Dual Hyperbolic Generalized Adrien Numbers

Abstract. In this paper, we introduce the generalized dual hyperbolic Adrien numbers. As special cases, we consider the dual hyperbolic Adrien and dual hyperbolic Adrien-Lucas numbers. We present their corresponding Binet formulas, generating functions, exponential function and summation formulas. Moreover, we provide associated matrix representations for these sequences.

2010 Mathematics Subject Classification. 11B39, 11B83.

Keywords. Adrien numbers, Adrien-Lucas numbers, dual hyperbolic Adrien numbers, dual hyperbolic Adrien-Lucas numbers.

1. Introduction

Hypercomplex number systems, as introduced by Kantor, [10], constitute algebraic extensions of the real number system.Among the commutative instances of such systems are the complex numbers, defined as:

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

as well as the hyperbolic numbers (also referred to as double or split-complex numbers), [8],

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

and dual numbers, [15],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Quaternions, introduced by Hamilton, extend complex numbers into a four-dimensional non-commutative algebra over the real numbers, [22],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

The octonions [11] and sedenions [24] exemplify higher-dimensional hypercomplex number systems.. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are structured as real algebras derived from the real number system $\mathbb{R}$ via a recursive doubling procedure known as the Cayley–Dickson Process.This iterative method allows for the construction of successive 2^n-dimensional algebras, extending beyond sedenions to form generalized entities referred to as 2^n-ions [5], [13], [7]).

Quaternions were first formulated by the Irish mathematician W. R. Hamilton (1805–1865) [22] as a non-commutative generalization of the complex numbers. In 1848, J. Cockle introduced the notion of hyperbolic numbers with complex coefficients [12].Later, H. H. Cheng and S. Thompson [9] extended this framework by defining dual numbers with complex coefficients, which they termed complex dual numbers. More recently, Akar, Yüce, and Şahin [14] proposed the algebraic system of dual hyperbolic numbers, further enriching the landscape of generalized number systems.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and addition of dual hyperbolic numbers is defined as componentwise.

The algebra of dual hyperbolic numbers forms a commutative ring, a real vector space, and a real algebra. However, the structure denoted by $\mathbb{H}_{\mathbb{D}}$does not constitute a field, as not every dual hyperbolic number possesses a multiplicative inverse. For further details on the algebraic properties and construction of dual hyperbolic numbers, the reader is referred to[14].

To lay the groundwork for subsequent analysis, we first recount the established definition of generalized Adrien numbers, as presented in the existing literature.

A generalized Adrien sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4},\ n \geq 4, \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n. Soykan has conducted a study on this particular sequence, for more details, see [16].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 3z^3 + z^2 + 1 = (z^3 - 2z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\delta &= 1.
\end{aligned}$$

Where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 3, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Note also that

$$\begin{aligned}
\alpha + \beta + \gamma &= 2, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

Using the roots and the recurrence relation of $\{W_n\}$ the Binet's formula for the generalized Adrien numbers can be expressed for all integers n as follows

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{p_2\beta^n}{4\beta^2 + 3\beta - 1} + \frac{p_3\gamma^n}{4\gamma^2 + 3\gamma - 1} + \frac{p_4\delta^n}{3} \\
&= S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4\delta^n.
\end{aligned} \tag{1.2}$$

Where p_1, p_2, p_3 and p_4 are given below

$$\begin{aligned}
p_1 &= (\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - 1W_0),\\
p_2 &= (\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - 1W_0),\\
p_3 &= (\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - 1W_0),\\
p_4 &= -(W_3 - 2W_2 - W_1 - W_0).
\end{aligned}$$

And

$$\begin{aligned}
S_1 &= \frac{p_1}{4\alpha^2 + 3\alpha - 1},\\
S_2 &= \frac{p_2}{4\beta^2 + 3\beta - 1},\\
S_3 &= \frac{p_3}{4\gamma^2 + 3\gamma - 1},\\
S_4 &= -\frac{(W_3 - 2W_2 - W_1 - W_0)}{3}.
\end{aligned} \tag{1.3}$$

Binet's formula of Adrien and Adrien-Lucas sequences are

$$A_n = \frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3},$$

and

$$B_n = \alpha^n + \beta^n + \gamma^n + 1.$$

respectively.

If we set $W_0 = 0, W_1 = 1, W_2 = 3, W_3 = 8$ then $\{W_n\}$ is the well-known Adrien sequence and if we set $W_0 = 4, W_1 = 3, W_2 = 7, W_3 = 18$ then $\{W_n\}$ is the well-known Lucas sequence. In other words, Adrien sequence $\{A_n\}_{n\geq 0}$ and Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations as;

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,\ A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8,\ n \geq 4, \tag{1.4}$$

$$B_n = 3B_{n-1} - B_{n-2} - B_{n-4}\ ,\ B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18,\ n \geq 4. \tag{1.5}$$

The sequences $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, can be extended to negative subscripts by defining,

$$\begin{aligned}
A_{-n} &= -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)},\\
B_{-n} &= -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)}.
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.4),(1.5) hold for all integer n. Binet's formulas as follows.

The first few generalized Adrien numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Adrien numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_2 - W_1 - W_3$
2	W_2	$3W_1 - W_0 - W_2$
3	W_3	$3W_0 - 3W_2 + W_3$
4	$3W_3 - W_2 - W_0$	$10W_2 - 6W_1 - 3W_3$
5	$8W_3 - W_1 - 3W_2 - 3W_0$	$10W_1 - 6W_0 - 3W_2$
6	$21W_3 - 3W_1 - 9W_2 - 8W_0$	$10W_0 + 3W_1 - 18W_2 + 6W_3$
7	$54W_3 - 8W_1 - 24W_2 - 21W_0$	$3W_0 - 28W_1 + 36W_2 - 10W_3$
8	$138W_3 - 21W_1 - 62W_2 - 54W_0$	$33W_1 - 28W_0 - W_2 - 3W_3$

After then we can write the generating function of generalized Adrien numbers,

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + W_0)x^2 + (W_3 - 3W_2 + W_1)x^3}{1 - 3x + x^2 + x^4}.$$

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

For more details about generalized Adrien numbers, see [16].

LEMMA 1. *[4].Suppose that* $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Adrien sequence* $\{W_n\}$.

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by:

$$\begin{aligned}\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} e^{\gamma x} + (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x}.\end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

COROLLARY 2. *Exponential generating function of Adrien and Adrien-Lucas numbers are given by:*

a): $\sum\limits_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} ((\frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3})\frac{x^n}{n!}$

$= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1} e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1} e^{\beta x} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} e^{\gamma x} - \frac{1}{3} e^{x}).$

b): $\sum\limits_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1)\frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^{x}.$

Subsequently, we provide an overview of the existing literature concerning published studies on hyperbolic and dual hyperbolic numbers.

- Cockle [12] presented the hyperbolic numbers with complex coefficients.
- Akar at al [14] introduced the dual hyperbolic numbers.
- Cheng and Thompson[9] studied dual numbers with complex coefficients.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan at al [17] introduced dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2},\ (n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Cihan at al [2] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$

$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}.$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1, L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1$.

- Soykan at al [18] introduced dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}, J_0 = a, J_1 = b$.

- Bród at al [1] studied dual hyperbolic generalized Balancing given by

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}, B_0 = 0, B_1 = 1$.

- Yılmaz and Soykan [23] introduced dual hyperbolic generalized Guglielmo numbers given by

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$.

- Dikmen [3] introduced dual hyperbolic generalised Leonardo numbers given by

$$\widehat{l}_0 = l_0 + jl_1 + \varepsilon l_2 + j\varepsilon l_3$$

$l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3$.

