Original Research Article

Edge Induced V_4-Magic Labeling of Subdivision Graphs

Abstract

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0 and $G = (V(G), E(G))$ be a graph. Let $f : E(G) \to V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \to V_4$ denote the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $v \in V(G)$. Then f^+ again induces an edge labeling $f^{++} : E(G) \to V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$, for all $uv \in E(G)$. Then a graph $G = (V(G), E(G))$ is said to be an edge induced V_4-magic graph if f^{++} is a constant function. The function f, so obtained is called an Edge Induced V_4-Magic Labeling (EIML) of G. This Paper deals with Edge Induced V_4-Magic Labeling of subdivision graphs.

Keywords: Klein 4-group, Induced V_4-magic graphs, Subdivision graphs.

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1 Introduction

The present paper intends to deal exclusively with simple, connected, finite and undirected graphs. Also note that, the Klein 4-group is denoted by $V_4 = \{0, a, b, c\}$ which is a non cyclic abelian group of order 4 with every non identity element has order 2. Let $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$. The reader may check [7] for the standard terminology and notations related to Graph theory. Let $f : E(G) \to V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \to V_4$ denote the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $v \in V(G)$. Then f^+ again induces an edge labeling, say, $f^{++} : E(G) \to V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$, for all $uv \in E(G)$. Then a graph $G = (V(G), E(G))$ is said to be an edge induced V_4-magic graph if $f^{++}(e)$ is a constant for all $e \in E(G)$. If this constant is x, then x is said to be the induced edge sum of the graph G and the function f, so obtained is called an edge induced V_4-magic labeling of G. This paper aims to discuss edge induced V_4-magic labeling of some subdivision graphs which belongs to the following categories:

(i) $\sigma_a(V_4)$:= Set of all edge induced V_4-magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = a$ for all $u \in V$.

(ii) $\sigma_0(V_4)$:= Set of all edge induced V_4-magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = 0$ for all $u \in V$.

(iii) $\sigma(V_4) := \sigma_a(V_4) \bigcap \sigma_0(V_4)$.

1.1 PRELIMINARIES

The bistar $B_{m,n}$ [4] is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge. A flag graph is denoted by Fl_n [3] and it is obtained by joining one vertex of C_n to an extra vertex called the root. The sun graph [3] on $m = 2n$ vertices, denoted by Sun_n, is the graph obtained by attaching a pendant vertex to each vertex of a $n-$cycle. Jelly fish graph $J(m, n)$ [3] is obtained from a $4-$cycle $v_1v_2v_3v_4v_1$ by joining v_1 and v_3 with an edge and appending the central vertex of $K_{1,m}$ to v_2 and appending the central vertex of $K_{1,n}$ to v_4. A triangular snake graph TS_n [3] is obtained from a path $v_1, v_2, v_3, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \cdots, n-1$. The join [7] of the graphs C_n and K_1 is called a wheel graph [7] and it is denoted by W_n, that is $W_n = C_n \vee K_1$. The corona [1] of P_n and K_1 is called the comb graph CB_n [1], that is $CB_n = P_n \circ K_1$

Theorem 1.1. [5]. Let $G = (V, E)$ be a graph with either each vertex is of odd degree or even degree then $G \in \sigma_0(V_4)$.

Theorem 1.2. [5] **(Induced edge sum theorem)**
For any graph G, f is an edge induced V_4-Magic labeling of G if and only if the induced edge sum

$$x = f^{++}(uv) = \sum_{u\alpha\in E,\ \alpha\neq v} f(u\alpha) + \sum_{\beta v\in E,\ \beta\neq u} f(\beta v),\ \text{for all } (u, v) \in E \tag{1.1}$$

The Equation (1.1) corresponding to an edge uv in G, is called induced edge sum equation of the edge uv.

Theorem 1.3. [5] For the path graph P_n, we have the following:

(i) $P_2 \in \sigma_0(V_4)$ and $P_2 \notin \sigma_a(V_4)$.

(ii) $P_3 \in \sigma_a(V_4)$ and $P_3 \notin \sigma_0(V_4)$.

(iii) $P_4 \in \sigma_a(V_4)$ and $P_4 \notin \sigma_0(V_4)$.

(iv) P_n is not an edge induced magic graph for any $n \geq 5$.

