INVESTIGATION OF

ONE-POINT COMPACTIFICATION IN SEMI-NORMAL SPACES

Abstract

Let *X* be a non-empty set and (*X, τ* ) be a semi-normal space. On this paper, we investigated the relationship be- tween one-point compactiﬁcation and semi-normal spaces. In addition, we in particular proved that if (*X, τ* ) is a semi- normal space, then its one-point compactiﬁcation, *X ∗* is also semi-normal. We also extended our work on establish- ing that, if (*X, τ* ) is a semi-normal space, then its one-point

compactiﬁcation, *X ∗* of *X* is compact if and only if (*X, τ* )

is also Hausdorﬀ.

Key Words and Phrases:Topological space, semi-open set, open set and semi-normal space

1 Introduction

A lot of research on the properties of topological spaces has been done and many results established. The properties (normality[1], regularity [1], semi regularity [7], connectedness[9] , compactiﬁca- tion [5], etc.) have been studied and internal characterization of some spaces like Tychonoﬀ spaces, established. Frink [2] described compactiﬁcation with regard to Wallman base in a Wallman space *w*(*Z* ). Later, Piekorsz [6] characterized One-point compactiﬁcation with regard to Wallman base *C* under the framework of generalized topological spaces. In this note, we have characterized one-point compactication with respect to semi-normal spaces.

2 Preliminary Notes

Deﬁnition 1. Consider a space (*X, τ* ), where *X* is non-empty set and *τ* is a topology on *X* if it satisﬁes the following properties:

(i). *∅, X ∈ τ* .

(ii). The abitrary union of sets in *τ* belong to *τ* . (iii). Any ﬁnite intersection of sets of *τ* belong to *τ* .

We made some deﬁnitions which are instrumental to this present paper.

Deﬁnition 2. ([4], Deﬁnition 3.1)

Let (*X, τ* ) be a topological space. If every open covering in *X* has a countable sub-covering, then the space is called Lindelof space.

Deﬁnition 3. ([8], Deﬁnition 7.1.7)

A subset *A* of a topological space (*X, τ* ) is said to be compact if, every open covering of *A* has a ﬁnite sub-covering. If the compact subset *A* equals to *X* then, (*X, τ* ) is said to be compact space.

Deﬁnition 4. , ([3], Deﬁnition1)

A topological space (*X, τ* ) is said to be semi-normal if for each pair of disjoint semi-closed sets *A, B ⊆ X* , there exist disjoint semi-open sets *U, V ⊆ X* such that *A ⊆ U* and *B ⊆ V* .

Deﬁnition 5. ([10], Deﬁnition 1.3)

A topological space (*X, τ* ) is said to be locally compact at a point

*x ∈ X* if, *x* lies in the interior of some compact subset of *X* .

3 One-Point compactiﬁcation

We begin our results with proposition 5 which is a basis for the main results.

Proposition 6. One-Point Compactiﬁcation *X ∗* of locally compact Semi-Normal Space is Compact Normal Space.

*Proof.* Let *X* be a locally compact semi normal space and *A* and *B* be disjoint closed subsets of *X* , such that there exist disjoint open sets *U* and *V* in *X* such that *A*¯*x ⊆ U* and *B*¯ *x ⊆ V* . Let *X ∗* be one-point compactiﬁcation of *X* with *∞* representing the point at *∞*. Considering two scenarios of *U* and *V* with respect to *∞*: (i). One or both *U* and *V* contain *∞*. In this case, extending them with *{∞}* does not aﬀect their disjointness. ie. If *U ∩ {∞}* = *∅* or *V ∩ {∞}* = *∅* i.e. *∞* belong to either *U* or *V* , then extending *U* and *V* with *{∞}* does not change their relative positions. *A ∪ {∞} ⊆ U ∪ {∞}* and *B ∪ {∞} ⊆ V ∪ {∞}*. Since *U* and *V* were disjoint *U ∪ {∞}* and *V ∪ {∞}* remain disjoint open sets in *X ∗* .

(ii). Neither *U* nor *V* contain *∞*: Utilize the local compactness of *X* . Since *X* is locally compact *∀ x ∈ X* has compact neighborhood. i.e *x ∈ X \* (*U ∪ V* ), *∃* a compact neighborhood *Kx* such that *Kx ⊆ X \* (*U ∪ V* ) and does not contain *∞*. Let *Ku* = *U ∩* (*∪{Kx* : *x ∈ X \ U }*) and *Kv* = *V ∩* (*∪{Kx* : *x ∈ X \ V }*). These sets are compact because they are intersections of compact sets and moreover, *Ku ⊆ U \ {∞}* and *Kv ⊆ V \ {∞}*. These guarantee the existence of disjoint open sets in *X ∗* i.e (*U ∪ {∞}*) *∪ Ku* and (*V ∪ {∞}*) *∪ Kv* .