- Eren and Soykan [6] introduced dual hyperbolic generalized Woodall numbers given by

$$\widehat{R}_0 = R_0 + jR_1 + \varepsilon R_2 + j\varepsilon R_3$$

where $R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}$, $R_0 = -1, R_1 = 1, R_2 = 7.$

In this paper, we define the dual hyperbolic generalized Adrien numbers in the next section and give some properties of them.

2. Dual Hyperbolic Generalized Adrien Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual hyperbolic generalized Adrien numbers and present generating functions and Binet formulas for them. We now define dual hyperbolic generalized Adrien numbers over $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Adrien number is

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}, \tag{2.1}$$

with the initial values $\widehat{W}_0, \widehat{W}_1, \widehat{W}_2, \widehat{W}_3$, (2.1) can be written to negative subscripts by defining,

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3}, \tag{2.2}$$

so identity (2.1) holds for all integers n.

As special cases, the nth dual hyperbolic Adrien numbers and the nth dual hyperbolic Adrien-Lucas numbers are given as.

$$\widehat{A}_n = A_n + jA_{n+1} + \varepsilon A_{n+2} + j\varepsilon A_{n+3}, \tag{2.3}$$

and

$$\widehat{B}_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}, \tag{2.4}$$

respectively. It can be easily shown that

$$\widehat{A} = 3\widehat{A}_{n-1} - \widehat{A}_{n-2} - \widehat{A}_{n-4} \ ,$$

and

$$\widehat{B} = 3\widehat{B}_{n-1} - \widehat{B}_{n-2} - \widehat{B}_{n-4}.$$

The sequence $\{\widehat{W}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{A}_{-n} = -\widehat{A}_{-(n-2)} + 3\widehat{A}_{-(n-3)} - \widehat{A}_{-(n-4)}.$$

$$\widehat{B}_{-n} = -\widehat{B}_{-(n-2)} + 3\widehat{B}_{-(n-3)} - \widehat{B}_{-(n-4)}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.1) holds for all integer n.

The first few dual hyperbolic generalized Adrien numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few dual hyperbolic generalized Adrien numbers

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$3\widehat{W}_2-\widehat{W}_1-\widehat{W}_3$
2	$\widehat{W}_2$	$3\widehat{W}_1-\widehat{W}_0-\widehat{W}_2$
3	$\widehat{W}_3$	$3\widehat{W}_0-3\widehat{W}_2+\widehat{W}_3$
4	$3\widehat{W}_3-\widehat{W}_2-\widehat{W}_0$	$10\widehat{W}_2-6\widehat{W}_1-3\widehat{W}_3$
5	$8\widehat{W}_3-\widehat{W}_1-3\widehat{W}_2-3\widehat{W}_0$	$10\widehat{W}_1-6\widehat{W}_0-3\widehat{W}_2$

Note that

$$\begin{aligned}
\widehat{W}_0 &= W_0+jW_1+\varepsilon W_2+j\varepsilon W_3 = W_0+jW_1+\varepsilon W_2+j\varepsilon W_3, \\
\widehat{W}_1 &= W_1+jW_2+\varepsilon W_3+j\varepsilon W_4 = W_1+jW_2+\varepsilon W_3+j\varepsilon(3W_3-W_2-W_0). \\
\widehat{W}_2 &= W_2+jW_3+\varepsilon(3W_3-W_2-W_0)+j\varepsilon(8W_3-W_1-3W_2-3W_0).
\end{aligned}$$

Now, we define some special cases of dual hyperbolic generalized Adrien numbers. The nth dual hyperbolic Adrien numbers, the nth dual hyperbolic Adrien-Lucas numbers, respectively, are given as the nth dual hyperbolic Adrien numbers is given $\widehat{A}_n = A_n + jA_{n+1} + \varepsilon A_{n+2} + j\varepsilon A_{n+3}$, with the initial values

$$\begin{aligned}
\widehat{A}_0 &= A_0+jA_1+\varepsilon A_2+j\varepsilon A_3, \\
\widehat{A}_1 &= A_1+jA_2+\varepsilon A_3+j\varepsilon A_4, \\
\widehat{A}_2 &= A_2+jA_3+\varepsilon A_4+j\varepsilon A_5,
\end{aligned}$$

the nth dual hyperbolic Adrien-Lucas numbers is given $\widehat{B}_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$ with the initial values

$$\begin{aligned}
\widehat{B}_0 &= B_0+jB_1+\varepsilon B_2+j\varepsilon B_3, \\
\widehat{B}_1 &= B_1+jB_2+\varepsilon B_3+j\varepsilon B_4, \\
\widehat{B}_2 &= B_2+jB_3+\varepsilon B_4+j\varepsilon B_5.
\end{aligned}$$

For dual hyperbolic Adrien numbers (taking $W_n = A_n,\ A_0 = 0, A_1 = 1,\ A_2 = 3, A_3 = 8,\ \ n \geq 4$) we get

$$\begin{aligned}
\widehat{A}_0 &= j+3\varepsilon+8\varepsilon j \\
\widehat{A}_1 &= 1+3j+8\varepsilon+21j\varepsilon \\
\widehat{A}_2 &= 3+8j+21\varepsilon+54j\varepsilon
\end{aligned}$$

and for dual hyperbolic Adrien-Lucas numbers (taking $W_n = B_n,\ B_0 = 4, B_1 = 3, B_0 = 7, B_0 = 18, \quad n \geq 4$) we get

$$\begin{aligned}\widehat{B}_0 &= 4 + 3j + 7\varepsilon + 18j\varepsilon \\ \widehat{B}_1 &= 3 + 7j + 18\varepsilon + 43j\varepsilon \\ \widehat{B}_2 &= 7 + 18j + 43\varepsilon + 108j\varepsilon\end{aligned}$$

A few dual hyperbolic Adrien numbers and dual hyperbolic Adrien-Lucas numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4.

Table 3. Dual hyperbolic Adrien numbers

n	$\widehat{A}_n$	$\widehat{A}_{-n}$
0	$j + 3\varepsilon + 8\varepsilon j$	$1 + 3\varepsilon + 8\varepsilon j$
1	$1 + 3j + 8\varepsilon + 21j\varepsilon$	$\varepsilon + 3\varepsilon j$
2	$3 + 8j + 21\varepsilon + 54j\varepsilon$	$j\varepsilon$
3	$8 + 21j + 54\varepsilon + 138j\varepsilon$	-1
4	$21 + 54j + 138\varepsilon + 352j\varepsilon$	$-j$
5	$54 + 138j + 352\varepsilon + 897j\varepsilon$	$1 - \varepsilon$

Table 4. Dual hyperbolic Adrien-Lucas numbers

n	$\widehat{B}_n$	$\widehat{B}_{-n}$
0	$4 + 3j + 7\varepsilon + 18j\varepsilon$	$4 + 3j + 7\varepsilon + 18j\varepsilon$
1	$3 + 7j + 18\varepsilon + 43j\varepsilon$	$4j + 3\varepsilon + 7j\varepsilon$
2	$7 + 18j + 43\varepsilon + 108j\varepsilon$	$-2 + 4\varepsilon + 3j\varepsilon$
3	$18 + 43j + 108\varepsilon + 274j\varepsilon$	$9 - 2j + 4j\varepsilon$
4	$43 + 108j + 274\varepsilon + 696j\varepsilon$	$-2 + 9j - 2\varepsilon$
5	$108 + 274j + 696\varepsilon + 1771j\varepsilon$	$-15 - 2j + 9\varepsilon - 2j\varepsilon$