Theorem 1.4. [5] For the cycle graph C_n, we have the following:

(i) $C_n \in \sigma_0(V_4)$ for all n.

(ii) $C_n \in \sigma_a(V_4)$ if and only if n is a multiple of 4.

Definition 1.1. The subdivision of an edge $e = uv$ in the graph G gives a new graph obtained by replacing the edge $e = uv$ by two edges $e_1 = uw$ and $e_2 = wv$. A subdivision of a graph G or simply a subdivision graph is a graph which is denoted by $S(G)$ and is obtained from the subdivision of all edges in G.

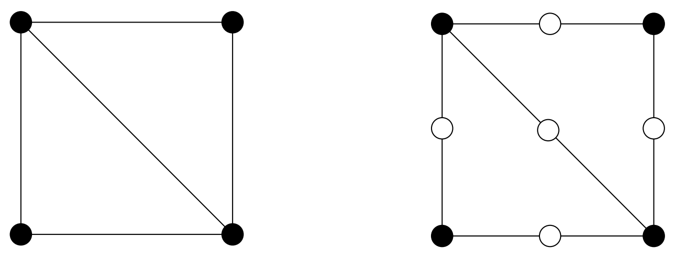

Figure 1: A Graph and its subdivision graph

2 Main Resuts

Theorem 2.1. *Let G be graph with every vertex is of odd degree, then $S(G) \in \sigma_a(V_4)$.*

Proof. Suppose G is a graph with every vertex is of odd degree. Then define $f : E(S(G)) \to V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(G))$.
Let $uv \in E(G)$ and α be the inserted vertex on the edge uv in $S(G)$. Then $f(u\alpha) = f(v\alpha) = a$. Therefore, $f^+(u) = f^+(v) = deg\ (u)a = a$, since $deg\ (u)$ is odd and $f^+(\alpha) = deg\ (\alpha)a = 0$, since $deg\ (\alpha) = 2$. Thus $f^{++}(u\alpha) = a$ and $f^{++}(v\alpha) = a$. Since uv is an arbitrary edge in $S(G)$, we can conclude that $f^{++}(e) = a$ for all $e \in S(G)$. Thus $S(G) \in \sigma_a(V_4)$.
Hence the proof. □

Theorem 2.2. *Let G be graph with every vertex is of even degree, then $S(G) \in \sigma_0(V_4)$.*

Proof. Suppose G is a graph with every vertex is of even degree. Then define $f : E(S(G)) \to V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(G))$.

Let $uv \in E(G)$ and α be the inserted vertex on the edge uv in $S(G)$. Then $f(u\alpha) = f(v\alpha) = a$. Therefore, $f^+(u) = f^+(v) = deg\ (u)a = 0$, since $deg\ (u)$ is even and $f^+(\alpha) = deg\ (\alpha)a = 0$, since $deg\ (\alpha) = 2$. Thus $f^{++}(u\alpha) = 0$ and $f^{++}(v\alpha) = 0$. Since uv is an arbitrary edge in $S(G)$, we can conclude that $f^{++}(e) = 0$ for all $e \in S(G)$. Thus $S(G) \in \sigma_0(V_4)$.
Hence the proof. □

Theorem 2.3. *For the path graph $P_n, n \geq 2$ we have the following:*

(i) *$S(P_2) \in \sigma_a(V_4)$ and $S(P_2) \notin \sigma_0(V_4)$.*

(ii) *$S(P_n) \notin \sigma_a(V_4)$ and $S(P_n) \notin \sigma_0(V_4)$ for any $n \geq 3$.*

Proof. Since $S(P_2) = P_3$ proof of (i) follows directly from Theorem 1.3 (ii).
Also we have, $S(P_n) = P_{2n-1}$ and if $n \geq 3$ then $2n - 1 \geq 5$, therefore, the proof of (ii) follows directly from Theorem 1.3 (iv). □

Theorem 2.4. *For the cycle graph C_n, we have the following:*

(a) *$S(C_n) \in \sigma_0(V_4)$ for all n.*

(b) *$S(C_n) \in \sigma_a(V_4)$ if and only if n is even.*

(c) *$S(C_n) \in \sigma(V_4)$ if and only if n is even.*

Proof. Since $S(C_n) = C_{2n}$, proof of (a) follows from Theorem 1.4 (i). Similarly, we have, $S(C_n) = C_{2n}$, therefore, proof of (b) follows directly from Theorem 1.4 (ii). Note that, the proof of (c) follows from (a) and (b) . □

