Original Research Article

Global boundedness in a two-species predator-prey chemo-taxis model with indirect signal production

Abstract

This paper is devoted to investigate the global boundedness of the following predator-prey chemotaxis model with indirect signal production

$$\begin{cases} u_t = \Delta u + \chi_1 \nabla \cdot (u \nabla z) + \mu_1 u(1 - u - e_1 v), & x \in \Omega, t > 0, \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla z) + \mu_2 v(-1 + e_2 u - v), & x \in \Omega, t > 0, \\ w_t = \Delta w - w + u + v, & x \in \Omega, t > 0, \\ z_t = \Delta z - z + w, & x \in \Omega, t > 0 \end{cases}$$

under the homogeneous Neumann boundary conditions in a bounded smooth domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$. The parameter are positive constants. With some supplementary conditions imposed on the parameters $\chi_1, \chi_2, \mu_1, \mu_2, e_1, e_2$, it is proved that the model has a unique global bounded classical solution.

Keywords: Predator-prey; Chemotaxis; Indirect signal production; Uniform boundedness

2010 Mathematics Subject Classification: 35B45;35K55;92C17

1 Introduction

In this paper, we focus on studying the following predator-prey chemotaxis model with indirect signal production

$$\begin{cases} u_t = d_1 \Delta u + \chi_1 \nabla \cdot (u \nabla z) + \mu_1 u \left(1 - u - e_1 v\right), & x \in \Omega, t > 0, \\ v_t = d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla z) + \mu_2 v \left(-1 + e_2 u - v\right), & x \in \Omega, t > 0, \\ w_t = \Delta w - w + u + v, & x \in \Omega, t > 0, \\ z_t = \Delta z - z + w, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \\ w(x,0) = w_0(x), z(x,0) = z_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, the parameter $\chi_1, \chi_2, d_1, d_{2,}\mu_1, \mu_2,$ e_1, e_2 are positive constants, $u(x,t)$ and $v(x,t)$ denote the density of prey and predator, respectively,

$w(x,t)$ and $z(x,t)$ stand for the concentration of the chemical substance, $z(x,t)$ is secreted by $w(x,t), w(x,t)$ is secreted by $u(x,t)$ and $v(x,t)$. d_1 and d_2 are called random diffusion coefficients which represent the natural dispersive force of movement of the prey and predator, respectively. χ_1 and χ_2 are chemotactic sensitivities, moreover, the term $+\chi_1\nabla\cdot(u\nabla z)$ implies that prey move away from chemicals secreted by predators at higher concentrations (chemorepulsion), $-\chi_2\nabla\cdot(v\nabla z)$ describes movement of predators toward chemicals secreted by prey at higher concentrations (chemoattraction). μ_1 and μ_2 denote the growth rates of two species. e_1 and e_2 measure interaction between two species.

If $\chi_1 < 0$ and $e_2 < 0$, the model (1.1) refers to the two-competing-species chemotaxis model, which has been proposed and studied by Xiang et al. (2022). They established a unique bounded classical solution for $n \leq 2$. In addition, the asymptotic stability of the global bounded solution to the model is discussed when μ_i and a_i satisfy certain conditions. Specially, Zheng et al. (2022Pan) investigated further this model with $d_1 = d_2 = 1$ in a bounded domain $\Omega \subset \mathbb{R}^3$. They proved that the model has a unique global classical solution under some suitable conditions of parameters, moreover, the asymptotic stability of the solution is obtained. Recently, Wang et al. (2023) continued to investigate the model with singular sensitivity and they explored that the global existence and boundedness of classical solutions depends on the coefficients of the model and spatial dimension n.

In order to better study model (1.1), it is necessary to review relevant works in this direction. Next, we recall the following predator-prey chemotaxis model with direct signal production

$$\begin{cases} u_t = d_1\Delta u + \chi_1\nabla\cdot(u\nabla w) + \mu_1 u\left(1-u-e_1v\right), & x\in\Omega, t>0,\\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + \mu_2 v\left(1+e_2u-v\right), & x\in\Omega, t>0,\\ w_t = d_3\Delta w + \alpha u + \beta v - \gamma w, & x\in\Omega, t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), w(x,0)=w_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

Where $d_1, d_2, d_3, \chi, \xi, \mu_1, \mu_2, e_1, e_2, \alpha, \beta, \gamma$ are positive constants, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. The model was proposed and studied by Fu and Miao (2020). They proved that if the parameters satisfy some suitable conditions, the model has a unique global classical solution. Furthermore, the global asymptotic stability of the equilibria was explored by constructing Lyapunov functions. Subsequently, in the case when $n=3$ and $d_1=d_2=d_3=1$, the global existence and boundedness of classical solution to model (1.2) was established in (Miao et al., 2021). Recently, the model (1.2) with nonlinear production has been studied by Gnanasekaran et al. (2022), they obtained the global existence of classical solution in higher dimensions.

For convenience of statement, in this paper, we always assume that $d_1 = d_2 = 1$ and the initial data (u_0, v_0, w_0, z_0) satisfy

$$\begin{cases} u_0 \in C^0(\bar{\Omega}) & \text{with } u_0 \geq, \not\equiv 0 \text{ in } \Omega,\\ v_0 \in C^0(\bar{\Omega}) & \text{with } v_0 \geq, \not\equiv 0 \text{ in } \Omega,\\ w_0, z_0 \in W^{1,q}(\bar{\Omega}) & \text{for some } q>3, \text{ with } w_0, z_0 \geq 0 \text{ in } \Omega. \end{cases} \tag{1.3}$$

Our main result is the following theorem which describe the global bounded of classical solution to the model (1.1) in a three-dimensional bounded domain.

Theorem 1.1. *Suppose that the parameters $\chi_1, \chi_2, \mu_1, \mu_2, e_1, e_2$ be positive constants and satisfy*

$$\mu_2 e_2^2 \chi_1 \leq 4\mu_1 e_1 \chi_2, \tag{1.4}$$

$$\mu_1\mu_2 e_2 \geq \chi_1\left(4+\mu_2^2 e_2^2\right), \tag{1.5}$$

$$\mu_1\mu_2 e_1 \geq \chi_2\left(4+\mu_1^2 e_1^2\right). \tag{1.6}$$

Then the model (1.1) has a unique global classical solution (u, v, w, z) *with*

$$\begin{aligned}
&u \in C^0(\bar{\Omega} \times [0,\infty)) \times C^{2,1}(\bar{\Omega} \times (0,\infty)) \\
&v \in C^0(\bar{\Omega} \times [0,\infty)) \times C^{2,1}(\bar{\Omega} \times (0,\infty)) \\
&w \in C^0(\bar{\Omega} \times [0,\infty)) \times C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty_{loc}\left([0,\infty); W^{1,q}(\Omega)\right) \\
&z \in C^0(\bar{\Omega} \times [0,\infty)) \times C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty_{loc}\left([0,\infty); W^{1,q}(\Omega)\right)
\end{aligned} \tag{1.7}$$

for some $q > 3$*. Moreover, one can get*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t > 0 \tag{1.8}$$

with some constant $C > 0$ *that is independent of* t*.*

Remark 1.1. In the case that $n > 3$, we have to leave an open question about global existence, which shall be solved in the future work.

The rest of this paper is organized as follows. Some preliminaries are collected in Section 2. Section 3 is devoted to studying the global existence and uniform boundedness of the classical solution to the model (1.1) in a three-dimensional bounded domain.

