**Gamma-Exponential Distribution (GED): Theory and its Properties**

**Abstract**

Research on probability distribution functions and their characteristics is necessary, even though these functions are essential for modeling random processes. Studies have shown, however, that in some circumstances, it is found that specific combinations of two or more random variables with known probability distributions follow the distributions of some real-world data that the currently used standard distributions are unable to adequately represent.
This study concentrated on developing the two-parameter Gamma-Exponential Distribution (GED), which was achieved by applying the Transformed-Transformer method, in order to produce a new distribution that combines the gamma and exponential distributions. Expressions for its survival and hazard functions, as well as the probability density function, cumulative distribution function, rth non-central moments, first four moments, variances, skewness and kurtosis and maximum likelihood estimator were among its attributes.
***Keywords- Probability distribution, Gamma-x family, Gamma-Exponential distribution, Maximum likelihood estimator, Survival function.***

1. **INTRODUCTION**

Advances in developing new distributions to imitate natural processes have led to the development and exploration of numerous new distributions. These include the earlier works of [1], [2], and [6]. Furthermore, some studies have shown that integrated random variable distributions are more versatile, perform better, and have a greater range of applications [7], [22], and [23]. In addition to introducing a new family of continuous distributions called the beta transmuted-H family, which extends the transmuted family [11], [9] proposed a new family of univariate distributions derived from the Weibull random variable, which they called the new Weibull-X family of distributions. Additionally, in [12], the Kumaraswamy Marshal-Olkin generalized family of distributions was presented, which is a new family of continuous distributions. In [13], a new family of distributions known as the exponentiated T-X distribution was created, and some of its characteristics and special situations were examined. [14] extended the fundamental Weibull, Gamma, Gumbel, and Inverse Gaussian distributions among a number of well-known distributions to produce a new family of generalized distributions (referred to as "Kw" distributions). On the basis of the daily maximum readings, Rodrigues and Achcar created a Markov chain model for ozone air pollution. They went on to say that the behavior of ozone showed no signs of a time-homogeneous feature. Rather, we want to use the T-X technique to combine two distinct continuous probability distributions to create a new one known as the Gamma-Exponential distribution. This will allow us to generalize a more flexible and controllable probability distribution.

1. **METHODS**

The pdf and cdf of the Gamma-Exponential Distribution is derived in this section as a class of Gamma-X family of generalized distributions.

**Theorem 1:** Let the pdf of a gamma distribution beand X Exponential distributed random variable with pdf and cdf .

Then, the pdf of the Gamma-Exponential distribution is given by:

;  (1)

Where is the gamma function.

**Proof:**

The pdf of the Gamma-X family of distribution is given by:

 (2)

Let X follow the Exponential distribution with pdf and cdf ,

such that . Then, the pdf in (2) becomes:

 =  (3)

= 

=  (4)

=  (5)

 =  (6)

Any random variable X that has the probability density function given in (6) is said to have the Gamma-Exponential distribution with shape parameter α and β and scale parameter  and written as *X* ~ GED(*α, β,*).

**Theorem 2**

Required to prove that  

 (7)

Let 

so that

Then (7) reduces to:

  (8)

 (9)

 (10)

**Theorem 3**

The cumulative distribution function (cdf) of the Gamma-Exponential distribution is given by:

F(x) = , x, > 0 (11)

Where is the lower incomplete gamma function and

is the complete gamma function.

**Proof:**

CDF = F(x) =  (12)

F(x) =  (13)

F(x) =  (14)

Let u = 

F(x) =  (15)

F(x) =  (16)

Hence, CDF = F(x) =  (17)

**Statistical Properties of the Gamma-Exponential Distribution (GED)**

This section presents the foundational statistical properties of the Gamma-Exponential Distribution (GED), focusing specifically on the first four moments, variance, coefficient of variation, moment-generating function, characteristic function, skewness, and kurtosis.

