_{n+1} & GC_n & GC_{n-1} \\ GC_{n+1} & GC_n & GC_{n-1} & GC_{n-2} \\ GC_n & GC_{n-1} & GC_{n-2} & GC_{n-3} \end{vmatrix} = -352 + 352i.$$

6. SUM FORMULAS

In this section, we identify some sum formulas of generalized Gaussian Pierre numbers

THEOREM 6.1. *For all integers $n \geq 0$, we obtain sum formulas below*

a)$\sum_{k=0}^{n} GW_k = \dfrac{1}{2}(-(n+3)W_{n+3} + (n+4)W_{n+2} + (n+3)W_{n+1} + (n+4)W_n + 3W_3 - 4W_2 - 3W_1 - 2W_0).$

b) $\sum_{k=0}^{n} GW_{2k} = \dfrac{1}{2}(-(n+2)W_{2n+2} + (n+3)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 2W_2 - 3W_1 - W_0).$

c)$\sum_{k=0}^{n} GW_{2k+1} = \dfrac{1}{2}(-(n+1)W_{2n+2} + (n+3)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 3W_2 - W_1 - 2W_0).$

Proof. It is given in Soykan [19, Theorem 3.10]. □

As a specialized case of the Theorem 6.1, we give following corollary.

COROLLARY 6.2. *For all integers $n \geq 0$, we have sum formulas below:*

a)$\sum_{k=0}^{n} GP_k = \frac{1}{2}\left(-(n+3)\right)P_{n+3} + (n+4)P_{n+2} + (n+3)P_{n+1} + (n+4)P_n + 1 + 2i)$.
b) $\sum_{k=0}^{n} GP_{2k} = \frac{1}{2}\left(-(n+2)\right)P_{2n+2} + (n+3)P_{n+2} + (n+3)P_{2n+1} + (n+3)P_{2n} + (n+2)P_{2n-1} + 1 + 2i)$.
c)$\sum_{k=0}^{n} GP_{2k+1} = \frac{1}{2}\left(-(n+1)\right)P_{2n+2} + (n+3)P_{2n+1} + (n+2)P_{2n} + (n+2)P_{2n-1} + 1 + i)$.
As a specialized case of the Theorem 6.1, we give following corollary.

Corollary 6.3. *For all integers $n \geq 0$, we have sum formulas below:*

a)$\sum_{k=0}^{n} GC_k = \frac{1}{2}\left(-(n+3)\right)C_{n+3} + (n+4)C_{n+2} + (n+3)C_{n+1} + (n+4)C_n - 6 - 8i)$.
b)$\sum_{k=0}^{n} GC_{2k} = \frac{1}{2}\left(-(n+2)\right)C_{2n+2} + (n+3)C_{n+2} + (n+3) + C_{2n+1} + (n+3)C_{2n} + (n+2) + C_{2n-1} - 2 - 8i)$.
c)$\sum_{k=0}^{n} GC_{2k+1} = \frac{1}{2}\left(-(n+1)\right)C_{2n+2} + (n+3)C_{2n+1} + (n+2)C_{2n} + (n+2)C_{2n-1} - 6 - 2i)$.

Next, we present the ordinary generating functions of some special cases of Gaussian generalized Pierre numbers.

Theorem 6.4. *The ordinary generating functions of the sequences W_{2n}, W_{2n+1} are shown as follows:*

a)$\sum_{n=0}^{\infty} GW_{2n}x^n = \dfrac{2x^2W_3 + (x^3 - 4x^2 + x)W_2 - 2x^3W_1 + (x^2 - 4x + 1)W_0}{x^4 + 2x^2 - 4x + 1}$.
b)$\sum_{n=0}^{\infty} GW_{2n+1}x^n = \dfrac{(x^3 + x)W_3 - 2x^3W_2 + (x^2 - 4x + 1)W_1 - 2x^2W_0}{x^4 + 2x^2 - 4x + 1}$.

