Stability of the Damped Three-Dimensional Incompressible Boussinesq Equations *

Abstract. In this paper, we investigate the global well-posedness on the three-dimensional (3D) Boussinesq equation near a equilibrium, where the velocity equation and temperature equation involve damping terms, respectively. Without temperature, the corresponding velocity equations is governed by a 3D incompressible anisotropic Navier-Stokes equation, and the stability is still unknown. However, when the velocity fluid is coupled temperature. Employing time-global uniform *a priori* estimates, we first establish the global well-posedness of the Boussinesq equations in $H^3(R^3)$. Additionally, we also obtain the explicit decay rates for the system.

Keywords. Boussinesq equations; equilibrium; stability; large-time behavior.

MSC: 35Q35; 35B35; 35B65;

1 Introduction

The three-dimensional (3D) incompressible Euler equations have been thoroughly investigated and the resolution of the global (in time) existence and uniqueness issue is currently in a satisfactory status. In contrast, the global regularity problem concerning the 3D inviscid Boussinesq equations remains widely open. This paper examines the global (in time) existence and uniqueness problem on the incompressible 3D Boussinesq equations with damping.

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P + (\nu u_h, 0)^\top = \theta e_3, \\ \partial_t \theta + u \cdot \nabla \theta + u_3 + \lambda \theta = 0, \\ \nabla \cdot U = 0. \end{cases} \tag{1.1}$$

where u represents the fluid velocity, p the pressure, e_3 the unit vector in the third direction, the temperature in thermal convectionand $\nu > 0$ and $\lambda > 0$ are real parameters. In this article, let $\nu = \lambda = 1$. If θ is identically zero, (1.1) degenerates to the 3D incompressible Navier-Stokes equations.

The Boussinesq equations model many geophysical flows such as atmospheric fronts and ocean circulations[11, 17, 22, 23]. Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. Abidi and Hmidi [1] studies the global well-posedness for Boussinesq system. Adhikari-Cao-Wu-Xu [3] obtained small global solutions to the damped two-dimensional Boussinesq equations. When suitable partial dissipation or

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fractional Laplacian dissipation with sufficiently large index is added, the vortex stretching can be controlled and the global regularity can be established. Danchin and Paicu [15] obtained global existence results for the anisotropic Boussinesq system in two dimension. Bian and Pu [6] established global smooth axisymmetic solutions of the Boussinesq equations for magneto-hydrodynamic convection. Readers can refer the studies in [2, 4, 5, 8, 9] and the references therein.

As we know, the issue of global existence and uniqueness relies crucially on whether or not one can obtain global bounds on the solutions. If the fluid is affected by the temperature, then the Navier-Stokes equation will become Boussinesq system. Castro-Cordoba-Lear [7] studied the stbility effect of hydrostatic equilibrium of temperature. Doering-Wu-Zhao-Zheng [13] obtained the long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion. Lai-Wu obtained the stability and the optimal decay rates for the system without the magnetic field in [20] and [21], respectively. Ji-Li-Wu [19] established optimal decay for the 3D anisotropic Boussinesq equations near the hydrostatic balance. For more results about stability of Boussinesq equations can refer to [10, 12, 14, 16, 18, 24].

The goal of this paper is to establish the global well-posedness of the Boussinesq equations in $H^3(R^3)$ and obtain the explicit decay rates for the system. Our main results can then be stated as follows.

Theorem 1.1. *Consider (1.1) with the initial data $(u_0, \theta_0) \in H^3(\mathbb{R}^3)$ satisfies $\nabla \cdot u_0 = 0$. Then there exists a positive constant $\varepsilon>0$, such that if*

$$\|(u_0, \theta_0)\|_{H^3} \leq \varepsilon,$$

then the system (1.1) has a unique global solution for any $t>0$, satisfying

$$\|(u, \theta)(t)\|_{H^3}^2 + \int_0^t \|u_h\|_{H^3}^2 + \|u_3\|_{H^3}^2 + \|\theta\|_{H^3}^2 d\tau \leq C\varepsilon^2, \tag{1.2}$$

where $C>0$ is a generic positive constant independent of ε and t.