2 Preliminaries and local existence

Before presenting the main results, we recall some known conclusions that will be utilized in the subsequent proofs. Firstly, we introduce the following lemma (refer to Lemma 3.4 in (Winkler, 2010)).

Lemma 2.1. *Let* $a > 0$ *and* $d > 0$*,and suppose that* y: $[0, T) \to [0, \infty)$ *is absolutely continuous. If there exists a nonnegative function* $h \in L^1_{loc}([0, T))$ *satisfying*

$$\int_t^{t+1} h(s)ds \le d \quad \text{for all } t \in [0, T-1)$$

and

$$y'(t) + ay(t) \le h(t),$$

we can get

$$y(t) \le \max\{y(0) + d, \frac{d}{a} + 2d\}.$$

Next, we present several properties of the Neumann heat semigroup. For the detailed proof, please refer to Lemma 1.3 in (Winkler, 2010).

Lemma 2.2. *Let* $\left(\mathrm{e}^{\mathrm{t}\Delta}\right)_{\mathrm{t}\ge 0}$ *be the Neumann heat semigroup in* Ω*, and* $\lambda_1 > 0$ *denotes the first nonzero eigenvalue of* $-\Delta$ *in* Ω*, under the Neumann boundary condition, then we obtain the following estimate with constants*

(I) If $1 \le q \le p \le \infty$*, then*

$$\left\|e^{t\Delta}\varphi\right\|_{L^p(\Omega)} \le k_1\left(1 + t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)e^{-\lambda_1 t}\|\varphi\|_{L^q(\Omega)} \quad \text{for all } t > 0$$

holds for all $\varphi \in L^q(\Omega)$ *with* $\int_\Omega \varphi = 0$*.*

(II) If $1 \le q \le p \le \infty$*, then*

$$\left\|\nabla e^{t\Delta}\varphi\right\|_{L^p(\Omega)} \le k_2\left(1 + t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)e^{-\lambda_1 t}\|\varphi\|_{L^q(\Omega)} \quad \text{for all } t > 0$$

are true for each $\varphi \in L^q(\Omega)$*.*

Then, we prepare the following generalization of the Gagliardo-Nirenberg inequality (see (Wu et al., 2018), Lemma 2.4).

Lemma 2.3. *Let $u \in L^p(\Omega)$ and $D^k u \in L^q(\Omega)$ where $p, q \in [1, \infty]$. Then for the derivatives $D^i u, i \in [0, k)$, there exists a constant $C > 0$ such that*

$$\left\| D^i u \right\|_h \leq C \left(\left\| D^k u \right\|_q^\lambda \|u\|_p^{1-\lambda} + \|u\|_m \right), \tag{2.1}$$

where

$$\frac{1}{h} - \frac{i}{n} = \lambda \left(\frac{1}{q} - \frac{k}{n} \right) + (1 - \lambda) \frac{1}{p}, \quad m > 0,$$

and λ satisfies

$$\frac{i}{k} \leq \lambda \leq 1.$$

Moreover, if $q \in (1, \infty)$ and $k - i - \frac{n}{q}$ is a non-negative integer, then the Gagliardo-Nirenberg inequality (2.1) holds for

$$\frac{i}{k} \leq \lambda < 1.$$

The following estimate plays an important role in removing the convexity of domains. The proof is indicated by Mizoguchi-Souplet (see (Mizoguchi et al., 2014), Lemma 4.2).

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary. If $\psi \in C^2(\bar{\Omega})$ satisfies $\partial \psi / \partial \nu = 0$, then*

$$\frac{\partial |\nabla \psi|^2}{\partial \nu} \leq c_\Omega |\nabla \psi|^2,$$

where $c_\Omega > 0$ is a constant depending only on the curvatures of $\partial \Omega$.

Now, we collect the trace theorem proved by Ishida et al. (2014), which is used to deal with the boundary integral term.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary. Then for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\int_{\partial \Omega} \varphi^2 \leq \varepsilon \int_\Omega |\nabla \varphi|^2 + C_\varepsilon \left(\int_\Omega |\varphi| \right)^2$$

for all $\varphi \in W^{1,2}(\Omega)$.

Finally, we state the local-in-time existence result of the classical solution of (1.1), which can be obtained through standard techniques that combining Banach's fixed point theorem and the parabolic regularity theory (refer to Lemma 2.1 in (Winkler, 2010) and Lemma 3.1 in (Horstmann et al., 2005)).

Lemma 2.6. *Assume that $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, and $\chi_1, \chi_2, d_1, d_2, \mu_1, \mu_2, e_1, e_2$ are positive constants, $q > \max\{2, n\}$ and exists $T_{\max} \in (0, \infty]$. Then for all (u_0, v_0, w_0, z_0) satisfying (1.3), the solution (u, v, w, z) of (1.1) is unique, which fulfills*

$$\begin{aligned}
&u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \\
&v \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \\
&w \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty_{loc}\left([0, T_{\max}); W^{1,q}(\Omega)\right) \\
&z \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty_{loc}\left([0, T_{\max}); W^{1,q}(\Omega)\right).
\end{aligned}$$

Furthermore, if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \left(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \right) = \infty.$$

Without loss of generality, we assume from this point forward that $T_{\max} > 1$, where $T_{\max}$ represents the maximal existence time derived from Lemma 2.6. We assume that C , C_i and k_i are positive constants, independent of both t and T throughout the subsequent analysis.

3 Global existence

We know that the uniform boundedness of $||u(\cdot,t)||_{L^2(\Omega)}$ and $||v(\cdot,t)||_{L^2(\Omega)}$ are sufficient to ensure the global boundedness of the classical solution to (1.1) in $\Omega\subset\mathbb{R}^3$. In this section, we shall establish a series of energy estimates on the functionals $\frac{3}{\chi_1}\int_\Omega u^2+\frac{3}{\chi_2}\int_\Omega v^2$ and $\int_\Omega(\mu_2e_2u+\mu_1e_1v)|\nabla z|^2$ as well as $\int_\Omega|\nabla z|^4$. The ideas used mainly come from (Pan et al., 2022).

To begin with, let us state the L^1-boundedness of u and v. The proof is similar to Lemma 2.2 in (Fu et al., 2020), we omit the details here.

Lemma 3.1. *Let $n\geq 1$. The solution of model (1.1) has the following properties*

$$\int_\Omega u(x,t)dx\leq k_1:=\max\left\{\|u_0\|_{L^1(\Omega)},|\Omega|\right\},\quad \textit{for all } t\in(0,T_{\max}),$$

$$\int_\Omega v(x,t)dx\leq k_2:=\frac{C_1}{\mu_1e_1},\quad \textit{for all } t\in(0,T_{\max}),$$

$$\int_t^{t+1}\int_\Omega u^2(x,s)dxds\leq k_3:=k_1+\frac{k_1}{\mu_1},\quad \textit{for all } t\in[0,T_{\max}-1),$$

$$\int_t^{t+1}\int_\Omega v^2(x,s)dxds\leq k_4:=\frac{1+\max\{\mu_1,\mu_2\}}{\mu_1\mu_2e_1}C_1,\quad \textit{for all } t\in[0,T_{\max}-1),$$

where $C_1:=\max\left\{\mu_2e_2\parallel u_0\parallel_{L^1(\Omega)}+\mu_1e_1\parallel v_0\parallel_{L^1(\Omega)},\frac{\mu_1\mu_2(e_1+e_2)|\Omega|}{\min\{\mu_1,\mu_2\}}\right\}$.