From the last Theorem, we get the following Corollary which gives sum formula of Gaussian Pierre numbers (Take $W_n = GP_n$ with $GP_0 = 0, GP_1 = 1, GP_2 = 2 + i, GP_3 = 4 + 2i$)

Corollary 6.5. *For $n \geq 0$ Gaussian Pierre numbers get the following properties.*

a)$\sum_{n=0}^{\infty} GP_{2n}x^n = \dfrac{ix^3 + (2+i)x}{x^4 + 2x^2 - 4x + 1}$.
b)$\sum_{n=0}^{\infty} GP_{2n+1}x^n = \dfrac{x^2 + 2ix + 1}{x^4 + 2x^2 - 4x + 1}$.

7. Matrix Formulation of $\mathbf{GW}_n$

We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detA = 1$. Notice that

$$A^n = \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [22].

Then we obtain the following lemma.

Lemma 7.1. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} GW_{n+3} \\ GW_{n+2} \\ GW_{n+1} \\ GW_{n} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix}. \tag{7.1}$$

Proof. The identitiy (7.1) can be proved by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{0} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix}$$

which is true. We assume that the identity (7.1) holds for $n = k$. So the following identitiy is true

$$\begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_{k} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix}..$$

For $n = k + 1$, we obtain

$$\begin{aligned} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix} &= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_{k} \end{pmatrix} \\ &= \begin{pmatrix} GW_{k+4} \\ GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by mathematical induction on n, the proof completed. □

We define

$$N_{Gw} = \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix}, \tag{7.2}$$

$$E_{Gw} = \begin{pmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1-} & GW_{n-2} & GW_{n-3} \end{pmatrix}. \tag{7.3}$$

Now, we get the following theorem with N_{Gw} and E_{Gw}

THEOREM 7.2. *Using N_{Gw} and E_{Gw} , we get*

$$A^n N_{Gw} = E_{Gw}.$$

Proof. Notice that using Corollary 3.6,

$$\begin{aligned} A^n N_{Gw} &= \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned}
a_{11} &= P_{n+1}GW_3 - P_{n-2}GW_2 - P_{n-1}GW_1 - P_nGW_0 = GW_{n+3} \\
a_{12} &= P_{n+1}GW_2 - P_{n-2}GW_1 - P_{n-1}GW_0 - P_nGW_{-1} = GW_{n+2} \\
a_{13} &= P_{n+1}GW_1 - P_{n-2}GW_0 - P_{n-1}GW_{-1} - P_nGW_{-2} = GW_{n+1} \\
a_{14} &= P_{n+1}GW_0 - P_{n-2}GW_{-1} - P_{n-1}GW_{-2} - P_nGW_{-3} = GW_n \\
a_{21} &= P_nGW_3 - P_{n-3}GW_2 - P_{n-2}GW_1 - P_{n-1}GW_0 = GW_{n+2} \\
a_{22} &= P_nGW_2 - P_{n-3}GW_1 - P_{n-2}GW_0 - P_{n-1}GW_{-1} = GW_{n+1} \\
a_{23} &= P_nGW_1 - P_{n-3}GW_0 - P_{n-2}GW_{-1} - P_{n-1}GW_{-2} = GW_n \\
a_{24} &= P_nGW_0 - P_{n-3}GW_{-1} - P_{n-2}GW_{-2} - P_{n-1}GW_{-3} = GW_{n-1} \\
a_{31} &= P_{n-1}GW_3 - P_{n-4}GW_2 - P_{n-3}GW_1 - P_{n-2}GW_0 = GW_{n+1} \\
a_{32} &= P_{n-1}GW_2 - P_{n-4}GW_1 - P_{n-3}GW_0 - P_{n-2}GW_{-1} = GW_n \\
a_{33} &= P_{n-1}GW_1 - P_{n-4}GW_0 - P_{n-3}GW_{-1} - P_{n-2}GW_{-2} = GW_{n-1} \\
a_{34} &= P_{n-1}GW_0 - P_{n-4}GW_{-1} - P_{n-3}GW_{-2} - P_{n-2}GW_{-3} = GW_{n-2} \\
a_{41} &= P_{n-2}GW_3 - P_{n-5}GW_2 - P_{n-4}GW_1 - P_{n-3}GW_0 = GW_n \\
a_{42} &= P_{n-2}GW_2 - P_{n-5}GW_1 - P_{n-4}GW_0 - P_{n-3}GW_{-1} = GW_{n-1} \\
a_{43} &= P_{n-2}GW_1 - P_{n-5}GW_0 - P_{n-4}GW_{-1} - P_{n-3}GW_{-2} = GW_{n-2} \\
a_{44} &= P_{n-2}GW_0 - P_{n-5}GW_{-1} - P_{n-4}GW_{-2} - P_{n-3}GW_{-3} = GW_{n-3}
\end{aligned}$$