Now, we will state Binet's formula for the dual hyperbolic generalized Adrien numbers and in the rest of the paper, we fix the following notations:

$$\widehat{\alpha} = 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \tag{2.5}$$

$$\widehat{\beta} = 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3. \tag{2.6}$$

$$\widehat{\gamma} = 1 + j\gamma + \varepsilon\gamma^2 + j\varepsilon\gamma^3 \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + j + \varepsilon + j\varepsilon, \tag{2.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha} &= 1+j\alpha+\varepsilon\alpha^2+j\varepsilon\alpha^3,\\
\widehat{\beta} &= 1+j\beta+\varepsilon\beta^2+j\varepsilon\beta^3,\\
\widehat{\alpha}^2 &= 1+\alpha^2+2\alpha j+2\alpha^2\left(\alpha^2+1\right)\varepsilon+4\alpha^3 j\varepsilon,\\
\widehat{\beta}^2 &= 1+\beta^2+2j\beta+2\beta^4\varepsilon+2\beta^2\varepsilon+4j\beta^3\varepsilon,\\
\widehat{\alpha}\widehat{\beta} &= 1+\alpha(\beta+j)+j\beta+\varepsilon(\alpha^2+\beta^2)+j\alpha\beta^2\varepsilon+j\alpha^2\beta\varepsilon+\beta^3(\alpha\varepsilon+j\varepsilon)+\alpha^3(\beta\varepsilon+j\varepsilon),\\
\widehat{\alpha}^2\widehat{\beta} &= 1+\alpha^2+\beta^2+\alpha^2\beta^2+2\left(\alpha\beta+1\right)\left(\alpha+\beta\right)j+2\left(\alpha^2+\beta^2+\alpha^2\beta^2+4\alpha\beta+1\right)\left(\alpha^2+\beta^2\right)\varepsilon\\
&\quad +4\left(\alpha+\beta\right)\left(\alpha^2+\beta^2+\alpha\beta^3\right)j\varepsilon,\\
\widehat{\alpha}\widehat{\beta}^2 &= 1+\beta^2+2\alpha\beta+\left(\alpha+2\beta+\alpha\beta^2\right)j+\left(\beta^2+2\alpha\beta+1\right)\left(\alpha^2+2\beta^2\right)\varepsilon+\left(\alpha+2\beta+\alpha\beta^2\right)\left(\alpha^2+2\beta^2\right)j\varepsilon,\\
\widehat{\alpha}^2\widehat{\beta}^2 &= 1+\beta^2+\alpha^2+\alpha^2\beta^2+4\alpha\beta+2\left(\alpha\beta+1\right)\left(\alpha+\beta\right)j+2\left(\alpha^2+\beta^2+\alpha^2\beta^2+4\alpha\beta+1\right)\left(\alpha^2+\beta^2\right)\varepsilon\\
&\quad +4\left(\alpha\beta+1\right)\left(\alpha+\beta\right)\left(\alpha^2+\beta^2\right)j\varepsilon.
\end{aligned}$$

Theorem 3. *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Adrien number is*

$$\widehat{W}_n=\widehat{\alpha}S_1\alpha^n+\widehat{\beta}S_2\beta^n+\widehat{\gamma}S_3\gamma^n+\widehat{\delta}S_4, \tag{2.9}$$

$\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$, $\widehat{\delta}$ are given as (2.5), (2.6), (2.7) ,(2.8).

Proof. Using Binet's formula of the generalized Adrien numbers given below

$$W_n=S_1\alpha^n+S_2\beta^n+S_3\gamma^n+S_4,$$

where S_1, S_2, S_2, S_4 are given (1.3) we get

$$\begin{aligned}
\widehat{W}_n &= W_n+jW_{n+1}+\varepsilon W_{n+2}+j\varepsilon W_{n+3}\\
&= S_1\alpha^n+S_2\beta^n+S_3\gamma^n+S_4\\
&\quad +j(S_1\alpha^{n+1}+S_2\beta^{n+1}+S_3\gamma^{n+1}+S_4)\\
&\quad +\varepsilon(S_1\alpha^{n+2}+S_2\beta^{n+2}+S_3\gamma^{n+2}+S_4)\\
&\quad +j\varepsilon(S_1\alpha^{n+3}+S_2\beta^{n+3}+S_3\gamma^{n+3}+S_4)\\
&= S_1\alpha^n(1+j\alpha+\varepsilon\alpha^2+j\varepsilon\alpha^3)+S_2\beta^n(1+j\beta+\varepsilon\beta^2+j\varepsilon\beta^3)\\
&\quad +S_3\gamma^n(1+j\gamma+\varepsilon\gamma^2+j\varepsilon\gamma^3)+S_4(1+j+\varepsilon+j\varepsilon)\\
&= \widehat{\alpha}S_1\alpha^n+\widehat{\beta}S_2\beta^n+\widehat{\gamma}S_3\gamma^n+\widehat{1}^nS_4.
\end{aligned}$$

This proves (2.9). □

As special cases, for any integer n, the Binet's Formula of nth dual hyperbolic Adrien number is

$$\widehat{A}_n=\frac{(2\alpha^2+\alpha+1)\alpha^n\widehat{\alpha}}{4\alpha^2+3\alpha-1}+\frac{(2\beta^2+\beta+1)\beta^n\widehat{\beta}}{4\beta^2+3\beta-1}+\frac{(2\gamma^2+\gamma+1)\gamma^n\widehat{\gamma}}{4\gamma^2+3\gamma-1}-\frac{\widehat{1}}{3}, \tag{2.10}$$

and the Binet's Formula of nth dual hyperbolic Adrien-Lucas number is

$$\widehat{B}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + 1. \tag{2.11}$$

Next, we present generating function of the dual hyperbolic genralized Adrien numbers.

THEOREM 4. *The generating function for the dual hyperbolic generalized Adrien numbers is*

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)x^2 + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)x^3}{1 - 3x + x^2 + x^4}. \tag{2.12}$$

Proof. Let

$$f_{\widehat{W}_n}(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n$$

be generating function of the dual hyperbolic generalized Adrien numbers. Then, using the definition of the dual hyperbolic generalized Adrien numbers, and substracting $xf_{\widehat{W}_n}(x)$ and $x^2 f_{\widehat{W}_n}(x)$ from $f_{\widehat{W}_n}(x)$, we obtain $(1 - 3x + x^2 + x^4)f_{\widehat{W}_n}(x)$

$$\begin{aligned}
(1 - 3x + x^2 + x^4)f_{\widehat{W}_n}(x) &= \sum_{n=0}^{\infty} \widehat{W} x^n - 3x \sum_{n=0}^{\infty} \widehat{W} x^n + x^2 \sum_{n=0}^{\infty} \widehat{W} x^n + x^4 \sum_{n=0}^{\infty} \widehat{W} x^n, \\
&= \sum_{n=0}^{\infty} \widehat{W} x^n - 3 \sum_{n=0}^{\infty} \widehat{W} x^{n+1} + \sum_{n=0}^{\infty} \widehat{W} x^{n+2} + \sum_{n=0}^{\infty} \widehat{W} x^{n+4}, \\
&= \sum_{n=0}^{\infty} \widehat{W} x^n - 3 \sum_{n=1}^{\infty} \widehat{W}_{(n-1)} x^n + \sum_{n=2}^{\infty} \widehat{W}_{(n-2)} x^n + \sum_{n=4}^{\infty} \widehat{W}_{(n-4)} x^n, \\
&= (\widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2 + \widehat{W}_3 x^3) - 3(\widehat{W}_0 x + \widehat{W}_1 x^2 + \widehat{W}_2 x^3) + 3(\widehat{W}_0 x^2 + \widehat{W}_1 x^3), \\
&\quad + \sum_{n=4}^{\infty} (\widehat{W}_n - \widehat{3W}_{n-1} + \widehat{W}_{n-2} + \widehat{W}_{n-4}) x^n, \\
&= \widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2 + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)x^3.
\end{aligned}$$