Now, we provide the following important lemmas that are crucial to validate L^2-boundedness of u and v.

Lemma 3.2. *Let $n\geq 1$. Then there exist $C_2>0$ and $C_3>0$ such that*

$$\int_\Omega|\nabla w(x,t)|^2\leq C_2\quad \textit{for all } t\in(0,T_{\max})\tag{3.1}$$

as well as

$$\int_\Omega w^2\leq C_3\quad \textit{for all } t\in(0,T_{\max}-1).\tag{3.2}$$

Proof. Multiplying the third equation in (1.1) by $-\Delta w$, integrating on Ω by parts and using Young's inequality, we derive

$$\frac12\frac{d}{dt}\int_\Omega|\nabla w|^2+\int_\Omega|\Delta w|^2+\int_\Omega|\nabla w|^2=-\int_\Omega u\Delta w-\int_\Omega v\Delta w\leq\int_\Omega|\Delta w|^2+\frac12\int_\Omega u^2+\frac12\int_\Omega v^2$$

for all $t\in(0,T_{\max})$. Define $y_1(t):=\int_\Omega|\nabla w(x,t)|^2dx$, then it satisfies $y'(t)+2y(t)\leq f(t)$ for all $t\in(0,T_{\max})$, where

$$f(t):=\int_\Omega u^2+\int_\Omega v^2.$$

By Lemma 3.1, we obtain

$$\int_t^{t+1}f(s)ds\leq C_4:=k_3+k_4\quad \text{for all } t\in(0,T_{\max}-1).$$

Accordingly, in view of Lemma 2.1, one can get

$$y_1(t)\leq C_5:=\max\left\{\int_\Omega|\nabla w_0|^2+C_4,\frac{C_4}{2}+2C_4\right\}\quad \text{for all } t\in(0,T_{\max}),$$

with $C_5 > 0$. Multiplying the third equation in (1.1) by w, integrating on Ω by parts and using Young's inequality, we can find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 = -\int_\Omega w^2 + \int_\Omega uw + \int_\Omega vw \leq -\frac{1}{2}\int_\Omega w^2 + \int_\Omega u^2 + \int_\Omega v^2$$

for all $t \in (0, T_{\max})$. Therefore, we have

$$\frac{d}{dt}\int_\Omega w^2 + \int_\Omega w^2 \leq 2\int_\Omega u^2 + 2\int_\Omega v^2 \quad \text{for all } t \in (0, T_{\max}).$$

Using Lemma 2.1, and there exists $C_6 > 0$ such that

$$\int_\Omega w^2 \leq C_6 := \max\left\{\int_\Omega w_0^2 + 2C_4, 6C_4\right\}. \tag{3.3}$$

Which implies (3.2). □

Lemma 3.3. *Let $n \geq 1$, and there exists $C_7 > 0$ such that the solution of (1.1) satisfies*

$$\int_\Omega |\nabla z(x,t)|^2 \leq C_7, t \in [0, T_{\max} - 1). \tag{3.4}$$

Proof. Multiplying the fourth equation in (1.1) by $-\Delta z$, integrating on Ω by parts and using Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla z|^2 + \int_\Omega |\nabla z|^2 + \int_\Omega |\Delta z|^2 = -\int_\Omega w\Delta z \leq \frac{1}{2}\int_\Omega w^2 + \frac{1}{2}\int_\Omega |\Delta z|^2 \quad \text{for all } t \in (0, T_{\max}),$$

which indicates

$$\frac{d}{dt}\int_\Omega |\nabla z|^2 + 2\int_\Omega |\nabla z|^2 + \int_\Omega |\Delta z|^2 \leq \int_\Omega w^2 \quad \text{for all } t \in (0, T_{\max}).$$

Let $y_2(t) := \int_\Omega |\nabla z(x,t)|^2 dx$, we obtain

$$y_2'(t) + 2y_2(t) + \int_\Omega |\Delta z(x,t)|^2 \leq \int_\Omega w^2(x,t) \quad \text{for all } t \in (0, T_{\max}).$$

Using (3.3) and Lemma 2.1, we arrive at

$$y_2(t) \leq C_8 := \max\left\{\int_\Omega |\nabla z_0(x)|^2 + C_6, \frac{5}{2}C_6\right\} \quad \text{for all } t \in (0, T_{\max}).$$

□

Now, we provide the L^2-boundedness of u and v, which are crucial for proving the L^∞-boundedness of u and v.

Lemma 3.4. *Let the conditions of Theorem 1.1 hold, the solution (u, v, w, z) of (1.1) satisfies*

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{3}{\chi_1}\int_\Omega u^2 + \frac{3}{\chi_2}\int_\Omega v^2\right) + \frac{3}{\chi_1}\int_\Omega |\nabla u|^2 + \frac{3}{\chi_2}\int_\Omega |\nabla v|^2 \\
\leq &3\chi_1\int_\Omega u^2|\nabla z|^2 + 3\chi_2\int_\Omega v^2|\nabla z|^2 + \frac{6\mu_1}{\chi_1}\int_\Omega u^2 - \frac{6\mu_1}{\chi_1}\int_\Omega u^3 \\
&-\frac{6\mu_2}{\chi_2}\int_\Omega v^2 - \frac{6\mu_2}{\chi_2}\left(1 - \frac{\mu_2 e_2^2\chi_1}{4\mu_1 e_1\chi_2}\right)\int_\Omega v^3.
\end{aligned} \tag{3.5}$$

Proof. Multiplying the first equations of (1.1) by $2u$, integrating by parts and using Young's inequality, we obtain

$$\frac{d}{dt}\int_\Omega u^2+2\int_\Omega|\nabla u|^2=-2\chi_1\int_\Omega u\nabla u\cdot\nabla z+2\mu_1\int_\Omega u^2\left(1-u-e_1v\right). \tag{3.6}$$

Applying Young's inequality, we derive

$$-2\chi_1\int_\Omega u\nabla u\cdot\nabla z\le\int_\Omega|\nabla u|^2+\chi_1^2\int_\Omega u^2|\nabla z|^2. \tag{3.7}$$

Substituting (3.7) into (3.6), one can get

$$\frac{d}{dt}\int_\Omega u^2+\int_\Omega|\nabla u|^2\le\chi_1^2\int_\Omega u^2|\nabla z|^2+2\mu_1\int_\Omega u^2\left(1-u-e_1v\right),\quad t\in\left(0,T_{\max}\right). \tag{3.8}$$

Similarly, we arrive at

$$\frac{d}{dt}\int_\Omega v^2+\int_\Omega|\nabla v|^2\le\chi_2^2\int_\Omega v^2|\nabla z|^2+2\mu_2\int_\Omega v^2\left(-1+e_2u-v\right),\quad t\in\left(0,T_{\max}\right). \tag{3.9}$$