Using the theorem 4.6 the proof is done. □

By taking $GW_n = GP_n$ with GP_0, GP_1, GP_2, GP_3 in (7.2) and (7.3)

$GW_n = GC_n$ with GC_0, GC_1, GC_2, GC_3 in (7.2) and (7.3)

respectively, we obtain:

$$N_{GP} = \begin{pmatrix} 4+2i & 2+i & 1 & 0 \\ 2+i & 1 & 0 & 0 \\ 1 & 0 & 0 & -i \\ 0 & 0 & -i & -1 \end{pmatrix}, \ E_{GP} = \begin{pmatrix} GP_{n+3} & GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+2} & GP_{n+1} & GP_n & GP_{n-1} \\ GP_{n+1} & GP_n & GP_{n-1} & GP_{n-2} \\ GP_n & GP_{n-1} & GP_{n-2} & GP_{n-3} \end{pmatrix},$$

$$N_{GC} = \begin{pmatrix} 8+4i & 4+2i & 2+4i & 4 \\ 4+2i & 2+4i & 4 & 0 \\ 2+4i & 4 & 0 & 6i \\ 4 & 0 & 6i & 6-4i \end{pmatrix}, \ E_{GC} = \begin{pmatrix} GC_{n+3} & GC_{n+2} & GC_{n+1} & GC_n \\ GC_{n+2} & GC_{n+1} & GC_n & GC_{n-1} \\ GC_{n+1} & GC_n & GC_{n-1} & GC_{n-2} \\ GC_n & GC_{n-1} & GC_{n-2} & GC_{n-3} \end{pmatrix}.$$

From Theorem 7.2, we can write the following corollary.

Corollary 7.3. *The following identities are hold:*

- **a):** $A^n N_{GP} = E_{GP}$.
- **b):** $A^n N_{GC} = E_{GC}$.

8. Conclusions

Recurrence relations have been extensively studied in the literature and play a crucial role across diverse fields, including physics, engineering, architecture, nature, and art. Among these, second-order recurrence relations involving integer sequences—such as the Fibonacci, Lucas, Pell, and Jacobsthal sequences—are particularly well known. The Fibonacci sequence, perhaps the most famous example, was introduced by Leonardo of Pisa in his 1202 treatise Liber Abaci, in connection with a rabbit population growth problem. Both the Fibonacci and Lucas sequences are rich sources of elegant and intriguing mathematical identities. For a comprehensive account of the applications of these second-order recurrence relations in science and nature, see [9,10,11].

In this study, we introduce the Gaussian generalized Pierre numbers as a novel class of fourth-order recurrence sequences, along with two of their distinguished special cases. Building upon their foundational recurrence relation, we derive explicit Binet-type formulas and construct corresponding generating functions that encode their structural behavior. Furthermore, we develop Simson-type identities, closed-form summation formulas. We also examine their recurrence dynamics in depth and present matrix representations.

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