**A. Moment**

**Theorem 4:**

If X is a random variable distributed as a GED (), then the non-central moment is given by;

 (18)

**Proof:**

  =  (19)

 =  (20)

 =  (21)

Let u =  x =  dx = 

 =  (22)

 =  (23)

 =  (24)

 =  (25)

From (25), the first (mean), second, third and the fourth moments shall be obtained respectively as follows;

Substituting r = 1, then the **Mean** is obtained as;

 (26)

 (27)

=  (28)

Substituting r =2, then the **2nd moment** is obtained as;

 (29)

 (30)

 =  (31)

Substituting r = 3, then the **3rd moment** is obtained as;

 (32)

 (33)

=  (34)

Substituting r = 4, then the **4th moment** is obtained as;

 (35)

 (36)

 =  (37)

 The variance is given as;

Var (x) =  (38)

Var (x) =  (39)

Var (x) =  (40)

 (41)

The standard deviation of *X* is given by:

 (42)

=  (43)

**B. Coefficient of variation (C.V)**

This is a standardized measure of dispersion of a probability distribution and is given as;

C.V =  (44)

C.V =  (45)

C.V =  (46)

 (47)

**C. Moment Generating Function (M.G.F)**

**Theorem 5:** If X is a continuous random variable distributed as a GED (*x*; , then the moment generating function is given as; M*x*(t) =  (48)

**Proof:**

M*x*(t) = E(etx) =  (49)

M*x*(t) = dx (50)

M*x*(t) =  (51)

M*x*(t) =  (52)

Let u = 

 dx = 

M*x*(t) =  (53)

M*x*(t) =  (54)

M*x*(t) =  (55)

M*x*(t) =  (56)

M*x*(t) =  (57)

**D. Characteristic function (C.F)**

**Theorem 6:** If X is a random variable distributed as a GED (), then the characteristics function is defined as;

 (58)

**Proof:**

 (59)

 (60)

 (61)

 (62)

Let u =  x =  dx = 

 (63)

 (64)

 (65)

 (66)

 (67)

**E. Reliability Function**

The reliability function, often referred to as the survival function, calculates the probability that a patient, device, or other subjects of interest will continue to function or survive beyond a specified time. It is defined as follows:

S(x) = 1 – F(*x*) (68)

where is the cumulative distribution function of *x* then,

S(x) =  (69)

**F. Hazard function**

The hazard function represents the instantaneous risk of the event of interest occurring within a very short time interval. It is defined as follows:

h(x) =  (70)

Where and are pdf and survival function of GED then,

h(x) =  (71)

**G. Cumulative hazard function**

The cumulative hazard function is the integral of the hazard function. It can be interpreted as the probability of failure at time x given survival until time x, and it can be defined as;

 (72)

 (73)

 (74)

Let 

then so that the equation (3.80) is reduced to:

 (75)

Recall that Recall that is an incomplete gamma function.

**H.** **Reverse Hazard Function**

The reversed hazard rate, denoted by r(*x*), is defined as the ratio of a random variable's probability density function to its distribution function. This concept is particularly valuable for analyzing censored data and finds applications in fields such as forensic science, and is expressed as follows:

 (77)

Then, ,  (78)

**I. Skewness and Kurtosis**

Skewness quantifies the extent to which a distribution deviates from symmetry, whereas kurtosis measures the degree of peakedness or describes the overall shape of the probability distribution.

The skewness and kurtosis of a distribution are respectively given as:

Skewness = , (79)

Kurtosis =  (80)

Where,  and  (81)

For the Gamma-Exponential distribution,

 (82)



 (83)





 (84)

Then, Skewness =

 (85)

And the Kurtosis =

 (86)

**J. Method of Maximum Likelihood**

This section focuses on determining the Maximum Likelihood Estimates (MLE) for the parameters of the Gamma-Exponential Distribution (GED). The concept behind MLE is to identify the parameter values that maximize the likelihood of observing the sample data (Mood *et al*., 1974). This process starts with formulating the likelihood function based on the sample data. The likelihood represents the probability of obtaining the specific data set under a chosen probability model, which includes the unknown parameters. The parameter values that maximize this likelihood are referred to as the maximum likelihood estimates (Elgarhy, 2017).

**The Maximum Likelihood Estimator for Gamma-Exponential Distribution (GED)**

**Theorem 7:** Let X1, X2, …, Xn be a random sample of size n from Gamma-Exponential distribution (GED) with pdf;

, x>0,  (87)

then, the likelihood function of GED is given by;

L(x; ,  (88)

By taking the natural logarithm of (3.94), the log-likelihood function is obtained as;

 (89)

Therefore, the MLE which maximizes the above (89) must satisfy the following standard equations;

 = 0 (90)

 = 0 (91)

 (92)

(90), (91) and (92) will be solved simultaneously to obtain the estimates of. This cannot be done analytically, therefore, a numerical technique will be adopted.