Multiplying (3.8) by $\frac{3}{\chi_1}$ and (3.9) by $\frac{3}{\chi_2}$, respectively, we can compute

$$\begin{aligned}&\frac{d}{dt}\left(\frac{3}{\chi_1}\int_\Omega u^2+\frac{3}{\chi_2}\int_\Omega v^2\right)+\frac{3}{\chi_1}\int_\Omega|\nabla u|^2+\frac{3}{\chi_2}\int_\Omega|\nabla v|^2\\ \le\,&3\chi_1\int_\Omega u^2|\nabla z|^2+3\chi_2\int_\Omega v^2|\nabla z|^2+\frac{6\mu_1}{\chi_1}\int_\Omega u^2-\frac{6\mu_1}{\chi_1}\int_\Omega u^3-\frac{6\mu_1e_1}{\chi_1}\int_\Omega u^2v\\ &-\frac{6\mu_2}{\chi_2}\int_\Omega v^2-\frac{6\mu_2}{\chi_2}\int_\Omega v^3+\frac{6\mu_2e_2}{\chi_2}\int_\Omega uv^2.\end{aligned} \tag{3.10}$$

Next, using Young's inequality, we see that

$$\frac{6\mu_2e_2}{\chi_2}\int_\Omega uv^2\le\frac{6\mu_1e_1}{\chi_1}\int_\Omega u^2v+\frac{3\mu_2^2e_2^2\chi_1}{2\mu_1e_1\chi_2^2}\int_\Omega v^3,\quad t\in\left(0,T_{\max}\right). \tag{3.11}$$

By combining (3.10) with (3.11), we can get (3.5). □

Lemma 3.5. *Suppose that the conditions of Theorem 1.1 hold, the solution* (u,v,w,z) *of (1.1) satisfies*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(\mu_2e_2u+\mu_1e_1v\right)|\nabla z|^2+2\int_\Omega\left(\mu_2e_2u+\mu_1e_1v\right)|\nabla z|^2+2\int_\Omega\left(\mu_2e_2u+\mu_1e_1v\right)\left|D^2z\right|^2\\ \le&\frac{1}{\chi_1}\int_\Omega|\nabla u|^2+\frac{1}{\chi_2}\int_\Omega|\nabla v|^2+\frac{5\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)}{4}\int_\Omega|\nabla|\nabla z|^2|^2\\ &+\left(\chi_1+\mu_2^2e_2^2\chi_1-\mu_2e_2\mu_1\right)\int_\Omega u^2|\nabla z|^2+\left(\chi_2+\mu_1^2e_1^2\chi_2-\mu_1e_1\mu_2\right)\int_\Omega v^2|\nabla z|^2\\ &+\mu_1\mu_2\int_\Omega\left(e_2u-e_1v\right)|\nabla z|^2+\int_{\partial\Omega}\left(\mu_2e_2u+\mu_1e_1v\right)\frac{\partial|\nabla z|^2}{\partial\nu}+\left(\frac{1}{\chi_1}+\frac{1}{\chi_2}\right)\int_\Omega|\nabla w|^2\end{aligned} \tag{3.12}$$

for all $t\in\left(0,T_{\max}\right)$.

Proof. Note that

$$\frac{d}{dt}\int_\Omega\left(\mu_2e_2u+\mu_1e_1v\right)|\nabla z|^2=\int_\Omega\left(\mu_2e_2u+\mu_1e_1v\right)_t|\nabla z|^2+\int_\Omega\left(\mu_2e_2u+\mu_1e_1v\right)\left(|\nabla z|^2\right)_t \tag{3.13}$$

Due to the first and second equations of (1.1), the first term on the right-hand side of (3.13) can be developed as

$$\begin{aligned}&\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)_t \, |\nabla z|^2 \\ =&-\int_\Omega \nabla(\mu_2 e_2 u + \mu_1 e_1 v)\cdot\nabla|\nabla z|^2 - \int_\Omega (\chi_1\mu_2 e_2 u - \chi_2\mu_1 e_1 v)\nabla z\cdot\nabla|\nabla z|^2 \\ &+\int_\Omega (\mu_2 e_2 \mu_1 u(1-u) + \mu_1 e_1 \mu_2 v(-1-v))\,|\nabla z|^2.\end{aligned} \tag{3.14}$$

By using the pointwise identity $2\nabla z\cdot\nabla\Delta z = \Delta|\nabla z|^2 - 2\left|D^2 z\right|^2$, one can get

$$\frac{d}{dt}|\nabla z|^2 = \Delta|\nabla z|^2 - 2\left|D^2 z\right|^2 + 2\nabla z\cdot\nabla w - 2|\nabla z|^2.$$

Then, the second term on the right-hand side of (3.13) can be rewritten as

$$\begin{aligned}&\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)\left(|\nabla z|^2\right)_t \\ =&-\int_\Omega \nabla(\mu_2 e_2 u + \mu_1 e_1 v)\cdot\nabla|\nabla z|^2 + \int_{\partial\Omega}(\mu_2 e_2 u + \mu_1 e_1 v)\frac{\partial|\nabla z|^2}{\partial\nu} \\ &-2\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)\left|D^2 z\right|^2 - 2\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)|\nabla z|^2 \\ &+2\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)\nabla z\cdot\nabla w.\end{aligned} \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)|\nabla z|^2 \\ \le&-2\int_\Omega \nabla(\mu_2 e_2 u + \mu_1 e_1 v)\cdot\nabla|\nabla z|^2 - \int_\Omega (\chi_1\mu_2 e_2 u - \chi_2\mu_1 e_1 v)\nabla z\cdot\nabla|\nabla z|^2 \\ &+\int_\Omega (\mu_2 e_2 \mu_1 u(1-u) + \mu_1 e_1 \mu_2 v(-1-v))|\nabla z|^2 - 2\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)|\nabla z|^2 \\ &+2\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)\nabla z\cdot\nabla w - 2\int_\Omega (\mu_2 e_2 u + \mu_1 e_1 v)\left|D^2 z\right|^2 \\ &+\int_{\partial\Omega}(\mu_2 e_2 u + \mu_1 e_1 v)\frac{\partial|\nabla z|^2}{\partial\nu}.\end{aligned} \tag{3.16}$$

By using Young's inequality, one can obtain that for all $t\in(0,T_{\max})$,

$$\begin{aligned}&-2\int_\Omega \nabla(\mu_2 e_2 u + \mu_1 e_1 v)\cdot\nabla|\nabla z|^2 \\ =&-2\mu_2 e_2\int_\Omega \nabla u\cdot\nabla|\nabla z|^2 - 2\mu_1 e_1\int_\Omega \nabla v\cdot\nabla|\nabla z|^2 \\ \le&\frac{1}{\chi_1}\int_\Omega |\nabla u|^2 + \frac{1}{\chi_2}\int_\Omega |\nabla v|^2 + \left(\mu_2^2 e_2^2\chi_1 + \mu_1^2 e_1^2\chi_2\right)\int_\Omega |\nabla|\nabla z|^2|^2,\end{aligned} \tag{3.17}$$