Note that, using the recurrence relation $\widehat{A} = 3\widehat{A}_{n-1} - \widehat{A}_{n-2} - \widehat{A}_{n-4}$ and rearranging above equation the (2.12) has been obtained. □

Now we can write the generating functions of the dual hyperbolic Adrien and Adrien-Lucas numbers as:

(a): $f_{\widehat{A}_n}(x) = \sum\limits_{n=0}^{\infty} \widehat{A}_n x^n = \frac{1}{1-3x+x^2+x^4}(j + 3\varepsilon + 8\varepsilon j) + (1 - 3\varepsilon + 8\varepsilon j)x + (6j - 9\varepsilon j)x^2 + (-\varepsilon - 3\varepsilon j)x^3,$

(b): $f_{\widehat{B}_n}(x) = \sum\limits_{n=0}^{\infty} \widehat{B}_n x^n = \frac{1}{1-3x+x^2+x^4}(4 + 3j + 7\varepsilon + 18\varepsilon j) + (-9 - 2j - 3\varepsilon - 11\varepsilon j)x$
$+(2 - 4\varepsilon - 3\varepsilon j)x^2 + (-4j - 3\varepsilon - 7\varepsilon j)x^3.$

LEMMA 5. *Suppose that* $f_{\widehat{W}_n}(x) = \sum\limits_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!}$ *is the exponential generating function of the dual hyperbolic generalized Adrien sequence* $\{\widehat{W}_n\}$.

Then $\sum\limits_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!} = S_1 e^{\alpha x}\widehat{\alpha} + S_2 e^{\beta x}\widehat{\beta} + S_3 e^{\gamma x}\widehat{\gamma} + S_4 e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as(2.5), (2.6), (2.7) ,(2.8).Using Binet's formula of dual hiperbolic generalized Adrien numbers or exponential generating function of the generalized Adrien sequence we get the required identity.Let' s get the details.

Proof. Using Binet' s formula

$$W_n = S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4,$$

where A_1, A_2, A_3, A_4 are given in (1.3) we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}W_n\frac{x^n}{n!} + j\sum_{n=0}^{\infty}W_{n+1}\frac{x^n}{n!} + \varepsilon\sum_{n=0}^{\infty}W_{n+2}\frac{x^n}{n!} + j\varepsilon\sum_{n=0}^{\infty}W_{n+3}\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}(S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4)\frac{x^n}{n!} + j\sum_{n=0}^{\infty}(S_1\alpha^{n+1} + S_2\beta^{n+1} + S_3\gamma^{n+1} + S_4)\frac{x^n}{n!} \\
&\quad +\varepsilon\sum_{n=0}^{\infty}(S_1\alpha^{n+2} + S_2\beta^{n+2} + S_3\gamma^{n+2} + S_4)\frac{x^n}{n!} + j\varepsilon\sum_{n=0}^{\infty}(S_1\alpha^{n+3} + S_2\beta^{n+3} + S_3\gamma^{n+3} + S_4)\frac{x^n}{n!} \\
&= (S_1e^{\alpha x} + S_2e^{\beta x} + S_3e^{\gamma x} + S_4e^{x}) + j(S_1\alpha e^{\alpha x} + S_2\beta e^{\beta x} + S_3\gamma e^{\gamma x} + S_4e^{x}) \\
&\quad +\varepsilon(S_1\alpha^2e^{\alpha x} + S_2\beta^2e^{\beta x} + S_3\gamma^2e^{\gamma x} + S_4e^{x}) + j\varepsilon(S_1\alpha^3e^{\alpha x} + S_2\beta^3e^{\beta x} + S_3\gamma^3e^{\gamma x} + S_4e^{x}) \\
&= S_1e^{\alpha x}(1 + j\alpha + +\varepsilon\alpha^2 + j\varepsilon\alpha^3) + S_2e^{\beta x}(1 + j\beta + +\varepsilon\beta^2 + j\varepsilon\beta^3) \\
&\quad +S_3e^{\gamma x}(1 + j\gamma + +\varepsilon\gamma^2 + j\varepsilon\gamma^3) + S_4e^{x}(1 + j + +\varepsilon + j\varepsilon) \\
&= S_1e^{\alpha x}\widehat{\alpha} + S_2e^{\beta x}\widehat{\beta} + S_3e^{\gamma x}\widehat{\gamma} + S_4e^{x}\widehat{1}. \square
\end{aligned}$$

□

The previous Lemma gives the following results as particular examples.

Corollary 6. *Exponential generating function of dual hiperbolic Adrien and dual hiperbolic Adrien-Lucas numbers are given by:*

a):

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{A}_n\frac{x^n}{n!} &= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^{x}) \\
&\quad +j(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^{x}) \\
&\quad +\varepsilon(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^{x}) \\
&\quad +j\varepsilon(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^{x}). \\
&= \frac{(2\alpha^2+\alpha+1)\widehat{\alpha}}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)\widehat{\beta}}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)\widehat{\gamma}}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{\widehat{1}}{3}e^{x}
\end{aligned}$$

b):

$$\sum_{n=0}^{\infty}\widehat{B}_n\frac{x^n}{n!} = e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^{x}+j(\alpha e^{\alpha x}+\beta e^{\beta x}+\gamma e^{\gamma x}+e^{x})$$
$$+\varepsilon(\alpha^2 e^{\alpha x}+\beta^2 e^{\beta x}+\gamma^2 e^{\gamma x}+e^{x})$$
$$+j\varepsilon(\alpha^3 e^{\alpha x}+\beta^3 e^{\beta x}+\gamma^3 e^{\gamma x}+e^{x}).$$
$$= e^{\alpha x}\widehat{\alpha}+e^{\beta x}\widehat{\beta}+e^{\gamma x}\widehat{\gamma}+e^{x}\widehat{1}.$$

3. Obtaining Binet Formula From Generating Function

We next find Binet formula of dual hyperbolic generalized Adrien number $\{\widehat{W}_n\}$ by the use of generating function for $\widehat{W}_n$.

THEOREM 7. *(Binet formula of dual hyperbolic generalized Adrien numbers)*

$$\widehat{W}_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{3.1}$$

where

$$p_1 = \widehat{W}_0\alpha^3+(\widehat{W}_1-3\widehat{W}_0)\alpha^2+(\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\alpha+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1),$$
$$p_2 = \widehat{W}_0\beta^3+(\widehat{W}_1-3\widehat{W}_0)\beta^2+(\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\beta+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1),$$
$$p_3 = \widehat{W}_0\gamma^3+(\widehat{W}_1-3\widehat{W}_0)\gamma^2+(\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\gamma+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1),$$
$$p_4 = \widehat{W}_0\delta^3+(\widehat{W}_1-3\widehat{W}_0)\delta^2+(\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\delta+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1).$$

Proof. Let

$$h(x) = 1-3x+x^2+x^4.$$

Then for some α,β,γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x),$$

i.e.,

$$1-3x+x^2 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x), \tag{3.2}$$

Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α,β,γ and δ as the roots of

$$h(\frac{1}{x}) = 1-\frac{3}{x}+\frac{1}{x^2}+\frac{1}{x^4} = 0.$$

This implies $x^4-3x^3+x^2+u=0$. Now, by it follows that

$$\sum_{n=0}^{\infty} \widehat{W} x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)x^2 + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\begin{aligned} & \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)x^2 + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} \\ = & \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}. \end{aligned} \tag{3.3}$$

So

$$\begin{aligned} & \widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)x^2 + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)x^3 \\ = & B_1(1-\beta x)(1-\gamma x)(1-\delta x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\ & + B_3(1-\alpha x)(1-\beta x)(1-\delta x) + B_3(1-\alpha x)(1-\beta x)(1-\gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)\frac{1}{\alpha} + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)\frac{1}{\alpha^2} + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)\frac{1}{\alpha^3}$ $= B_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})(1 - \frac{\delta}{\alpha})$.

