A Study on $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$: Eulerian Conditions, Kernel, and Domination

Abstract

In this research paper, we undertake a detailed study of a subdigraph of $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$ for $n \geq 2$, formed by removing the vertex 0. This resulting subdigraph is denoted by $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$. We establish the necessary and sufficient conditions under which $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is Eulerian. Furthermore, we investigate the kernel of this digraph and analyze domination and twin domination properties within the framework of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$

Keywords: Directed Power graph $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$, $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$, Eulerian Graph, Kernel, Domination, Twin dominating number.

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1 Introduction

In today's context, digraphs corresponding to algebraic structures such as groups and rings are highly significant. The directed power graph $\overrightarrow{\mathcal{G}}(G)$ of a group G, introduced by Kelarev et al. in (1), is a digraph with vertex set G and for any $a, b \in G$, there is a directed edge from a to b in $\overrightarrow{\mathcal{G}}(G)$ if and only if $a^k = b$, where $k \in \mathbb{N}$. In (6) Manuel et al. defined the concept of a directed power graph associated with the finite cyclic group $\mathbb{Z}_n$ as a simple digraph with a vertex set $\mathbb{Z}_n$ and two distinct vertices in $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$ are joined by a directed edge or an arc $\overrightarrow{uv}$ from u to v if and only if there exists a non-negative integer r such that $v \equiv ru (\bmod\, n)$. They delve into the $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$, the directed power graph of the cyclic group $\mathbb{Z}_n$ using the help of congruence and the definition of cyclic subgroups. Through this exploration, we unveil several characteristics of $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$, leveraging the notions of kernel, domination, and connectedness. In this paper, we examine a vertex-deleted sub-digraph of $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$ formed by its non-zero elements. I

Definition 1.1. $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is a subdigraph of $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$ obtained by removing the vertex 0. That is, $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is obtained from $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$ by removing the vertex 0 and the $n-1$ arcs incident with 0.

Figure 1 gives $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ for $n = 3$ and $n = 4$. $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_2)$ is a digraph with only one vertex and no arcs.

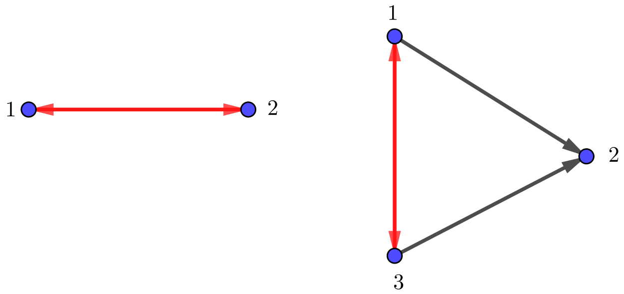

Figure 1: $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_3)$ and $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_4)$.

Theorem 1.1. *(6) Let $n \in \mathbb{Z}$ and let $1 < m_1 < m_2 < \cdots < m_r = n$ be the divisors of n. Then the number of arcs a in $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$ is*

$$a = \sum_{i=1}^{r} \left[2\binom{\phi(m_i)}{2} + \phi(m_i)[m_i - \phi(m_i)] \right]. \tag{1.1}$$

From Theorem 1.1, the number of arcs in $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is

$$\sum_{i=1}^{r} \left[2\binom{\phi(m_i)}{2} + \phi(m_i)(m_i - \phi(m_i)) \right] - (n-1).$$

In particular, for a prime number p, the number of arcs in $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is $(p-1)(p-2)$. $\overrightarrow{\mathcal{G}}(\mathbb{Z}_n)$ can not be Eulerian, since $od(0) = 0$. But $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is $Eulerian$ when n is a prime number. The following Theorem gives a characterization of Eulerian $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$.

Theorem 1.2. *$\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is Eulerian if and only if n is a prime number.*

Proof. Suppose that $n = p$ is a prime number. Then for each of its vertices v, $od(v) = id(v)$. Hence $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_p)$ is Eulerian. More precisely, let $\{1, 2, \cdots, p-1\}$ be the vertices of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_p)$. Also, $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_p)$ has $(p-1)(p-2)$ arcs. Then $1\,2\,3 \cdots (p-1)\,(p-2)\,(p-3) \cdots 2\,1\,3\,1\,4 \cdots 1\,(p-1)\,2\,4\,2\,5 \cdots 2\,(p-1)\,3\,5\,3\,6 \cdots 3\,(p-1) \cdots i\,(i+2)\,i\,(i+3) \cdots i\,(p-1) \cdots (p-3)\,(p-1)\,1$ is an Euler tour $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_p)$.

Conversely, suppose that n is not a prime number. Let a be a generator of $\mathbb{Z}_n$, then $id(a) = \phi(n) - 1$ and $od(a) = n - 1$, where $\phi(n)$ is the number of generators of $\mathbb{Z}_n$. Since n is not a prime number, $od(a) \neq id(a)$ and consequently $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is not Eulerian. □

Figure 2 gives an Euler tour in $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_5)$. Here $1\,2\,3\,4\,3\,2\,1\,3\,1\,4\,2\,4\,1$ is the Euler tour.