$$\begin{aligned}&-\int_\Omega (\mu_2 e_2\chi_1 u - \mu_1 e_1\chi_2 v)\nabla z\cdot\nabla|\nabla z|^2 \\ =&-\mu_2 e_2\chi_1\int_\Omega u\nabla z\cdot\nabla|\nabla z|^2 + \mu_1 e_1\chi_2\int_\Omega v\nabla z\cdot\nabla|\nabla z|^2 \\ \le&\chi_1\int_\Omega u^2|\nabla z|^2 + \chi_2\int_\Omega v^2|\nabla z|^2 + \frac{\mu_2^2 e_2^2\chi_1 + \mu_1^2 e_1^2\chi_2}{4}\int_\Omega |\nabla|\nabla z|^2|^2,\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&2\int_\Omega \left(\mu_2 e_2 u + \mu_1 e_1 v\right) \nabla z \cdot \nabla w \\
=&2\mu_2 e_2 \int_\Omega u \nabla z \cdot \nabla w + 2\mu_1 e_1 \int_\Omega v \nabla z \cdot \nabla w \\
\le&\frac{1}{\chi_1}\int_\Omega |\nabla w|^2 + \frac{1}{\chi_2}\int_\Omega |\nabla w|^2 + \mu_2^2 e_2^2 \chi_1 \int_\Omega u^2 |\nabla z|^2 + \mu_1^2 e_1^2 \chi_2 \int_\Omega v^2 |\nabla z|^2,
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\mu_2 e_2 \mu_1 \int_\Omega u(1-u)|\nabla z|^2 + \mu_1 e_1 \mu_2 \int_\Omega v(-1-v)|\nabla z|^2 \\
=&\mu_1\mu_2 \int_\Omega \left(e_2 u - e_1 v\right)|\nabla z|^2 - \mu_2 e_2 \mu_1 \int_\Omega u^2 |\nabla z|^2 - \mu_1 e_1 \mu_2 \int_\Omega v^2 |\nabla z|^2.
\end{aligned} \tag{3.20}$$

By combining (3.16) with (3.17)-(3.20), (3.12) is obtained immediately. □

Lemma 3.6. *Suppose that the conditions of Theorem 1.1 hold, the solution* (u, v, w, z) *of (1.1) satisfies*

$$\frac{d}{dt}\int_\Omega |\nabla z|^4 + \int_\Omega |\nabla|\nabla z|^2|^2 + 4\int_\Omega |\nabla z|^4 \le 7\int_\Omega w^2 |\nabla z|^2 + 2\int_{\partial\Omega} |\nabla z|^2 \frac{\partial |\nabla z|^2}{\partial \nu} \tag{3.21}$$

for all $t \in (0, T_{\max})$.

Proof. Due to the third equation of (1.1), we obtain the pointwise identity

$$\frac{1}{2}\frac{d}{dt}|\nabla z|^2 = \frac{1}{2}\Delta|\nabla z|^2 - \left|D^2 z\right|^2 + \nabla z \cdot \nabla w - |\nabla z|^2 \quad \text{for all } t \in (0, T_{\max}).$$

Multiplying the above equation by $|\nabla z|^2$ and integrating over Ω, we derive

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla z|^4 =& -\frac{1}{2}\int_\Omega |\nabla|\nabla z|^2|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla z|^2 \frac{\partial|\nabla z|^2}{\partial\nu} - \int_\Omega |\nabla z|^2 \left|D^2 z\right|^2 \\
&- \int_\Omega |\nabla z|^4 - \int_\Omega w\Delta z |\nabla z|^2 - \int_\Omega w \nabla z \cdot \nabla|\nabla z|^2
\end{aligned} \tag{3.22}$$

for all $t \in (0, T_{\max})$. Using Young's inequality to the last two terms of (3.22) and applying the inequality $|\Delta z| \le \sqrt{3}\left|D^2 z\right|$, we have

$$-\int_\Omega w\Delta z|\nabla z|^2 \le \int_\Omega |\nabla z|^2 \left|D^2 z\right|^2 + \frac{3}{4}\int_\Omega w^2 |\nabla z|^2 \tag{3.23}$$

and

$$-\int_\Omega w\nabla z \cdot \nabla|\nabla z|^2 \le \frac{1}{4}\int_\Omega |\nabla|\nabla z|^2|^2 + \int_\Omega w^2 |\nabla z|^2 \tag{3.24}$$

for all $t \in (0, T_{\max})$.Combining with (3.22)-(3.24), we obtain

$$\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla z|^4 \le -\frac{1}{4}\int_\Omega |\nabla|\nabla z|^2|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla z|^2 \frac{\partial|\nabla z|^2}{\partial\nu} - \int_\Omega |\nabla z|^4 + \frac{7}{4}\int_\Omega w^2 |\nabla z|^2.$$

for all $t \in (0, T_{\max})$. Which implies (3.21). □

Lemma 3.7. *Suppose that the conditions of Theorem 1.1 hold, the solution* (u,v,w,z) *of (1.1) satisfies*

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{3}{\chi_1}\int_\Omega u^2+\frac{3}{\chi_2}\int_\Omega v^2+\int_\Omega(\mu_2e_2u+\mu_1e_1v)|\nabla z|^2+\frac{3\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)}{2}\int_\Omega|\nabla z|^4\right)\\
&+\frac{2}{\chi_1}\int_\Omega|\nabla u|^2+\frac{2}{\chi_2}\int_\Omega|\nabla v|^2+6\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)\int_\Omega|\nabla z|^4+2\int_\Omega(\mu_2e_2u+\mu_1e_1v)\left|D^2z\right|^2\\
&+\left(\mu_1\mu_2e_1-4\chi_2-\mu_1^2e_1^2\chi_2\right)\int_\Omega v^2|\nabla z|^2+2\int_\Omega(\mu_2e_2u+\mu_1e_1v)|\nabla z|^2\\
&+\frac{\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)}{4}\int_\Omega|\nabla|\nabla z|^2|^2+\left(\mu_1\mu_2e_2-4\chi_1-\mu_2^2e_2^2\chi_1\right)\int_\Omega u^2|\nabla z|^2\\
\leq&\mu_1\mu_2\int_\Omega(e_2u-e_1v)|\nabla z|^2+\frac{6\mu_1}{\chi_1}\int_\Omega u^2-\frac{6\mu_1}{\chi_1}\int_\Omega u^3-\frac{6\mu_2}{\chi_2}\int_\Omega v^2+\left(\frac{1}{\chi_1}+\frac{1}{\chi_2}\right)\int_\Omega|\nabla w|^2\\
&+3\int_{\partial\Omega}\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)|\nabla z|^2\cdot\frac{\partial|\nabla z|^2}{\partial\nu}+\int_{\partial\Omega}(\mu_2e_2u+\mu_1e_1v)\cdot\frac{\partial|\nabla z|^2}{\partial\nu}\\
&+\frac{21}{2}\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)\int_\Omega w^2|\nabla z|^2-\frac{6\mu_2}{\chi_2}\left(1-\frac{\mu_2e_2^2\chi_1}{4\mu_1e_1\chi_2}\right)\int_\Omega v^3,
\end{aligned}
\tag{3.25}
$$

for all $t\in(0,T_{\max})$.

Now, we establish the uniform boundedness of $\|u(\cdot,t)\|_{L^2(\Omega)}$ and $\|v(\cdot,t)\|_{L^2(\Omega)}$.