This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3(\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)\frac{1}{\alpha} + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)\frac{1}{\alpha^2} + (\widehat{W}_3 - 6\widehat{W}_2 + \widehat{W}_1)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{\widehat{W}_0\alpha^3 + (\widehat{W}_1 - \widehat{W}_0)\alpha^2 + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)\alpha + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{\widehat{W}_0\beta^3 + (\widehat{W}_1 - 3\widehat{W}_0)\beta^2 + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)\beta + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{\widehat{W}_0\gamma^3 + (\widehat{W}_1 - 3\widehat{W}_0)\gamma^2 + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)\gamma + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{\widehat{W}_0\delta^3 + (\widehat{W}_1 - 3\widehat{W}_0)\delta^2 + (\widehat{W}_2 - 3\widehat{W}_1 + \widehat{W}_0)\delta + (\widehat{W}_3 - 3\widehat{W}_2 + \widehat{W}_1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} \widehat{W} x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{W} x^n &= B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n \\ &= \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n)x^n. \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$\widehat{W} = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

and then we get (3.1). □

We can get an identitiy related to dual hyperbolic Adrien numbers given below.

THEOREM 8. *For all integers m, n the following identities hold:*

$\widehat{W}_{m+n} = A_{m-2}\widehat{W}_{n+3} + (-A_{m-3} - A_{m-5})\widehat{W}_{n+2} + (-A_{m-4})\widehat{W}_{n+1} - A_{m-3}\widehat{W}_n.$

Proof. First we assume that $m, n \geq 0$ then (8) can be proved by mathematical induction on m. If $m = 0$ we get

$$\widehat{W}_n = A_{-2}\widehat{W}_{n+3} + (-A_{-3} - A_{-5})\widehat{W}_{n+2} + (-A_{-4})\widehat{W}_{n+1} - A_{-3}\widehat{W}_n.$$

which is true since $A_{-2} = 0$, $A_{-3} = -1$, $A_{-4} = 0$, $A_{-5} = 1$. Assume that the equality holds for $m \leq k$. For $m = k + 1$, we get

$$\begin{aligned} \widehat{W}_{k+1+n} &= 3\widehat{W}_{n+k} - \widehat{W}_{n+k-1} - \widehat{W}_{n+k-3}, \\ &= 3A_{k-2}\widehat{W}_{n+3} + (-A_{k-3} - A_{k-5})\widehat{W}_{n+2} \\ &\quad +3(-A_{k-4})\widehat{W}_{n+1} - A_{k-3}\widehat{W}_n \\ &\quad -(A_{k-3}\widehat{W}_{n+3} + (-A_{k-4} - A_{k-6})\widehat{W}_{n+2} + (-A_{-5})\widehat{W}_{n+1} - A_{k-4}\widehat{W}_n) \\ &\quad -A_{k-5}\widehat{W}_{n+3} + (-A_{k-6} - A_{k-8})\widehat{W}_{n+2} + (-A_{k-6})\widehat{W}_{n+1} - A_{k-6}\widehat{W}_n. \end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem (8).

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $\widehat{W}_n = A_n$ or $\widehat{W}_n = B_n$ in above Theorem, respectively, we get:

COROLLARY 9.

$$\begin{aligned} \widehat{A}_{m+n} &= A_{m-2}\widehat{A}_{n+3} + (-A_{m-3} - A_{m-5})\widehat{A}_{n+2} + (-A_{m-4})\widehat{A}_{n+1} - A_{m-3}\widehat{A}_n, \\ \widehat{B}_{m+n} &= A_{m-2}\widehat{B}_{n+3} + (-A_{m-3} - A_{m-5})\widehat{B}_{n+2} + (-A_{m-4})\widehat{B}_{n+1} - A_{m-3}\widehat{B}_n. \end{aligned}$$

4. Simson's Formulas

In this section, we present Simson's formula for the dual hyperbolic generalized Adrien numbers . This is a special case of [21,Theorem 4.1].

THEOREM 10. *For all integers n, we have*

$$\begin{vmatrix} \widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} & \widehat{W}_{n-3} \end{vmatrix} = (\widehat{W}_0+\widehat{W}_1+2\widehat{W}_2-\widehat{W}_3)(-\widehat{W}_3^3+5\widehat{W}_2^3+\widehat{W}_1^3+\widehat{W}_0^3-(\widehat{W}_0+3\widehat{W}_1-7\widehat{W}_2)\widehat{W}_3^2$$

$+(3\widehat{W}_0-4\widehat{W}_1-14\widehat{W}_3)\widehat{W}_2^2+(2\widehat{W}_0+\widehat{W}_2-6\widehat{W}_3)\widehat{W}_1^2-(\widehat{W}_1+2\widehat{W}_3)\widehat{W}_0^2+13\widehat{W}_1\widehat{W}_2\widehat{W}_3+\widehat{W}_0\widehat{W}_2\widehat{W}_3+5\widehat{W}_0\widehat{W}_1\widehat{W}_3-7\widehat{W}_0\widehat{W}_1\widehat{W}_2).$

Proof. Take $r=3, s=-1, t=0, u=-1$. □

Corollary 11. *For all integers n, the Simson's formulas of dual hyperbolic generalized Adrien number and dual hyperbolic generalized Adrien-Lucas numbers are given as:*

$$\begin{vmatrix} \widehat{A}_{n+3} & \widehat{A}_{n+2} & \widehat{A}_{n+1} & \widehat{A}_n \\ \widehat{A}_{n+2} & \widehat{A}_{n+1} & \widehat{A}_n & \widehat{A}_{n-1} \\ \widehat{A}_{n+1} & \widehat{A}_n & \widehat{A}_{n-1} & \widehat{A}_{n-2} \\ \widehat{A}_n & \widehat{A}_{n-1} & \widehat{A}_{n-2} & \widehat{A}_{n-3} \end{vmatrix} = 18j-97\varepsilon+139j\varepsilon-12,$$

$$\begin{vmatrix} \widehat{B}_{n+3} & \widehat{B}_{n+2} & \widehat{B}_{n+1} & \widehat{B}_n \\ \widehat{B}_{n+2} & \widehat{B}_{n+1} & \widehat{B}_n & \widehat{B}_{n-1} \\ \widehat{B}_{n+1} & \widehat{B}_n & \widehat{B}_{n-1} & \widehat{B}_{n-2} \\ \widehat{B}_n & \widehat{B}_{n-1} & \widehat{B}_{n-2} & \widehat{B}_{n-3} \end{vmatrix} = -2349j-16\,443\varepsilon-16\,443j\varepsilon-2349,$$

respectively.

5. Linear Sums

In this section, we give the summation formulas of the dual hyperbolic generalized Adrien numbers with positive and negatif subscripts. Now, we present the summation formulas of the generalized Adrien numbers.