Theorem 2.5. *For the the complete graph K_n with n vertices, we have the following:*

(i) $S(K_n) \in \sigma_0(V_4)$ *for n odd.*

(ii) $S(K_n) \in \sigma_a(V_4)$ *for n even.*

Proof. Consider the subdivision graph of the complete graph $S(K_n)$. Let vu be an edge in $S(K_n)$, where $v \in V(K_n)$ and u be an inserted vertex in $S(K_n)$. Define $f : E(S(K_n)) \to V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(K_n))$. Then $f^+(v) = (n-1)a$ and $f^+(u) = a+a = 0$. Therefore, $f^{++}(vu) = (n-1)a$. Since the vertices u and v are arbitrary, we have $f^{++}(vu)$ is a constant.

Case (i) n is an odd integer.
In this case, $f^{++}(vu) = (n-1)a = 0$. Therefore, $S(K_n) \in \sigma_0(V_4)$.

Case (ii) n is an even integer.
In this case, $f^{++}(vu) = (n-1)a = a$. Therefore, $S(K_n) \in \sigma_a(V_4)$.

Hence the proof. □

Theorem 2.6. *For the star graph $K_{1,n}$, we have the following:*

(i) $S(K_{1,n}) \in \sigma_a(V_4)$ *if and only if n is odd.*

(ii) $S(K_{1,n}) \notin \sigma_0(V_4)$ *for any n.*

Proof. Consider $K_{1,n}$ with vertex set $\{v, v_1, v_2, v_3, \ldots, v_n\}$, where $vv_i \in E(K_{1,n})$ for $i = 1, 2, 3, \ldots, n$. Let u_i be the inserted vertices on the edge vv_i for $i = 1, 2, 3, \ldots, n$ in $S(K_{1,n})$.

Let $f : E(S(K_{1,n})) \to V_4 \setminus \{0\}$ with $f(vu_i) = x_{i1}$, and $f(u_iv_i) = x_{i2}$ for $i = 1, 2, 3, \ldots, n$. Then from the induced edge sum equation of each edge we have the following equation.

$$\begin{aligned} x_{11} = x_{21} = x_{31} = \cdots = x_{n1} &= x_{21} + x_{31} + x_{41} + \cdots + x_{n1} + x_{12} \\ &= x_{11} + x_{31} + x_{41} + \cdots + x_{n1} + x_{22} \\ &= x_{11} + x_{21} + x_{31} + \cdots + x_{n1} + x_{32} \\ &\vdots \\ &= x_{11} + x_{31} + x_{41} + \cdots + x_{(n-1)1} + x_{n2}. \end{aligned}$$

Let $x = x_{11} = x_{21} = x_{31} = \cdots = x_{n1}$ then above equations become

$$\begin{aligned} x &= (n-1)x + x_{12} \\ &= (n-1)x + x_{22} \\ &= (n-1)x + x_{32} \\ &\vdots \\ &= (n-1)x + x_{n2}. \end{aligned}$$

Note that the above system implies that $x_{12} = x_{22} = x_{32} = \cdots = x_{n2} = y$(say).
Then the above system of equations reduces to $x = (n-1)x + y$.

$$\text{That is } (n-2)x + y = 0.$$

Case (i) n is an odd integer.
In this case, the equation $(n-2)x + y = 0$ reduces to $x + y = 0$, that is $x = y$. Thus by taking $x = y = a$ that is, by defining $f : E(S(K_{1,n})) \to V_4 \setminus \{0\}$ as $f(e_i) = a$, for all $e_i \in E(S(K_{1,n}))$ we can prove that $S(K_{1,n}) \in \sigma_a(V_4)$.

Case (ii) n is an even integer.
In this case, the equation $(n-2)x+y=0$ reduces to $y=0$. That is $f(u_iv_i)=x_{i2}=0$, which is a contradiction to the choice for f. Therefore, in this case, $S(K_{1,n})$ is not an edge induced magic graph.