**III. Conclusion**

The creation of a new distribution using the gamma and exponential distributions was the main goal of this research. Using the Transformed-Transformer approach, the resulting two-parameter Gamma-Exponential Distribution (GED) was generated. The resulting distribution's properties were determined, including its probability density function, cumulative distribution function, variances, rth non-central moments, skewness, kurtosis, and maximum likelihood estimator. Additionally, expressions for its hazard and survival functions were provided.

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**IV. References**

[1] S. Nadarajah (2005). *Exponentiated pareto distributions.* Statistics Journal, 39: 255-260Alzaatreh A. Lee C. and

 Famoye F. (2013). A new method for generating families of continuous distributions. International Journal of

 Statistics. 71(1): 63-79.

[2] A.A. Adeleke, E.E Akarawak and R.O. Okafor (2015). On the distribution of the ratio of independent Gamma

 and Rayleigh random variables. Journal of Scientific Research and Developments. 15(1): 54-63.

[3] G.M. Cordeiro, M. Alizadeh and ADC Nascimento (2016). The exponentiated Gompertz generated family of

 distributions: Properties and

 applications. Chill J Stat; 7: 29-50.

[4] A.A. Adeleke and R.O. Okafor (2017). The Gamma-Rayleigh Distribution and applications to survival data.

 Nigerian Journal of Basic and Applied Science. 25(2): 130-142.

[5] B.O. Oluyede, Pu S., B. Makubate (2018). The Gamma-Weibull-G family of distributions with applications.

 Austrian J Stat, 47: 45-76

[6] O.D. Ogunwale, T.M. Ayeni and O.A. Adewusi (2019). Exponential-Gamma Distribution and its

 Application. International Journal of Research and Innovation in Applied Science, IV(XII): 116-120.

[7] T.M Ayeni, O.D Ogunwale and O.A. Adewusi (2020). Application of Exponential-Gamma Distribution in

 Modeling Queuing Data. International Journal of Trend in Scientific Research and Development.

 4(2): 839-842.

[8] Abdul-Monem and I.B. and Seham, M. (2015). Transmuted Gompertz distribution. Comput.

 Appl. Math. J. 1, 88-96

[9] Afify, A., Yousof, H. and Nadarajah, S. (2017). The beta transmuted-H family for lifetime data. Stat. its

 Interface 10, 505-520.

[10] Ahmed, Z. (2018). The Zubair-G family of distributions: properties and applications. Ann. Data Sci., 1-14.

[11] Ahmad Z., Elgarhy M. and Hamedani G.G. (2018). A new weibull-x family of distributions: properties,

 characterizations and applications. J. Stat. Distrib. Appl. 5, 5.

[12] Alizadeh M, Tahir M.H, Cordeiro G.M, Mansoor M., Zubair M. and Hamedani G.G. (2015). The

 Kumaraswamy Marshal-Olkin family of distributions. J. Egyptian Math. Soc. **23**, 546- 557.

[13] Alzaghal A., Famoye F. and Lee C. (2013Exponentiated T-X family of distributions with some applicatons.

 International Journal of Statistics and Probability **2**, 31.

[14] Cordeiro G.M. and De Castro M. (2011). A new family of generalized distributions. J. Stat.

 Comput. Simul. **81**, 883-898

[15] Eghwerido J.T., Nzei L.C and Agu F.I. (2020). Alpha power Gompertz distribution: properties and

 applications. Sankhya A.

[16] El-GoharyA., Alshamrani A. and Al-Otaibi A.N. (2013). The generalized Gompertz distribution. Appl. Math.

 Model. **37**, 13-24.

[17] Khan M.S., King R. and Hudson I.L. (2016a) Transmuted Gompertz distribution; application and estimation

 Pakistan J. Stat. **32**, 161-182.

[18] Khan M.S., King R. and Hudson I.L. (2016). Transmuted Generalized Gompertz distribution with application.

 J. Stat. **16**, 65-80.

 [19] Gupta R.D. and Kundu D. (2001). Generalized Exponential Distribution: Different Method of Estimations.

 Journal of Statistical Computation and Simulation, 69, 315-337.

[20] Rodrigues E & Achcar J. (2013). Application Discrete-time Markov Chains and Poisson Processes to Air

 Pollution Modeling and Studies.10.1007/978-1-461