Theorem 12. *For the generalized Adrien numbers with positive and negative subscript, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_k = \frac{1}{3}(-(n+3)W_{n+3}+(2n+7)W_{n+2}+(n+2)W_{n+1}+(n+4)W_n+3W_3-7W_2-2W_1-W_0).$

(b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(-(n+2)W_{2n+2}+(2n+5)W_{2n+1}+(n+3)W_{2n}+(n+2)W_{2n-1}+2W_3-4W_2-3W_1).$

(c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(n+1)W_{2n+2}+(2n+5)W_{2n+1}+(n+2)W_{2n}+(n+2)W_{2n-1}+2W_3-5W_2-2W_0).$

(d): $\sum_{k=1}^{n} W_{-k} = \frac{1}{3}(-(n+1)W_{-n+3}+(2n+1)W_{-n+2}+(n+2)W_{-n+1}+(n+3)W_{-n}+W_3-W_2-2W_1-3W_0).$

(e): $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(-(n+2)W_{-2n+2}+(2n+3)W_{-2n+1}+(n+4)W_{-2n}+(n+2)W_{-2n-1}+2W_3-4W_2-W_1-4W_0).$

(f): $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(n+3)W_{-2n+2}+2(n+3)W_{-2n+1}+(n+2)W_{-2n}+(n+2)W_{-2n-1}+2W_3-3W_2-4W_1-2W_0).$

Proof. For the proof, see Soykan [19]. □

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic numbers.

Theorem 13. *For the dual hyperbolic numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n}\widehat{W}_k=\frac{1}{3}(-(n+3)\widehat{W}_{n+3}+(2n+7)\widehat{W}_{n+2}+(n+2)\widehat{W}_{n+1}+(n+4)\widehat{W}_n+3\widehat{W}_3-7\widehat{W}_2-2\widehat{W}_1-\widehat{W}_0)$.

(b): $\sum\limits_{k=0}^{n}\widehat{W}_{2k}=\frac{1}{3}(-(n+2)\widehat{W}_{2n+2}+(2n+5)\widehat{W}_{2n+1}+(n+3)\widehat{W}_{2n}+(n+2)\widehat{W}_{2n-1}+2\widehat{W}_3-4\widehat{W}_2-3\widehat{W}_1)$.

(c): $\sum\limits_{k=0}^{n}\widehat{W}_{2k+1}=\frac{1}{3}(-(n+1)\widehat{W}_{2n+2}+(2n+5)\widehat{W}_{2n+1}+(n+2)\widehat{W}_{2n}+(n+2)\widehat{W}_{2n-1}+2\widehat{W}_3-5\widehat{W}_2-2\widehat{W}_0)$.

(d): $\sum\limits_{k=1}^{n}\widehat{W}_{-k}=\frac{1}{3}(-(n+1)\widehat{W}_{-n+3}+(2n+1)\widehat{W}_{-n+2}+(n+2)\widehat{W}_{-n+1}+(n+3)\widehat{W}_{-n}+\widehat{W}_3-\widehat{W}_2-2\widehat{W}_1-3\widehat{W}_0)$.

(e): $\sum\limits_{k=1}^{n}\widehat{W}_{-2k}=\frac{1}{3}(-(n+2)\widehat{W}_{-2n+2}+(2n+3)\widehat{W}_{-2n+1}+(n+4)\widehat{W}_{-2n}+(n+2)\widehat{W}_{-2n-1}+2\widehat{W}_3-4\widehat{W}_2-\widehat{W}_1-4\widehat{W}_0)$.

(f): $\sum\limits_{k=1}^{n}\widehat{W}_{-2k+1}=\frac{1}{3}(-(n+3)\widehat{W}_{-2n+2}+2(n+3)\widehat{W}_{-2n+1}+(n+2)\widehat{W}_{-2n}+(n+2)\widehat{W}_{-2n-1}+2\widehat{W}_3-3\widehat{W}_2-4\widehat{W}_1-2\widehat{W}_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widehat{W}_k=\sum_{k=0}^{n}W_k+j\sum_{k=0}^{n}W_{k+1}+\varepsilon\sum_{k=0}^{n}W_{k+2}+j\varepsilon\sum_{k=0}^{n}W_{k+3}$$

and using Theorem (12) then proof is straightforward.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{2k}=\sum_{k=0}^{n}W_{2k}+j\sum_{k=0}^{n}W_{2k+1}+\varepsilon\sum_{k=0}^{n}W_{2k+2}+j\varepsilon\sum_{k=0}^{n}W_{2k+3}$$

and using Theorem (12) the proof is easily attainable.

(c): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{2k+1}=\sum_{k=0}^{n}W_{2k+1}+j\sum_{k=0}^{n}W_{2k+2}+\varepsilon\sum_{k=0}^{n}W_{2k+3}+j\varepsilon\sum_{k=0}^{n}W_{2k+4}$$

and using Theorem (12) the proof is straightforward.

Proof.

(d): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-k}=\sum_{k=0}^{n}W_{-k}+j\sum_{k=0}^{n}W_{-k+1}+\varepsilon\sum_{k=0}^{n}W_{-k+2}+j\varepsilon\sum_{k=0}^{n}W_{-k+3}$$

and using Theorem (12) the proof is easily attainable.

(e): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-2k}=\sum_{k=0}^{n}W_{-2k}+j\sum_{k=0}^{n}W_{-2k+1}+\varepsilon\sum_{k=0}^{n}W_{-2k+2}+j\varepsilon\sum_{k=0}^{n}W_{-2k+3}$$

and using Theorem (12) then proof is straightforward.

(f): Note that using (2.1), we get using Theorem (12), we get

$$\sum_{k=0}^{n}\widehat{W}_{-2k+1}=\sum_{k=0}^{n}W_{-2k+1}+j\sum_{k=0}^{n}W_{-2k+2}+\varepsilon\sum_{k=0}^{n}W_{2k+3}+j\varepsilon\sum_{k=0}^{n}W_{-2k+4}$$

and using Theorem (13) the proof is easily completed. □

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic Adrien numbers.

THEOREM 14. *For $n\geq 0$, dual hyperbolic generalized Adrien numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n}\widehat{A}_k=\frac{1}{3}(-(n+3)\widehat{A}_{n+3}+(2n+7)\widehat{A}_{n+2}+(n+2)\widehat{A}_{n+1}+(n+4)\widehat{A}_n+j-4\varepsilon-14j\varepsilon)$.

(b): $\sum\limits_{k=0}^{n}\widehat{A}_{2k}=\frac{1}{3}(-(n+2)\widehat{A}_{2n+2}+(2n+5)\widehat{A}_{2n+1}+(n+3)\widehat{A}_{2n}+(n+2)\widehat{A}_{2n-1}+1+j-3j\varepsilon)$.

(c): $\sum\limits_{k=0}^{n}\widehat{A}_{2k+1}=\frac{1}{3}(-(n+1)\widehat{A}_{2n+2}+(2n+5)\widehat{A}_{2n+1}+(n+2)\widehat{A}_{2n}+(n+2)\widehat{A}_{2n-1}-1+2j-3\varepsilon-10j\varepsilon)$.

(d): $\sum\limits_{k=1}^{n}\widehat{A}_{-k}=\frac{1}{3}(-(n+1)\widehat{A}_{-n+3}+(2n+1)\widehat{A}_{-n+2}+(n+2)\widehat{A}_{-n+1}+(n+3)\widehat{A}_{-n}+7j+8\varepsilon+18j\varepsilon)$.

(e): $\sum\limits_{k=1}^{n}\widehat{A}_{-2k}=\frac{1}{3}(-(n+2)\widehat{A}_{-2n+2}+(2n+3)\widehat{A}_{-2n+1}+(n+4)\widehat{A}_{-2n}+(n+2)\widehat{A}_{-2n-1}-1+7j+4\varepsilon+7j\varepsilon)$.

(f): $\sum\limits_{k=1}^{n}\widehat{A}_{-2k+1}=\frac{1}{3}(-(n+3)\widehat{A}_{-2n+2}+2(n+3)\widehat{A}_{-2n+1}+(n+2)\widehat{A}_{-2n}+(n+2)\widehat{A}_{-2n-1}+1+6j+7\varepsilon+14j\varepsilon)$.