Note that $S(K_{1,n}) \in \sigma_0(V_4)$ only when $x=0$. But $x=0$ is not possible. Therefore, $S(K_{1,n}) \notin \sigma_0(V_4)$ for any n.
Hence the proof. □

Theorem 2.7. *For the bistar graph $B_{m,n}$, $S(B_{m,n})$ is not an edge induced magic graph for any m and n.*

Proof. Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \ldots, v_m, u_1, u_2, u_3, \ldots, u_n\}$, where $uv, vv_i, uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. Also let w_i, t_j and w be the inserted vertices on the edge vv_i, uu_j and uv respectively for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$ in the graph $S(B_{m,n})$.
Let $f : E(S(B_{m,n})) \to V_4 \setminus \{0\}$ with $f(vw) = \gamma$, $f(wu) = \delta$, $f(vw_i) = x_i$, $f(w_iv_i) = \alpha_i$, $f(ut_j) = y_j$ and $f(t_ju_j) = \beta_j$, then by considering the induced edge sum equation of each edge we have the following equations.
The induced edge sum equation of the edges w_iv_i gives: $x_1 = x_2 = x_3 = \cdots = x_m = x$ (say).
Similarly the induced edge sum equation of the edges t_ju_j gives: $y_1 = y_2 = y_3 = \cdots = y_n = y$ (say).
The induced edge sum equation of the edges vw_i gives:

$$\begin{aligned} \alpha_1 + \gamma + (m-1)x &= \alpha_2 + \gamma + (m-1)x \\ &= \alpha_3 + \gamma + (m-1)x \\ &\vdots \\ &= \alpha_m + \gamma + (m-1)x. \end{aligned}$$

It should be noted that the above system of equations imply that $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_m = \alpha$ (say). Thus, each induced edge sum in above system reduces to $\alpha + \gamma + (m-1)x$.
Similarly by considering the induced edge sum equation of the edges ut_j, we get the induced induced edge sum is $\beta + \delta + (n-1)y$, where $\beta = \beta_1 = \beta_2 = \beta_3 = \cdots = \beta_n$.
Also we get, the induced edge sum of the edges vw is $mx + \delta$ and the edge sum of the edge wu is $ny + \gamma$.
Thus the edge sum equation of the graph $S(B_{m,n})$ is given by:

$$x = y = \alpha + \gamma + (m-1)x = \beta + \delta + (n-1)y = mx + \delta = ny + \gamma. \tag{2.1}$$

Case 1: m and n are even integers.
In this case, Equation (2.1) becomes

$$x = y = \alpha + \gamma + x = \beta + \delta + y = \delta = \gamma.$$

Therefore, $x = \gamma$, which implies that $\alpha = 0$, which is not possible. Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Case 2: m and n are odd integers.
In this case, Equation (2.1) becomes

$$x = y = \alpha + \gamma = \beta + \delta = x + \delta = y + \gamma.$$

Therefore, $x = x + \delta$ which implies that $\delta = 0$, which is not possible. Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Case 3: m is even and n is odd.

In this case, Equation (2.1) becomes

$$x = y = \alpha + \gamma + x = \beta + \delta = \delta = y + \gamma.$$

Therefore, $\beta + \delta = \delta$ which implies that $\beta = 0$, which is not possible. Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Case 4: m is odd and n is even.

In this case, Equation (2.1) becomes

$$x = y = \alpha + \gamma = \beta + \delta + y = x + \delta = \gamma.$$

Therefore, $\alpha + \gamma = \gamma$ which implies that $\alpha = 0$, which is not possible. Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Thus in all cases, we get $S(B_{m,n})$ is not an edge induced magic graph.
Hence the proof. □

Theorem 2.8. *For the flag graph Fl_n, we have the following.*

Case (i) $S(Fl_n) \notin \sigma_0(V_4)$ *for any n.*

Case (ii) $S(Fl_n) \in \sigma_a(V_4)$ *if and only if n is odd.*

Proof. Let $V(Fl_n) = \{v, v_1, v_2, v_3, \ldots, v_n\}$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of corresponding cycle graph C_n and v is the root vertex adjacent to the vertex v_1. Also let u be the inserted vertex on the edge $v_1 v$ and $u_1, u_2, u_3, \ldots, u_n$ be the inserted vertices on the edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_nv_1$ respectively in the graph $S(Fl_n)$.