We observe that (1.2) of Theorem 1.1 rigorously assesses any small initial perturbation leads to a unique global solution of (1.1) and remains consistently small in H^3. Since the local existence result can be shown via the standard method, we only need to establish a global prior estimates of the solutions. To use the bootstrapping argument, we introduce an energy functional specifically to achieve our desired estimates. Let

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t),$$

where

$$\mathcal{E}_1(t) = \sup_{0\leq\tau\leq t} \|(u, \theta)(\tau)\|_{H^3}^2 + 2\int_0^t \|u_h(\tau)\|_{H^3}^2 + \|\theta(\tau)\|_{H^3}^3 d\tau,$$
$$\mathcal{E}_2(t) = \int_0^t \|u_3(\tau)\|_{H^3}^2 d\tau.$$

The main goal of the proof is to establish the following estimate:

$$\mathcal{E}(t) \leq C\mathcal{E}(0) + C\mathcal{E}^{3/2}(t). \tag{1.3}$$

The proof of (1.3) is not obvious and requires significant effort. We need to establish the following three inequalities respectively, and there exists a generic positive constant C,

$$\mathcal{E}_1(t) \leq C\mathcal{E}_1(0) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t), \tag{1.4}$$

$$\mathcal{E}_2(t) \leq C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t). \tag{1.5}$$

For any $t>0$, adding (1.5) to (1.4) by the appropriate constant, then we can yield the estimate of (1.3). The bootstrapping argument implies that if

$$\mathcal{E}(0) = \|(u_0, \theta_0)\|_{H^3}^2 \leq \varepsilon^2$$

for suitable $\varepsilon>0$, then $\mathcal{E}(t)$ remains uniformly bounded for $0<t<\infty$,

$$\mathcal{E}(t) \leq C\varepsilon^2,$$

for some pure constant $C>0$. The more details are provided in Section 3.

Theorem 1.2. *Suppose that $(u_0, \theta_0) \in H^3$ with $\nabla \cdot u = 0$, where (u, θ) is the solution of (1.1). For (u, θ) satisfies*

$$\|(\nabla u, \nabla \theta)(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}, \tag{1.6}$$

The rest of this paper is divided into three sections. Section 2 presents several tool lemmas to be used in the proof of Theorem 1.1. The inequalities of (1.4), (1.5) and are established, and the proof of Theorem 1.1 is completed in Section 3. The last section proves Theorem 1.2.

2 Preliminaries

In this section, we provide several lemmas that will be very important in subsequent proofs. Lemma 2.1 can be obtained from the book on partial differential equations. Lemma 2.2 (see [20]) are helpful in obtaining some relevant results for large time behavior.

Lemma 2.1. *Assume f and ∇f all in $L^2(\mathbb{R}^3)$. it holds that*

$$\begin{aligned} &\|f\|_{L^\infty} \leq C\|f\|_{L^6}^{1/2}\|\nabla f\|_{L^6}^{1/2}, \\ &\|f\|_{L^6} \leq C\|\nabla f\|_{L^2}. \end{aligned} \tag{2.1}$$

Lemma 2.2. *For given positive constants $C_0 > 0$ and $C_1 > 0$, assume that $f = f(t)$ is a nonnegative function defined on $[0, \infty)$ and satisfies*

$$\int_0^\infty f(t)dt \leq C_0<\infty, \quad \textit{and} \quad f(t) \leq C_1 f(s), \quad \forall 0 \leq s < t.$$

Then there exists a positive constant $C_2 := \max\{2C_1 f(0), 4C_0C_1\}$ such that

$$f(t) \leq C_2(1+t)^{-1}, \quad \forall t \geq 0.$$

3 The global well-posedness

The main purpose of this section is to prove Theorem 1.1. In the following, we establish the validity of (1.4) and (1.5) respectively.