Lemma 3.8. *Suppose that the conditions of Theorem 1.1 hold, there exist* $C_9,C_{10}>0$ *such that*

$$
\|u(\cdot,t)\|_{L^2(\Omega)}\leq C_9\quad\textit{and}\quad\|v(\cdot,t)\|_{L^2(\Omega)}\leq C_{10}\quad\textit{for all } t\in(0,T_{\max}).
\tag{3.26}
$$

Proof. By the assumptions (1.4)-(1.6) and Lemma 3.7, we can see that

$$
y'(t)+y(t)+\zeta(t)\leq h(t)\quad\text{for all } t\in(0,T_{\max}),
\tag{3.27}
$$

where

$$
\begin{aligned}
y(t)&:=\int_\Omega\left(\frac{3}{\chi_1}u^2+\frac{3}{\chi_2}v^2+(\mu_2e_2u+\mu_1e_1v)|\nabla z|^2+\frac{3\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)}{2}|\nabla z|^4\right),\\
\zeta(t)&:=\int_\Omega\left(\frac{2}{\chi_1}|\nabla u|^2+\frac{2}{\chi_2}|\nabla v|^2\right)+\frac{\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)}{4}\int_\Omega|\nabla|\nabla z|^2|^2,\\
h(t)&:=\frac{3(2\mu_1+1)}{\chi_1}\int_\Omega u^2+\frac{3(-2\mu_2+1)}{\chi_2}\int_\Omega v^2-\frac{6\mu_1}{\chi_1}\int_\Omega u^3-\frac{6\mu_2}{\chi_2}\left(1-\frac{\mu_2e_2^2\chi_1}{4\mu_1e_1\chi_2}\right)\int_\Omega v^3\\
&\quad+\left(\frac{1}{\chi_1}+\frac{1}{\chi_2}\right)\int_\Omega|\nabla w|^2+\frac{21}{2}\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)\int_\Omega w^2|\nabla z|^2\\
&\quad+\mu_1\mu_2\int_\Omega(e_2u-e_1v)|\nabla z|^2+3\int_{\partial\Omega}\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)|\nabla z|^2\cdot\frac{\partial|\nabla z|^2}{\partial\nu}\\
&\quad+\int_{\partial\Omega}(\mu_2e_2u+\mu_1e_1v)\cdot\frac{\partial|\nabla z|^2}{\partial\nu}.
\end{aligned}
\tag{3.28}
$$

Let

$$
\begin{aligned}
I_1&=\left(\frac{1}{\chi_1}+\frac{1}{\chi_2}\right)\int_\Omega|\nabla w|^2,\\
I_2&=\frac{21}{2}\left(\mu_2^2e_2^2\chi_1+\mu_1^2e_1^2\chi_2\right)\int_\Omega w^2|\nabla z|^2,
\end{aligned}
$$

$$I_3 = \mu_1\mu_2 \int_\Omega (e_2 u - e_1 v)\,|\nabla z|^2$$

and

$$I_4 = 3\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right) \int_{\partial\Omega} |\nabla z|^2 \cdot \frac{\partial |\nabla z|^2}{\partial \nu} + \int_{\partial\Omega} (\mu_2 e_2 u + \mu_1 e_1 v)\, \frac{\partial |\nabla z|^2}{\partial \nu}.$$

We estimate $I_1 - I_4$, respectively. For I_1, it follows from Lemma 3.2 that

$$I_1 \le C_{11} \tag{3.29}$$

with $C_{11} > 0$. For I_2, using Young's inequality, we derive

$$I_2 \le \frac{21}{2}\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right)\left(\int_\Omega w^4 + \frac{1}{4}\int_\Omega |\nabla z|^4\right). \tag{3.30}$$

By employing the estimate $\|w\|_{L^1(\Omega)} \le \|u\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)}$ and the Gagliardo-Nirenberg inequality, it follows from Lemmas 3.1 and 3.2 that there exist $C_{12}, C_{13} > 0$ such that

$$\int_\Omega w^4 = \|w\|_{L^4(\Omega)}^4 \le C_{12} \|\nabla w\|_{L^2(\Omega)}^{\frac{18}{5}} \cdot \|w\|_{L^1(\Omega)}^{\frac{2}{5}} + C_{12}\|w\|_{L^1(\Omega)}^4 \le C_{13}. \tag{3.31}$$

Substituting (3.31) in (3.30), it follows from Lemmas 2.3 and 3.3 and Young's inequality that there exist $C_{14}, C_{15}, C_{16}, C_{17}, C_{18} > 0$, such that

$$\begin{aligned} I_2 &\le \frac{21}{2}\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right)\left(\int_\Omega w^4 + \frac{1}{4}\int_\Omega |\nabla z|^4\right) \\ &\le \frac{21 C_{13}}{2}\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right) + \frac{21}{8}\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right)\int_\Omega |\nabla z|^4 \\ &\le C_{14} + C_{15}\left\|\nabla |\nabla z|^2\right\|_{L^2(\Omega)}^{\frac{6}{5}} \left\||\nabla z|^2\right\|_{L^1(\Omega)}^{\frac{4}{5}} + C_{15}\left\||\nabla z|^2\right\|_{L^1(\Omega)}^2 \\ &\le C_{16}\left\|\nabla |\nabla z|^2\right\|_{L^2(\Omega)}^{\frac{6}{5}} + C_{17} \\ &\le \frac{\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2}{12}\int_\Omega |\nabla |\nabla z|^2|^2 + C_{18}. \end{aligned} \tag{3.32}$$

Where $C_{18} = \frac{2}{5}\left(\frac{5\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right)}{36}\right)^{-\frac{3}{2}} C_{16}^{\frac{5}{2}} + C_{17}$. For I_3, applying Young's inequality and Lemma 2.3, one has

$$I_3 \le \frac{(\mu_1\mu_2 e_2)^2}{2}\int_\Omega u^2 + \frac{(\mu_1\mu_2 e_1)^2}{2}\int_\Omega v^2 + \frac{\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right)}{12}\int_\Omega |\nabla |\nabla z|^2|^2 + C_{19} \tag{3.33}$$

with $C_{19} > 0$. As for I_4, it follows from Young's inequality, Lemmas 2.4, 2.5, 3.1 and 3.3 that

$$\begin{aligned} I_4 \le& C_{20}\int_{\partial\Omega} |\nabla z|^4 + \frac{C_{20}}{2}\mu_2^2 e_2^2 \int_{\partial\Omega} u^2 + \frac{C_{20}}{2}\mu_1^2 e_1^2 \int_{\partial\Omega} v^2 \\ \le& \frac{\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right)}{12}\int_\Omega |\nabla |\nabla z|^2|^2 + \frac{2}{\chi_1}\int_\Omega |\nabla u|^2 + \frac{2}{\chi_2}\int_\Omega |\nabla v|^2 \\ &+ C_{21}\left(\int_\Omega u\right)^2 + C_{22}\left(\int_\Omega v\right)^2 + C_{23}\left(\int_\Omega |\nabla z|^2\right)^2 \\ \le& \frac{\left(\mu_2^2 e_2^2 \chi_1 + \mu_1^2 e_1^2 \chi_2\right)}{12}\int_\Omega |\nabla |\nabla z|^2|^2 + \frac{2}{\chi_1}\int_\Omega |\nabla u|^2 + \frac{2}{\chi_2}\int_\Omega |\nabla v|^2 + C_{24}, \end{aligned} \tag{3.34}$$

where $C_{20}, C_{21}, C_{22}, C_{23}, C_{24} > 0$. Combining (3.27), (3.28), (3.29) and (3.32)-(3.34), we deduce

$$\begin{aligned} y'(t) + y(t) \le& \left(\frac{3\,(2\mu_1 + 1)}{\chi_1} + \frac{(\mu_1\mu_2 e_2)^2}{2}\right)\int_\Omega u^2 + \left(\frac{3\,(1 - 2\mu_2)}{\chi_2} + \frac{(\mu_1\mu_2 e_1)^2}{2}\right)\int_\Omega v^2 \\ &- \frac{6\mu_1}{\chi_1}\int_\Omega u^3 - \frac{6\mu_2}{\chi_2}\left(1 - \frac{\mu_2 e_2^2 \chi_1}{4\mu_1 e_1 \chi_2}\right)\int_\Omega v^3 + C_{25}, \end{aligned}$$

for all $t \in (0, T_{\max})$, with $C_{25} > 0$, which by means of Young's inequality yields $C_{26} > 0$ such that

$$y'(t) + y(t) \le C_{26} \quad \text{for all } t \in (0, T_{\max}) .$$

By lemma 2.1, this ensures that $y(t) \le \max\{y(0), C_{26}\}$. we derive (3.26) directly. □

Next, we present the following lemma, which addresses the extensibility and regularity of solutions to model (1.1) in $L^\infty((0, T_{\max}); L^p(\Omega))$ for some $p > \frac{n}{2}$, which will be used to obtain the global existence and uniform boundedness for the case $n = 3$.