*⇒ X ∗* is compact by deﬁnition of one point compactiﬁcation and the existence of disjoint open sets (*U ∪{∞}*)*∪Ku* and (*V ∪{∞}*)*∪Kv* in *X ∗* , shows that *X ∗* is normal which completes the proof.

Theorem 7. *If* (*X, τ* ) *is a semi-normal space, then its one- point compactiﬁcation, X ∗ is also semi-normal. .*

*Proof.* Let *X* be a semi normal space with topology *τ* and *A, B ⊆ X* be disjoint closed sets. By proposition [5] we consider two scenarios: (i).*A ∩ {∞}* = *∅* and *B ∩ {∞}* = *∅*. ( i.e. *A* contains

*∞* and *B* is entirely in *X* ), then the semi-normality of *X* guarantees the existence of disjoint sets *U* and *V* in *X* such that *A*¯*x ⊆ U* and *B*¯*x ⊆ V* . Extending *U* and *V* with *{∞}* in *X ∗* creates disjoint sets used in *U ∪ {∞}* and *V* in *X ∗* that separates *A* and *B*.

*⇒ A ⊆ A ∪ {∞}* and *B ⊆ V* .

(ii). Both sets contain *∞*: Analyzing the open sets used in *X* , con- sider the case when the original open sets in are used to separate *A* and *B*, already contain *∞*, extending them with *{∞} ∈ X ∗* would maintain their disjoint property i.e, Let *U ′* and *V ′* be two original open sets in *X* . Then *U ′ ∪ {∞}* and *V ′ ∪ {∞}* are disjoint open sets separating *A* and *B*.

(iii). Both sets *A* and *B* are entirely contained in *X* i.e Neither *A* nor *B* contain *∞*: Since *X* is semi-normal, *∃* disjoint open sets *U* and *V* in *X* such that *A ⊆ U* , *B ⊆ V* , then *U* and *V* are also disjoint open in *X ∗* . Therefore, for a semi-normal space *X* , its One-point compactiﬁcation *X ∗* is semi-normal.

Theorem 8. *If* (*X, τ* ) *is a semi-normal space, then its one- point compactiﬁcation, X ∗ of X is compact if and only if* (*X, τ* ) *is also Hausdorﬀ.*

*Proof.* Let *X* be a Hausdorﬀ and semi-normal space. We want to show that its one point compactiﬁcation *X ∗* is compact. From deﬁnition [1.19], to show that *X ∗* is compact, we consider any open cover *{Ui }i∈I* of *X ∗* . We split this in to 2 cases:

Case 1: Some open set *Uj* in the cover contains *∞*. If one of the open set contain *∞*, the remaining set covers *X* . Moreover, since *X* is locally compact and Hausdorﬀ, and *X* is covered by open sets, *∃*a ﬁnite subcover that covers all points in *X* . Hence, combining the subcover with the set *Uj* that contain *∞*, we have a ﬁnite subcover for *X ∗* , proving that *X ∗* is compact in this case.

case 2: No open set *Ui* contains *∞*.

Here the open cover *{Ui }* only covers points *X ∗ \ {∞}*, which is

homeomorphic to *X* . Since *X* is locally compact and Haudorﬀ, ev-

ery point in *X* has a compact neighborhood. Because *X ∗ \ {∞}* is homeomorphic to *X* , *∃* a ﬁnite subcover that covers all of *X* . Since

*∞* is not covered by any *Ui* , this leads to a contradiction (as all open cover include *∞*).

*⇒* We can always ﬁnd a ﬁnite subcover for any open cover of *X ∗* . This shows that *X ∗* is compact.

Conversely:

Assume that the one-point compactiﬁcation *X ∗* of *X* is compact. We need to show that *X* is Hausdorﬀ. Suppose *X* is not Hausdorﬀ. Then, *∃* distinct points *x, y ∈ X* that cannot be separated by dis- joint open sets.

*⇒* no open neighborhoods of *x* and *y* are disjoint.

*⇒* the space *X* does not satisfy the separation property [3.2.1]. However, *X ∗* is compact and in a compact space, distinct points can always be separated by disjoint open sets because compact spaces are normal (by Tychonoﬀ theorem).

Since *X ∗* is compact and normal, the points *x* and *y ∈ X* must be

separable by disjoint open sets, contradicting our assumption that

*X* is not Hausdorﬀ.

Thus, if *X ∗* is compact, *X* must be Hausdorﬀ, completing the proof.

4 Conclusion

Finally, the results we discussed, outline clearly the relationship between semi normal space and one-point compactiﬁcation.

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