3.1 Proof of (1.4)

Proof. First, we take the L^2-inner product of (1.1) with (u,θ) to obtain

$$\frac{1}{2}\frac{d}{dt}\|(u,\theta)\|_{L^2}^2+\|u_h\|_{L^2}^2+\|\theta\|_{L^2}^2=0. \tag{3.1}$$

Next, to estimate the $\dot{H}^1$-norm, applying ∇ to (1.1) and dotting them with $(\nabla u,\nabla\theta)$ in L^2, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla\theta)\|_{L^2}^2+\|\nabla u_h\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\\&=\sum_{i=1}^{3}\int\partial_i(u\cdot\nabla\theta)\cdot\partial_i\theta dx\\&=\int\nabla u\cdot\nabla\theta\cdot\nabla\theta dx\\&\le C\|u\|_{H^3}\|\nabla\theta\|_{L^2}^2.\end{aligned} \tag{3.2}$$

where we used the significant fact that

$$\sum_{i=1}^{3}\int\partial_i(u\cdot\nabla u)\cdot\partial_i u dx=0.$$

To bound the $\dot{H}^2$-norm of (u,θ), applying $\partial_i^2(i=1,2,3)$ to (1.1) and dotting them with $(\partial_i^2 u,\partial_i^2\theta)$ in L^2, one can obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\|(\partial_i^2 u,\partial_i^2\theta)\|_{L^2}^2+\sum_{i=1}^{3}\|\partial_i^2 u_h\|_{L^2}^2+\sum_{i=1}^{3}\|\partial_i^2\theta\|_{L^2}^2\\&=-\sum_{i=1}^{3}\int\partial_i^2(u\cdot\nabla u)\cdot\partial_i^2 u dx-\sum_{i=1}^{3}\int\partial_i^2(u\cdot\nabla\theta)\cdot\partial_i^2\theta dx\\&:=A_1+A_2.\end{aligned}$$

Due to Newton-Leibniz formula and the fact of $\nabla\cdot u=0$, it follows

$$\begin{aligned}A_1&=-\sum_{i=1}^{3}\int\partial_i^2 u\cdot\nabla u\cdot\partial_i^2 u dx-2\sum_{i=1}^{3}\int\partial_i u\cdot\partial_i\nabla u\cdot\partial_i^2 u dx\\&\le C\|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2\\&\le C\|u\|_{H^3}\|\nabla^2 u\|_{L^2}^2.\end{aligned} \tag{3.3}$$

By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}A_2&=-\sum_{i=1}^{3}\int\partial_i^2 u\cdot\nabla\theta\cdot\partial_i^2\theta dx-2\sum_{i=1}^{3}\int\partial_i u\cdot\partial_i\nabla\theta\cdot\partial_i^2\theta dx\\&\le C\sum_{i=1}^{3}(\|\nabla\theta\|_{L^\infty}\|\partial_i^2 u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla\partial_i\theta\|_{L^2})\|\nabla\partial_i\theta\|_{L^2}\\&\le C(\|u\|_{H^3}+\|\theta\|_{H^3})(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2).\end{aligned} \tag{3.4}$$

Applying $\partial_i^3 (i=1,2,3)$ to the equations (1.1) and taking the L^2-inner product of the resulting equations with $(\partial_i^3 u, \partial_i^3 b)$

$$
\begin{aligned}
&\sum_{i=1}^{3} \frac{1}{2}\frac{d}{dt}\|(\partial_i^3 u, \partial_i^3 \theta)\|_{L^2}^2 + \sum_{i=1}^{3}\|\partial_i^3 u_h\|_{L^2}^2 + \sum_{i=1}^{3}\|\partial_i^3 \theta\|_{L^2}^2 \\
&= \sum_{i=1}^{3}\int \partial_i^3 (u\cdot\nabla u)\cdot \partial_i^3 u \,\underset{\cdot}{\mathrm{x}} - \sum_{i=1}^{3}\int \partial_i^3 (u\cdot\nabla\theta)\cdot\partial_i^3\theta \,\underset{\cdot}{\mathrm{x}} \\
&:= B_1 + B_2.
\end{aligned}
$$