Next, we give the ordinary generating functions of some special cases of dual hyperbolic generalized Adrien numbers.

THEOREM 15. *The ordinary generating functions of the sequences $\widehat{W}_{2n}$, $\widehat{W}_{2n+1}$ are given as follows:*

(a): $\sum_{n=0}^{\infty}\widehat{W}_{2n}x^n=\dfrac{3x^2\widehat{W}_3+(x^3-8x^2+x)\widehat{W}_2-3x^3\widehat{W}_1+(x^3+2x^2-7x+1)\widehat{W}_0}{x^4+2x^3+3x^2-7x+1}$

(b): $\sum_{n=0}^{\infty}\widehat{W}_{2n+1}x^n=\dfrac{(x^3+x^2+x)\widehat{W}_3-(3x^3+3x^2)\widehat{W}_2+(x^3+2x^2-7x+1)\widehat{W}_1-3x^2\widehat{W}_0}{x^4+2x^3+3x^2-7x+1}$

From the last Theorem, we have the following Corollary which gives sum formula of dual hyperbolic Adrien numbers (Take $\widehat{W}_n=\widehat{A}_n$ with

$\widehat{A}_0=j+3\varepsilon+8\varepsilon j$, $\widehat{A}_1=1+3j+8\varepsilon+21j\varepsilon$, $\widehat{A}_2=3+8j+21\varepsilon+54j\varepsilon$, $\widehat{A}_3=8+21j+54\varepsilon+138j\varepsilon$)

COROLLARY 16. *For $n\geq 0$ dual hyperbolic Adrien numbers have the following properties.*

(a): $\sum_{n=0}^{\infty}\widehat{A}_{2n}x^n=\frac{1}{x^4+2x^3+3x^2-7x+1}3x^2(21j+54\varepsilon+138j\varepsilon+8)-3x^3(3j+8\varepsilon+21j\varepsilon+1)+(3\varepsilon+8j\varepsilon+1)\left(x^3+2x^2-7x+1\right)+\left(x^3-8x^2+x\right)(8j+21\varepsilon+54j\varepsilon+3)$.

(b): $\sum_{n=0}^{\infty}\widehat{A}_{2n+1}x^n=\frac{1}{-(x^4+2x^3+3x^2-7x+1)}3x^2(3\varepsilon+8j\varepsilon+1)+\left(3x^3+3x^2\right)(8j+21\varepsilon+54j\varepsilon+3)-(3j+8\varepsilon+21j\varepsilon+1)\left(x^3+2x^2-7x+1\right)-\left(x^3+x^2+x\right)(21j+54\varepsilon+138j\varepsilon+8)$.

6. Matrices related with Dual Hyperbolic Generalized Adrien Numbers

In this part of our study we give some identities on some matrices linked to dual hyperbolic Adrien numbers.

By using the $\{A_n\}$ which is defined by the fourth-order recurrence relation as follows:

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,$$

with the initial conditions

$$A_0 = 0,\ A_1 = 1,\ A_2 = 3,\ A_3 = 8. \tag{6.1}$$

We define the square matrix M of order 4 as

$$M = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detM = 1$. Then, we give the following Lemma.

Lemma 17. *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} \widehat{W}_{n+3} \\ \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} \tag{6.2}$$

Proof. First, we prove the assertion for the case $n \geq 0$. Lemma 17 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$

which is true. We assume that (6.2) is true for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{W}_{k+4} \\ \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}.$$

Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^n = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}$$

For the proof see [20].

We define

$$N_{H_{\widehat{W}}} = \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix}, \tag{6.3}$$

$$E_{H_{\widehat{W}}} = \begin{pmatrix} \widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} & \widehat{W}_{n-3} \end{pmatrix}. \tag{6.4}$$

Now, we have the following theorem with $N_{\widehat{W}}$ and $E_{\widehat{W}}$.

THEOREM 18. *Using $N_{\widehat{W}}$ and $E_{\widehat{W}}$, we get*

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. Note that we get

$$
\begin{aligned}
A^n N_{\widehat{W}} &= \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix} \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
a_{11} &= A_{n+1}\widehat{W}_3 + (-A_n - A_{n-2})\widehat{W}_2 + (-A_{n-1})\widehat{W}_1 + (-A_n)\widehat{W}_0, \\
a_{12} &= A_{n+1}\widehat{W}_2 + (-A_n - A_{n-2})\widehat{W}_1 + (-A_{n-1})\widehat{W}_0 + (-A_n)\widehat{W}_{-1}, \\
a_{13} &= A_{n+1}\widehat{W}_1 + (-A_n - A_{n-2})\widehat{W}_0 + (-A_{n-1})\widehat{W}_{-1} + (-A_n)\widehat{W}_{-2}, \\
a_{14} &= A_{n+1}\widehat{W}_0 + (-A_n - A_{n-2})\widehat{W}_{-1} + (-A_{n-1})\widehat{W}_{-2} + (-A_n)\widehat{W}_{-3}, \\
a_{21} &= A_n\widehat{W}_3 + (-A_{n-1} - A_{n-3})\widehat{W}_2 + (-A_{n-2})\widehat{W}_1 + (-A_{n-1})\widehat{W}_0, \\
a_{22} &= A_n\widehat{W}_2 + (-A_{n-1} - A_{n-3})\widehat{W}_1 + (-A_{n-2})\widehat{W}_0 + (-A_{n-1})\widehat{W}_{-1}, \\
a_{23} &= A_n\widehat{W}_1 + (-A_{n-1} - A_{n-3})\widehat{W}_0 + (-A_{n-2})\widehat{W}_{-1} + (-A_{n-1})\widehat{W}_{-2}, \\
a_{24} &= A_n\widehat{W}_0 + (-A_{n-1} - A_{n-3})\widehat{W}_{-1} + (-A_{n-2})\widehat{W}_{-2} + (-A_{n-1})\widehat{W}_{-3}, \\
a_{31} &= A_{n-1}\widehat{W}_3 + (-A_{n-2} - A_{n-4})\widehat{W}_2 + (-A_{n-3})\widehat{W}_1 + (-A_{n-2})\widehat{W}_0, \\
a_{32} &= A_{n-1}\widehat{W}_2 + (-A_{n-2} - A_{n-4})\widehat{W}_1 + (-A_{n-3})\widehat{W}_0 + (-A_{n-2})\widehat{W}_{-1}, \\
a_{33} &= A_{n-1}\widehat{W}_1 + (-A_{n-2} - A_{n-4})\widehat{W}_0 + (-A_{n-3})\widehat{W}_{-1} + (-A_{n-2})\widehat{W}_{-2}, \\
a_{34} &= A_{n-1}\widehat{W}_0 + (-A_{n-2} - A_{n-4})\widehat{W}_{-1} + (-A_{n-3})\widehat{W}_{-2} + (-A_{n-2})\widehat{W}_{-3}, \\
a_{41} &= A_{n-2}\widehat{W}_3 + (-A_{n-3} - A_{n-5})\widehat{W}_2 + (-A_{n-4})\widehat{W}_1 + (-A_{n-3})\widehat{W}_0, \\
a_{42} &= A_{n-2}\widehat{W}_2 + (-A_{n-3} - A_{n-5})\widehat{W}_1 + (-A_{n-4})\widehat{W}_0 + (-A_{n-3})\widehat{W}_{-1}, \\
a_{43} &= A_{n-2}\widehat{W}_1 + (-A_{n-3} - A_{n-5})\widehat{W}_0 + (-A_{n-4})\widehat{W}_{-1} + (-A_{n-3})\widehat{W}_{-2}, \\
a_{44} &= A_{n-2}\widehat{W}_0 + (-A_{n-3} - A_{n-5})\widehat{W}_{-1} + (-A_{n-4})\widehat{W}_{-2} + (-A_{n-3})\widehat{W}_{-3}.
\end{aligned}
$$