If possible, let $g : E(S(Fl_n)) \to V_4 \setminus \{0\}$ be an edge label with $g^{++}(e) = 0$ for all edge in $S(Fl_n)$. Then consider the induced edge sum of the edge uv. Note that $g^{++}(uv) = g(uv_1)$. Therefore, $g(uv_1) = 0$, which is a contradiction and it proves (i).

Suppose n is an odd integer. In this case, define $f : E(S(Fl_n)) \to V_4 \setminus \{0\}$ as follows.

$$f(e) = \begin{cases} a & \text{if} \quad e = uv, uv_1 \\ b & \text{if} \quad e = u_1v_1, u_3v_3, u_5v_5, \ldots, u_{n-2}v_{n-2}, u_nv_n \\ c & \text{if} \quad e = u_2v_2, u_4v_4, u_6v_6, \ldots, u_{n-3}v_{n-3}, u_{n-1}v_{n-1} \\ b & \text{if} \quad e = u_1v_2, u_3v_4, u_5v_6, \ldots, u_{n-2}v_{n-1}, u_nv_1 \\ c & \text{if} \quad e = u_2v_3, u_4v_5, u_6v_7, \ldots, u_{n-3}v_{n-2}, u_{n-1}v_n. \end{cases}$$

Then $f^{++}(e) = a$ for all $e \in E(S(Fl_n))$. Thus $S(Fl_n) \in \sigma_a(V_4)$.

To prove the converse part, suppose n is an even integer. If possible, let $h : E(S(Fl_n)) \to V_4 \setminus \{0\}$ be an edge label with $h^{++}(e) = a$ for all edge in $S(Fl_n)$. Consider the induced edge sum of the edge uv. We have $h^{++}(uv) = h(uv_1)$. Similarly if we let $h(u_iv_{i+1}) = y_i$, for $i = 1, 2, 3, \ldots, n$ with $i + 1$ is taken modulo n. Then the induced edge sum of the edges v_iu_i for $i = 1, 2, 3, \ldots, n$ gives

$$y_n + y_1 + h(uv_1) = y_1 + y_2 = y_2 + y_3 = \cdots = y_{n-1} + y_n. \tag{2.2}$$

Since n is an even integer the above equation implies that $y_1 = y_3 = y_5 = \cdots = y_{n-1} = x$ (say) and $y_2 = y_4 = y_6 = \cdots = y_n = y$ (say). Thus the Equation (2.2) reduces to $x + y + h(uv_1) = x + y$, which implies that $h(uv_1) = 0$, which is not admissible. Hence there exists no such edge label h. Hence if n is an even integer, then $S(Fl_n) \notin \sigma_a(V_4)$.
Hence the proof. □

Corollary 2.9. $S(Fl_n) \in \sigma(V_4)$ *if and only if n is odd.*

Proof. Proof follows from the above Theorem 2.8. □

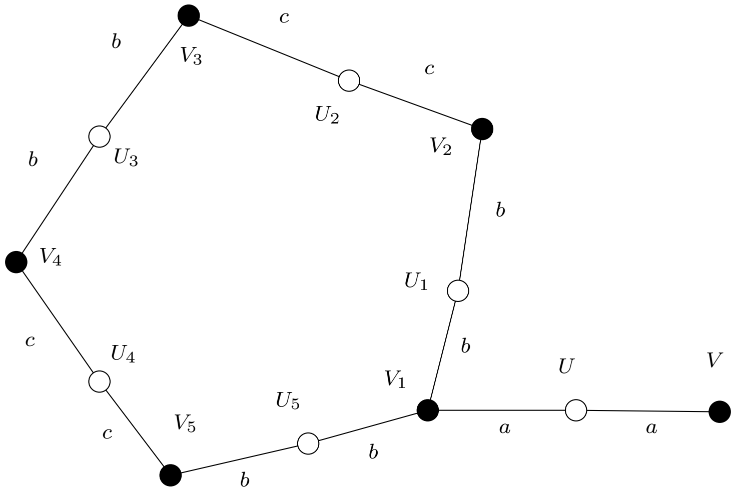

Figure 2: EIML of $S(Fl_5)$

Theorem 2.10. *For the sun graph Sun_n, we have $S(Sun_n) \in \sigma_a(V_4)$ for all n.*