Due to Newton-Leibniz formula and the fact of $\nabla\cdot u = 0$, it follows

$$
\begin{aligned}
B_1 &= -\sum_{i=1}^{3}\int \partial_i^3 (u\cdot\nabla u)\cdot\partial_i^3 u \,\underset{\cdot}{\mathrm{x}} \\
&= -\sum_{i=1}^{3}\int (\partial_i^3 u\cdot\nabla u + 3\partial_i^2 u\cdot\nabla\partial_i u + 3\partial_i u\cdot\nabla\partial_i^2 u)\cdot\partial_i^3 u \,\underset{\cdot}{\mathrm{x}} \\
&\le C\sum_{i=1}^{3}(\|\nabla u\|_{L^\infty}\|\partial_i^3 u\|_{L^2} + \|\partial_i^2 u\|_{L^4}\|\partial_i\nabla u\|_{L^4})\|\partial_i^3 u\|_{L^2} \\
&\le C\|u\|_{H^3}^3.
\end{aligned}
\tag{3.5}
$$

By Hölder's inequality, we can get

$$
\begin{aligned}
B_2 &= -\sum_{i=1}^{3}\int \partial_i^3 (u\cdot\nabla\theta)\cdot\partial_i^3\theta \,\underset{\cdot}{\mathrm{x}} \\
&= -\sum_{i=1}^{3}\int (\partial_i^3 u\cdot\nabla\theta + 3\partial_i^2 u\cdot\nabla\partial_i\theta + 3\partial_i u\cdot\nabla\partial_i^2\theta)\cdot\partial_i^3\theta \,\underset{\cdot}{\mathrm{x}} \\
&\le C\sum_{i=1}^{3}(\|\nabla\theta\|_{L^\infty}\|\partial_i^3 u\|_{L^2} + \|\partial_i^2 u\|_{L^4}\|\partial_i\nabla\theta\|_{L^4} + \|\partial_i u\|_{L^\infty}\|\partial_i^2\nabla\theta\|_{L^2})\|\partial_i^3\theta\|_{L^2} \\
&\le C(\|u\|_{H^3} + \|\theta\|_{H^3})(\|u\|_{H^3}^2 + \|\theta\|_{H^3}^2).
\end{aligned}
\tag{3.6}
$$

Combining (3.1)-(3.4), (3.5), with (3.6) and integrating it over $[0,t]$ yields

$$
\begin{aligned}
&\|(u,\theta)(t)\|_{H^3}^2 + 2\int_0^t (\|u_h\|_{H^3}^2 + \|\theta\|_{H^3}^2)d\tau \\
&\le C\|(u_0,\theta_0)\|_{H^3}^2 + C\sup_{0\le\tau\le t}\|(u,\theta)\|_{H^3}\int_0^t (\|u\|_{H^3}^2 + \|\theta\|_{H^3}^2)d\tau \\
&\le C\mathcal{E}_1(0) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t).
\end{aligned}
$$

The proof of (1.4) is therefore complete. □

3.2 Proof of (1.5)

In order to establish the bound of $\mathcal{E}_2(t)$, we need the following special structure of equation $(1.1)_2$:

$$u_3 = -\partial_t\theta - u\cdot\nabla\theta - \theta.$$

Proof. First, multiplying $(1.1)_2$ by u_3 and integrate over $\mathbb{R}^3$, it follows

$$\begin{aligned}\|u_3\|_{L^2}^2 &= -\int \partial_t\theta \cdot u_3 dx - \int u_3(u\cdot\nabla\theta)dx - \int u_3\cdot\theta dx\\ &:= M_1 + M_2 + M_3.\end{aligned}$$