Using the theorem (8) the proof is done. □

by taking $W_n = A_n$ with A_0, A_1, A_2, A_3 in (6.3) and (6.4)

$\widehat{W}_n = \widehat{W}B_n$ with $\widehat{W}B_0$, $\widehat{W}B_1$, $\widehat{W}B_2$, $\widehat{W}B_3$ in (6.3) and (6.4)

respectively, we get:

$$N_{\widehat{A}} = \begin{pmatrix} 8+21j+54\varepsilon+138j\varepsilon & 3+8j+21\varepsilon+54j\varepsilon & 1+3j+8\varepsilon+21j\varepsilon & j+3\varepsilon+8\varepsilon j \\ 3+8j+21\varepsilon+54j\varepsilon & 1+3j+8\varepsilon+21j\varepsilon & j+3\varepsilon+8\varepsilon j & \varepsilon+3\varepsilon j \\ 1+3j+8\varepsilon+21j\varepsilon & j+3\varepsilon+8\varepsilon j & \varepsilon+3\varepsilon j & j\varepsilon \\ j+3\varepsilon+8\varepsilon j & \varepsilon+3\varepsilon j & j\varepsilon & -1 \end{pmatrix},$$

$$E_{\widehat{A}} = \begin{pmatrix} \widehat{A}_{n+3} & \widehat{A}_{n+2} & \widehat{A}_{n+1} & \widehat{A}_n \\ \widehat{A}_{n+2} & \widehat{A}_{n+1} & \widehat{A}_n & \widehat{A}_{n-1} \\ \widehat{A}_{n+1} & \widehat{A}_n & \widehat{A}_{n-1} & \widehat{A}_{n-2} \\ \widehat{A}_n & \widehat{A}_{n-1} & \widehat{A}_{n-2} & \widehat{A}_{n-3} \end{pmatrix},$$

$$N_{\widehat{B}} = \begin{pmatrix} 18+43j+108\varepsilon+274j\varepsilon & 7+18j+43\varepsilon+108j\varepsilon & 3+7j+18\varepsilon+43j\varepsilon & 4+3j+7\varepsilon+18j\varepsilon \\ 7+18j+43\varepsilon+108j\varepsilon & 3+7j+18\varepsilon+43j\varepsilon & 4+3j+7\varepsilon+18j\varepsilon & 4j+3\varepsilon+7j\varepsilon \\ 3+7j+18\varepsilon+43j\varepsilon & 4+3j+7\varepsilon+18j\varepsilon & 4j+3\varepsilon+7j\varepsilon & -2+4\varepsilon+3j\varepsilon \\ 4+3j+7\varepsilon+18j\varepsilon & 4j+3\varepsilon+7j\varepsilon & -2+4\varepsilon+3j\varepsilon & 9-2j+4j\varepsilon \end{pmatrix},$$

$$E_{\widehat{B}} = \begin{pmatrix} \widehat{B}_{n+3} & \widehat{B}_{n+2} & \widehat{B}_{n+1} & \widehat{B}_n \\ \widehat{B}_{n+2} & \widehat{B}_{n+1} & \widehat{B}_n & \widehat{B}_{n-1} \\ \widehat{B}_{n+1} & \widehat{B}_n & \widehat{B}_{n-1} & \widehat{B}_{n-2} \\ \widehat{B}_n & \widehat{B}_{n-1} & \widehat{B}_{n-2} & \widehat{B}_{n-3} \end{pmatrix}.$$

From Theorem [18], we can write the following corollary.

Corollary 19. *The following identities are hold:*

- **a):** $A^n N_{\widehat{W}A} = E_{\widehat{W}A}$.
- **b):** $A^n N_{\widehat{W}B} = E_{\widehat{W}B}$.

References

[1] Bród D, Liana A, Wloch Two generalizations of dual-hyperbolic balancing numbers. Symmetry. 2020;12(11):1866.

[2] Cihan A, Azak AZ, Güngör MA, Tosun M.A study on dual hyperbolic Fibonacci and Lucas numbers. An Şt.Univ.Ovidius Constanta.2019;27(1):35-48.

[3] Dikmen, C. M. A study on dual hyperbolic generalised leonardo numbers. Asian Research Journal of mathematics, 21(6), 143-165, (2025).

[4] Demirci F, Soykan Y, Gaussian generalized Adrien numbers. Archives of Current Research International 25(7),469-491,2025.

[5] D.K. Biss, D. Dugger, D.C. Isaksen, Large annihilators in Cayley-Dickson algebras, Communication in Algebra, 36 (2), 632-664, 2008.

[6] Eren ,O. On dual hyperbolic generalized Woodall numbers, Archives of Current Research International, 24(11), 398-423 ,(2024). Science & Computational.

[7] G. Moreno, The zero divisors of the Cayley-Dickson algebras over the real numbers, Bol. Soc. Mat. Mexicana 3(4) (1998), 13-28.

[8] G. Sobczyk, The Hyperbolic Number Plane, The College Mathematics Journal, 26(4) (1995), 268-280.

[9] H. H. Cheng, S. Thompson, D.

[10] I. Kantor, A. Solodovnikov, Hypercomplex Numbers, Springer-Verlag, New York (1989).ual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanisms, Proc. of ASME 24th Biennial Mechanisms Conference, Irvine, CA, August, 1996, 19-22.

[11] J. Baez, The octonions, Bull. Amer. Math. Soc. 39(2) (2002), 145-205.

[12] J. Cockle, On a New Imaginary in Algebra, Philosophical magazine, London-Dublin-Edinburgh, 3(34) (1849), 37-47.

[13] K. Imaeda, M. Imaeda, Sedenions: algebra and analysis, Applied Mathematics and Computation, 115 (2000), 77-88.

[14] M. Akar, S. Yüce, Ş. Şahin, On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers, Journal of Computer Mathematics, 8(1) (2018), 1-6.

[15] P. Fjelstad, S.G. Gal, n-dimensional Hyperbolic Complex Numbers, Advances in Applied Clifford Algebras, 8(1) (1998), 47-68.

[16] Soykan, Y., Generalized Adrien Numbers, Applied Mathematic and Computer Science,7(1),37-51.

[17] Soykan Y, Gümüş M, Göcen M. A study on dual hyperbolic generalized Pell numbers. Malaya Journal of Matematik. 2021;09(03):99-116.

[18] Soykan Y, Taşdemir E, Okumuş İ. On dual hyperbolic numbers with generalized Jacobsthal numbers Components. Indian J Pure Appl Math.2023;27(1):35-48.

[19] Soykan, Y., Sums and Generating Functions of Special Cases of Generalized Tetranacci Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 12(4), 34-10, 2025.

[20] Soykan, Y., Properties of Generalized (r,s,t,u)-Numbers, Earthline Journal of Mathematical Sciences, 5(2), 297-327, 2021. https://doi.org/10.34198/ejms.5221.297327.

[21] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56, 2019.

[22] W.R. Hamilton, Elements of Quaternions, Chelsea Publishing Company, New York (1969).

[23] Yılmaz B, Soykan Y. On Dual Hyperbolic Guglielmo Numbers Journal of Advances in Mathematics and Computer Science. 2024;39(4):37-61.

[24] Y. Soykan, Tribonacci and Tribonacci-Lucas Sedenions. Mathematics 7(1) (2019), 74.