Now we give a characterization of symmetric $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$.

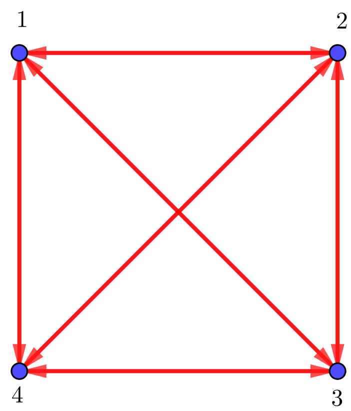

Figure 2: An Euler tour of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_5)$

Theorem 1.3. *For $n \geq 2$, $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is a symmetric digraph if and only if n is a prime number.*

Proof. Suppose $n = p$, a prime. Then there exist arcs between each distinct vertices $u, v \in V(\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n))$. Therefore $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is symmetric.

Conversely, suppose that $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ are symmetric digraphs. We claim that n, is prime. Since 1 is a generator of $\mathbb{Z}_n$, there exist arcs from 1 to u, for every $u \in V(\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n))$. Since $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is symmetric, there exists an arc from u to 1 also. This is possible only if u is a generator of $\mathbb{Z}_n$. Thus every non-zero element in $\mathbb{Z}_n$ is a generator and hence n is a prime. $\square$

Now Let us go through the following results relating kernel of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$.

Remark 1.1. If $n = p$, a prime number, the kernel of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_p)$ are the singleton subsets of non-zero elements of $\mathbb{Z}_p$.

Theorem 1.4. *No kernel of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ contains a generator of $\mathbb{Z}_n$, if $n \neq p$, a prime.*

Proof. Let K be a generator of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ and let a be a generator of $\mathbb{Z}_n$. Since K is an independent set $K = \{a\}$. Since n is not a prime, there exists $u \in V - K$, which is not a generator of $\mathbb{Z}_n$. Then there exists no arc from u to a which is a contradiction. Hence no kernel of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ contains a generator of $\mathbb{Z}_n$ if n is not a prime. $\square$

Let $C^0 = \{S_{a_1}, S_{a_2}, \cdots, S_{a_r}\}$ be the subset of C explained in notations **??**. By the above Theorem 1.4 we can prove the following characterizations on $kernel$ of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$.

Theorem 1.5. *For $n \neq p$, a prime, K be a kernel of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ if and only if K contains exactly one element from each member of C^0.*

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Proof. Suppose that K is a kernel of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$. Since $n \neq p$, by the above Theorem, K contains no generator. Since K is independent there exists at most one element from each member of C^0. If K contains no member from S_{a_i}, then there exists no arc from a_i to any element in K. Since K is a kernel, this is not possible.

Conversely, suppose that K contains exactly one element from each member of C^0. Then K is independent. Let $b \in V - K$, if b is a generator of $\mathbb{Z}_n$, then there exists an arc from b to elements of K. If b is not a generator, then $b \in S_b$. Let $b' \in S_b$ such that $b' \in K$, then there exists an arc bb'. Therefore K is a kernel. □

Theorem 1.6. *$S \subseteq V$ is a twin-dominating set of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ if and only if S contains at least one element from each strong component.*

Proof. Let $S_1, S_{a_1}, \cdots, \text{and } S_{a_r}$ are the strong components of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$. Let $S = \{b, b_1, b_2, \cdots, b_r\}$ be a subset of V containing exactly one element from each strong component of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$, where b is a generator of $\mathbb{Z}_n$. Now since b is a generator of $\mathbb{Z}_n$, there exist arcs $\overrightarrow{bv}$ for every $v \in V$, in particular for every $v \in V - S$. Therefore $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$ is out-dominated by b. Now it is enough to prove that S is an in-dominating set. For that let $v \in V - S$. If v is a generator, then v is adjacent to every vertex in S. If v is not a generator, then $v \in S_{a_i}$, for some $i = 1, 2, \cdots, r$. Since $b_i \in S_{a_i}$, there exists an arc from v to b_i. Thus S is a twin-dominating set.

Conversely, supposes S be a twin dominating set. We claim to prove that S contains at least one element from each strong component. If possible suppose that S contains no element from a strong component S_{a_i}, the generators of $< a_i >$. Let $v \in S_{a_i}$, then there is no arc from v to any vertex of S. Therefore S cannot be a twin-dominating set. So S must contain at least one element from each strong component. □

Remark 1.2. From the above Theorem, we can realize that the twin-dominating number, $\gamma^*(\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n))$ is the same as the number of strong components of $\overrightarrow{\mathcal{G}}_0(\mathbb{Z}_n)$.

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