Proof. Let $\{u_i, v_i : i = 1, 2, 3, \dots, n\}$ be the vertex set of CB_n, where v_i are the pendant vertex adjacent to u_i. Also let t_i and w_i, be the inserted vertices on the edge u_iu_{i+1}, u_iv_i, for $i = 1, 2, 3, \dots, n$ and $i + 1$ is taken modulo n.
Suppose $f : E(S(Sun_n)) \to V_4 \setminus \{0\}$ is an edge induced magic label of Sun_n with $f(u_it_i) = e_i$, $f(t_iu_{i+1}) = \alpha_i$, $f(u_iw_i) = \beta_i$ and $f(w_iv_i) = \gamma_i$.
Then using the induced edge sum equation of the edges w_iv_i, we get

$$\beta_1 = \beta_2 = \beta_3 = \cdots = \beta_n = \beta \text{ (say)}. \tag{2.3}$$

By the induced edge sum equation of the edges u_it_i, we get

$$\alpha_n + \alpha_1 + \beta = \alpha_1 + \alpha_2 + \beta = \alpha_2 + \alpha_3 + \beta = \cdots = \alpha_{n-1} + \alpha_n + \beta. \tag{2.4}$$

By the induced edge sum equation of the edges t_iu_{i+1}, we get

$$e_1 + e_2 + \beta = e_2 + e_3 + \beta = e_3 + e_4 + \beta = \cdots = e_n + e_1 + \beta. \tag{2.5}$$

By the induced edge sum equation of the edges u_iw_i, we get

$$\alpha_n + e_1 + \gamma_1 = \alpha_1 + e_2 + \gamma_2 = \alpha_2 + e_3 + \gamma_3 = \cdots = \alpha_{n-1} + e_n + \gamma_n. \tag{2.6}$$

Case (i) n is an odd integer.

In this case, Equation (2.4) and Equation(2.5) implies that

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = \alpha \ (say).$$
$$e_1 = e_2 = e_3 = \cdots = e_n = e \ (say).$$

Therefore, equation (2.6) implies that

$$\gamma_1 = \gamma_2 = \gamma_3 = \cdots = \gamma_n = \gamma \ (say).$$

Therefore, in this case, the induced edge sum equation of the graph $S(Sun_n)$ is given by:

$$\beta = 2\alpha + \beta = 2e + \beta = \alpha + e + \gamma.$$

Since $\alpha \in V_4$, the above equation reduces to $\beta = \alpha + e + \gamma$.
Therefore, in this case, if we choose $\alpha = e = b$ and $\beta = \gamma = a$, then we can easily prove that $S(Sun_n) \in \sigma_a(V_4)$.

Case (ii) n is an even integer.
In this case, Equation (2.4) implies

$$\alpha_1 = \alpha_3 = \alpha_5 = \cdots = \alpha_{n-1} = x_1 \ (say).$$
$$\alpha_2 = \alpha_4 = \alpha_6 = \cdots = \alpha_n = x_2 \ (say).$$

Also in this case Equation (2.5) implies

$$e_1 = e_3 = e_5 = \cdots = e_{n-1} = y_1 \ (say).$$
$$e_2 = e_4 = e_6 = \cdots = e_n = y_2 \ (say).$$

Therefore, Equation (2.6) reduces to

$$x_2 + y_1 + \gamma_1 = x_1 + y_2 + \gamma_2 = x_2 + y_1 + \gamma_3 = x_1 + y_2 + \gamma_4 = \cdots = x_1 + y_2 + \gamma_n. \quad (2.7)$$

Note that Equation (2.7) implies that

$$\gamma_1 = \gamma_3 = \gamma_5 = \cdots = \gamma_{n-1} = z_1 \ (say).$$
$$\gamma_2 = \gamma_4 = \gamma_6 = \cdots = \gamma_n = z_2 \ (say).$$

Therefore, in this case, the induced edge sum equation of the graph $S(Sun_n)$ is given by:

$$\beta = x_1 + x_2 + \beta = y_1 + y_2 + \beta = x_2 + y_1 + z_1 = x_1 + y_2 + z_2.$$

Therefore, in this case, if we choose $x_1 = x_2 = y_1 = y_2 = b$ and $\beta = z_1 = z_2 = a$ then we can easily prove that $f^{++}(e) = a$ for all $e \in E(S(Sun_n))$. Thus $S(Sun_n) \in \sigma_a(V_4)$.