Lemma 3.9. *Let $n \ge 1$, assume that there exists $p \ge 1$ such that $p > \frac{n}{2}$ and*

$$\sup_{t\in(0,T_{\max})} (||u(\cdot,t)||_{L^p(\Omega)} + ||v(\cdot,t)||_{L^p(\Omega)}) < \infty. \tag{3.35}$$

Then, $T_{\max} = \infty$, and

$$\sup_{t>0} \big(||u(\cdot,t)||_{L^\infty(\Omega)} + ||v(\cdot,t)||_{L^\infty(\Omega)} + ||w(\cdot,t)||_{L^\infty(\Omega)} + ||z(\cdot,t)||_{L^\infty(\Omega)}\big) < \infty.$$

Proof. We suppose $p \le n$, since $p > \frac{n}{2}$, we have $\frac{np}{n-p} > n$, so that we can fix $r > n$ such that

$$r < \frac{np}{n-p}$$

we can find $\theta > 1$ such that

$$2 \le r\theta \le \frac{np}{n-p}.$$

Now, for each $t \in (0, T_{\max})$, set

$$M_1(T) = \sup_{t\in(0,T)} \| u(\cdot,t) \|_{L^\infty(\Omega)} .$$

In order to reasonably estimate $M_1(T)$, we fix an arbitrary $t \in (0,T)$, set $t_0 = \max\{0, t-1\}$, and apply the variation-of-constants formula, we have

$$\begin{aligned} u(\cdot,t) =& e^{d_1(t-t_0)\Delta}u(\cdot,t_0) + \chi_1 \int_{t_0}^t e^{d_1(t-s)\Delta}\nabla\cdot\big(u(\cdot,s)\nabla w(\cdot,s)\big)ds \\ &+ \mu_1 \int_{t_0}^t e^{d_1(t-s)\Delta}u(\cdot,s)\big(1-u(\cdot,s)-e_1 v(\cdot,s)\big)ds \\ :=& \phi_1 + \phi_2 + \phi_3. \end{aligned}$$

For ϕ_1, when $t \le 1$, we have $t_0 = 0$, in light of the comparison principle, we can get

$$||e^{d_1(t-t_0)\Delta}u(\cdot,t_0)||_{L^\infty(\Omega)} \le ||u_0||_{L^\infty(\Omega)},$$

when $t > 1$, we use the order-preserving property of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge0}$ in Ω to find a $C_{27} > 0$, such that

$$||e^{d_1(t-t_0)\Delta}u(\cdot,t_0)||_{L^\infty(\Omega)} \le C_{27}(t-t_0)^{-\frac{n}{2}}||u(\cdot,t_0)||_{L^1(\Omega)} \le C_{27}k_1.$$

For ϕ_2, we recall by Lemma 2.2 that there exists $C_{28} > 0$ satisfying

$$||e^{\tau\Delta}\nabla\cdot\varphi||_{L^\infty(\Omega)} \le C_{28}\tau^{-\frac{1}{2}-\frac{n}{2r}}||\varphi||_{L^r(\Omega)},$$

for all $\tau \ge 0$ and each $\varphi \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that $\varphi \cdot v = 0$ on $\partial\Omega$, this allows us to recall ϕ_2, we have

$$\chi_1 \int_{t_0}^t ||e^{d_1(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla z(\cdot,s))||_{L(\Omega)}ds \le C_{28}\chi_1 \int_{t_0}^t (d_1(t-s))^{-\frac{1}{2}-\frac{n}{2r}} \cdot ||u(\cdot,s)\nabla z(\cdot,s)||_{L^r(\Omega)}ds.$$

Here, using the Hölder inequality and Lemma 3.1, and setting $\theta' := \frac{\theta}{\theta-1}$, we get

$$\begin{aligned} ||u(\cdot,s)\nabla z(\cdot,s)||_{L^r(\Omega)} &\le ||u(\cdot,s)||_{L^{r\theta}(\Omega)} \cdot ||\nabla z(\cdot,s)||_{L^{r\theta}(\Omega)} \\ &\le ||u(\cdot,s)||^{\rho_1}_{L^\infty(\Omega)} \cdot ||u(\cdot,s)||^{1-\rho_1}_{L^1(\Omega)} \cdot ||\nabla z(\cdot,s)||_{L^{r\theta}(\Omega)} \\ &\le M_1^{\rho_1}(T) \cdot m_1^{1-\rho_1} \cdot ||\nabla z(\cdot,s)||_{L^{r\theta}(\Omega)}, \end{aligned} \tag{3.36}$$

for all $s \in (t_0, t)$, with

$$\rho_1 := 1 - \frac{1}{r\theta} \in (0,1).$$

To estimate $||\nabla z(\cdot,s)||_{L^{r\theta}(\Omega)}$, by the fourth equation of (1.1), ∇z can be represented according to

$$\nabla z(\cdot,s) = e^{-s}\nabla e^{s\Delta} z_0 + \int_0^s e^{-(s-\sigma)}\nabla e^{(s-\sigma)\Delta} w(\cdot,\sigma)d\sigma \quad \text{for all } t \in (0,T),$$

applying by Lemma 2.2, we can find a constant $C_{29} > 0$, such that

$$\begin{aligned} ||\nabla z(\cdot,s)||_{L^{r\theta}(\Omega)} \le & e^{-s}||\nabla e^{s\Delta} z_0||_{L^{r\theta}(\Omega)} + \int_0^s e^{-(s-\sigma)}||\nabla e^{-(s-\sigma)\Delta} w(\cdot,\sigma)||_{L^{r\theta}(\Omega)}d\sigma \\ \le & C_{29}e^{-s}||\nabla z_0||_{L^{r\theta}(\Omega)} + C_{29}\int_0^s e^{-(s-\sigma)}(1+(s-\sigma))^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{r\theta})}||w(\cdot,\sigma)||_{L^p(\Omega)}d\sigma \end{aligned}$$

where we have applied that $r\theta \ge 2$, due to the fact that $r\theta \le \frac{np}{n-p}$, we know $C_{29}e^{-s}||\nabla z_0||_{L^{r\theta}(\Omega)} + C_{29}\int_0^s e^{-(s-\sigma)}(1+(s-\sigma))^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{r\theta})}||w(\cdot,\sigma)||_{L^p(\Omega)}d\sigma$ is finite.

Then, we need to estimate L^p bounds of w. Using the variation-of-constants formula, we have

$$w(\cdot,\sigma) = e^{s(\Delta-1)}w_0 + \int_0^s e^{-(s-\sigma)}e^{(s-\sigma)\Delta}\big(u(\cdot,\sigma)+v(\cdot,\sigma)\big)d\sigma \quad \text{for all } t \in (0,T_{\max}).$$

Applying the order-preserving property of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge 0}$ in Ω, we obtain

$$\begin{aligned} ||w(\cdot,s)||_{L^p(\Omega)} \le & e^{-s}||e^{s\Delta}w_0||_{L^p(\Omega)} + \int_0^s e^{-(s-\sigma)}||e^{(s-\sigma)\Delta}(u(\cdot,\sigma)+v(\cdot,\sigma))||_{L^p(\Omega)}d\sigma \\ := & J_1 + J_2. \end{aligned}$$

Next, we need estimate J_1 and J_2. Employing Lemma 2.2, we can find $C_{30} > 0$,$C_{31} > 0$ such that

$$J_1 \le C_{30}e^{-s}||w_0||_{L^1(\Omega)} \le C_{31}.$$

Using (3.35), there exists $C_{32} > 0$ such that$||u(\cdot,\sigma)+v(\cdot,\sigma)||_{L^p(\Omega)}d\sigma \le C_{32}$, we derive

$$J_2 \le C_{33}\int_0^s e^{-(s-\sigma)}(s-\sigma)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{p})}||u(\cdot,\sigma)+v(\cdot,\sigma)||_{L^p(\Omega)}d\sigma \le C_{34},$$

with $C_{33} > 0$, $C_{34} > 0$, we can find

$$||w(\cdot,s)||_{L^p(\Omega)} \le C_{31} + C_{34} := C_{35},$$

with $C_{35} > 0$, shows that

$$||\nabla z(\cdot,s)||_{L^{r\theta}(\Omega)} \le C_{35}.$$

Rewrite (3.36), we have

$$||u(\cdot,s)\nabla z(\cdot,s)||_{L^r(\Omega)} \le C_{35}M_1^{\rho_1}(T)\cdot m_1^{1-\rho_1} \quad \text{for all s } \in (t_0,t),$$

so that since $\frac{1}{2} + \frac{n}{2r} < 1$ because of $r > n$. As a consequence, we obtain

$$\phi_2 \le C_{36}M_1^{\rho_1}(T).$$

For the ϕ_3, we note that

$$u(1-u-e_1v) \le \frac{1}{4} \quad \text{in } \Omega \times (0, T_{\max}),$$

it follows that

$$\begin{aligned}\phi_3 &= \mu_1 \int_{t_0}^{t} ||e^{d_1(t-s)\Delta} u(\cdot,s)(1-u(\cdot,s)-e_1v(\cdot,s))||_{L^\infty(\Omega)} ds \\ &\le \mu_1 \int_{t_0}^{t} ||u(\cdot,s)(1-u(\cdot,s)-e_1v(\cdot,s))||_{L^\infty(\Omega)} ds \\ &\le \frac{\mu_1}{4},\end{aligned}$$

and $t - t_0 \le 1$. Combining this, since $t \in (0,T)$ is chosen arbitrarily, we infer that

$$||u(\cdot,t)||_{L^\infty(\Omega)} \le \max\{||u_0||_{L^\infty(\Omega)}, C_2k_1\} + C_{36}M_1^{\rho_1}(T) + \frac{\mu_1}{4} \quad \text{for all } t \in (0,T).$$

So for each $t \in (0,T)$, we have $M_1(T) \le C_{36}M_1^{\rho_1}(T) + C_{37}$, with $C_{37} = \max\{||u_0||_{L^\infty(\Omega)}, C_2k_1\} + \frac{\mu_1}{4}$, we derive

$$M_1(T) \le C_{38} := \max\left\{ \left(\frac{C_{37}}{C_{36}}\right)^{\frac{1}{\rho_1}}, (2C_{36})^{\frac{1}{1-\rho_1}} \right\}, \quad T \in (0, T_{\max}).$$

Similarly, for each $t \in (0,T)$ we define

$$M_2(T) = \sup_{t \in (0,T)} \| v(\cdot,t) \|_{L^\infty(\Omega)} .$$

To estimate $M_2(T)$, we fix an arbitrary $t \in (0,T)$, set $t_0 = \max\{0, t-1\}$, and apply the variation-of constants formula and the order-preserving property of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau \ge 0}$ in Ω, we obtain

$$\begin{aligned}\| v(\cdot,t) \|_{L^\infty(\Omega)} \le& \| e^{d_2(t-t_0)\Delta} v(\cdot,t_0) \|_{L^\infty(\Omega)} + \chi_2 \int_{t_0}^{t} \| e^{d_2(t-s)\Delta} \nabla \cdot (v(\cdot,s)\nabla z(\cdot,s)) \|_{L^\infty(\Omega)} ds \\ &+ \mu_2 \int_{t_0}^{t} \| e^{d_2(t-s)\Delta} v(\cdot,s)(-1+e_2u(\cdot,s)-v(\cdot,s))_+ \|_{L^\infty(\Omega)} ds,\end{aligned}$$

since $M_1(T) \le C_{38}$, we have

$$\begin{aligned}&\mu_2 \int_{t_0}^{t} \|e^{d_2(t-s)\Delta} v(\cdot,s)(-1+e_2u(\cdot,s)-v(\cdot,s))_+\|_{L^\infty(\Omega)} ds \\ \le& \mu_2 \int_{t_0}^{t} \| v(\cdot,s)(-1+e_2u(\cdot,s)-v(\cdot,s))_+ \|_{L^\infty(\Omega)} ds \\ \le& C_{39},\end{aligned}$$

and $t - t_0 \le 1$, with $C_{39} = \frac{\mu_2}{4}(1+e_2C_{38})^2$. The estimation of v residue is completely similar to that of u, there exist $C_{40} > 0, C_{41} > 0$ and $\rho_2 \in (0,1)$ such that for all $t \in (0, T_{\max})$ have

$$M_2(T) \le C_{40}M_2^{\rho_2}(T) + C_{41}.$$

We can get

$$M_2(T) \le \max\left\{ \left(\frac{C_{40}}{C_{41}}\right)^{\frac{1}{\rho_2}}, (2C_{40})^{\frac{1}{1-\rho_2}} \right\}, \quad T \in (0, T_{\max}).$$

This means that Lemma 3.9 holds. □

Proof of Theorem 1.1. In light of L^1-boundedness for u, v in Lemma 3.1 and L^2-boundedness for u, v in Lemma 3.8, adapting Lemma 3.9 by choosing $p = 2,\ n = 3$, which easily derives Theorem 1.1.

4 CONCLUSIONS

In this paper, we investigate predator-prey chemotaxis model with indirect signal production in a three-dimensional bounded domain. The result show that the model (1.1) has a unique global bounded classical solution under the assumption $\mu_2 e_2^2 \chi_1 \leq 4\mu_1 e_1 \chi_2$, $\mu_1 \mu_2 e_2 \geq \chi_1 \left(4 + \mu_2^2 e_2^2\right)$ and $\mu_1 \mu_2 e_1 \geq \chi_2 \left(4 + \mu_1^2\right)$.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Author has declared that no competing interests exist.

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