Hence the proof. □

Theorem 2.11. *Let $J(m,n)$ be the jelly fish graph. Then $S(J(m,n)) \in \sigma_a(V_4)$ for m and n are of same parity.*

Proof. Consider the jelly fish graph with $V(J(m,n)) = \{v_k : k = 1,2,3,4\} \cup \{u_i : i = 1,2,3,\dots,m\} \cup \{w_j : j = 1,2,3,\dots,n\}$, where v_k's are the vertices of the corresponding C_4, u_i, w_j are the vertices of the corresponding $K_{1,m}$ and $K_{1,n}$ respectively and α_i $(1 \le i \le m), \beta_j$ $(1 \le j \le n)$ are the inserted vertices at the edges v_2u_i, v_4w_j respectively and $\alpha, \beta, \gamma, \delta, \mu$ are the vertices inserted on the edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3$ respectively.

Case (i) m and n are even integers.
In this case, define $f : E(S(J(m,n))) \to V_4 \setminus \{0\}$ by

$$f(e) = \begin{cases} a & \text{if} \quad e = \ \alpha_i u_i,\ v_2\alpha_i,\ v_4\beta_j,\ \beta_j w_j,\ v_1\mu,\ v_3\mu \\ b & \text{if} \quad e = \ v_2\alpha,\ \alpha v_1,\ v_1\delta,\ \delta v_4 \\ c & \text{if} \quad e = \ v_4\gamma,\ \gamma v_3,\ v_3\beta,\ \beta v_2. \end{cases}$$

Case (ii) m and n are n odd integers.
In this case, define $f : E(S(J(m,n))) \to V_4 \setminus \{0\}$ by

$$f(e) = \begin{cases} a & \text{if} \quad e = \ v_2\alpha,\ \alpha v_1,\ v_3\beta,\ \beta v_2,\ v_2\alpha_i,\ v_4\beta_j,\ \alpha_i u_i,\ \beta_j w_j \\ b & \text{if} \quad e = \ v_1\delta,\ \delta v_4,\ v_4\gamma,\ \gamma v_3,,\ v_1\mu,\ v_3\mu. \end{cases}$$

Then in both cases we can verify that $f^{++}(e) = a$ for all $e \in E(S(J(m,n)))$. That is f is an EIML of $S(J(m,n))$. Thus in both cases $S(J(m,n)) \in \sigma_a(V_4)$.
Hence the proof. □

Theorem 2.12. *For the the triangular snake graph TS_n, we have $S(TS_n) \in \sigma_0(V_4)$ for all n.*

Proof. Since every vertex is of even degree, the proof follows from Theorem 2.2. □

Theorem 2.13. *For the wheel graph W_n, we have $S(W_n) \in \sigma_a(V_4)$ for n is odd.*

Proof. Suppose n is odd. Since every vertex is of odd degree, the proof follows from Theorem 2.1. □

Theorem 2.14. *For the comb graph CB_n, we have $S(CB_n)$ is not an edge induced magic graph, for any n.*

Proof. Let $\{u_i, v_i : 1, 2, 3, \ldots, n\}$ be the vertex set of CB_n, where v_i is the pendant vertex adjacent to u_i. Let w_i and t_j be the inserted vertices in the edges u_iv_i and u_ju_{j+1} for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, n-1$ respectively . If possible, suppose $f : E(S(CB_n)) \to V_4 \setminus \{0\}$ is an edge induced magic label of $S(CB_n)$. Then using the induced edge sum equation of the edges v_1w_1 and u_1t_1, we get $f(u_1w_1) = f(t_1u_2) + f(u_1w_1)$. That is $f(t_1u_2) = 0$, which is a contradiction. Hence $S(CB_n)$ is not an edge induced magic graph, for any n.
Hence the proof. □

3 CONCLUSIONS

This paper has attempted to investigate the key results pertaining to edge-induced V_4- magic labeling of graphs with same parity. Subsequently it establishes edge-induced V_4-magic labeling characteristics for a selection of graphs, such as $P_n, C_n, K_n, K_{1,n}$ as well as the Bi-star graph, Flag graph, Sun graph, Jelly graph, Triangular Snake graph, Wheel graph and Comb graph.

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