By integration by parts and applying the momentum equation in $(1.1)_1$,

$$\begin{aligned}M_1 &= \frac{d}{dt}\int \theta u_3 dx + \int \theta(\theta - \partial_3 P - u\cdot\nabla u_3)dx\\ &:= M_{11} + M_{12} + M_{13} + M_{14}.\end{aligned} \tag{3.7}$$

It is easily conclude that

$$\begin{aligned}M_{12} + M_{14} &= \int \theta(\theta - u\cdot\nabla u_3)dx\\ &\le \|\theta\|_{L^2}^2 + \|\theta\|_{L^\infty}\|u\|_{L^2}\|\nabla u_3\|_{L^2}\\ &\le \|\theta\|_{L^2}^2 + \|\theta\|_{H^2}(\|u\|_{L^2}^2 + \|\nabla u_3\|_{L^2}^2)\end{aligned} \tag{3.8}$$

Now, we need to estimate M_{13}, applying $\nabla\cdot$ to $(1.1)_1$, one can obtain

$$p = (-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u) - (-\Delta)^{-1}\partial_3 u_3 - (-\Delta)^{-1}\partial_3\theta. \tag{3.9}$$

Due to Hölder's inequality,

$$M_{13} = \int \theta\partial_3 p dx \le C\|\theta\|_{L^2}\|\partial_3 p\|_{L^2}, \tag{3.10}$$

where

$$\begin{aligned}\|\partial_3 p\|_{L^2} &\le \|(-\Delta)^{-1}\nabla\cdot\partial_3(u\cdot\nabla u)\|_{L^2} + \|(-\Delta)^{-1}\partial_3^2 u_3\|_{L^2} + \|(-\Delta)^{-1}\partial_3^2\theta\|_{L^2}\\ &:= M_{131} + M_{132} + M_{133}.\end{aligned}$$

Using the fact of Riesz operator $\partial_i(-\Delta)^{-1/2}$ with $i = 1, 2, 3$ is bounded in $L^r, 0<r<\infty$, one find

$$\begin{aligned}M_{131} &= \|(-\Delta)^{-1}\nabla\cdot\partial_3(u_1\partial_1 u + u_2\partial_2 u)\|_{L^2}\\ &\le C\|u\|_{L^\infty}\|\nabla u\|_{L^2}\\ &\le C\|u\|_{H^2}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}M_{132} + M_{133} &= \|(-\Delta)^{-1}\partial_3^2 u_3\|_{L^2} + \|(-\Delta)^{-1}\partial_3^2\theta\|_{L^2}\\ &\le C(\|\theta\|_{L^2} + \|u_3\|_{L^2}).\end{aligned}$$

Combining the estimates for M_{131}, M_{132} and M_{133}, we have

$$\|\partial_3 p\|_{L^2} \le C(\|\theta\|_{L^2} + \|u_3\|_{L^2} + \|u\|_{H^2}^2). \tag{3.11}$$

Putting (3.11) into (3.10), one can get

$$M_{13} \le C\|\theta\|_{L^2}^2 + \frac{1}{4}\|u_3\|_{L^2}^2 + \|\theta\|_{L^2}\|u\|_{H^2}^2. \tag{3.12}$$

Therefore, it follows from (3.7),(3.8) and (3.12),

$$M_1 \leq \frac{d}{dt}\int \theta u_3 dx + C\|\theta\|_{L^2}^2 + \frac{1}{4}\|u_3\|_{L^2}^2 + \|\theta\|_{H^2}\|u\|_{H^2}^2. \tag{3.13}$$

By Hölder's inequality and Young's inequality, one has

$$\begin{aligned} M_2 &= \int u \cdot \nabla\theta \cdot u_3 dx \\ &\leq C\|u\|_{L^\infty}\|\nabla\theta\|_{L^2}\|u_3\|_{L^2} \\ &\leq C\|u\|_{H^2}(\|\nabla\theta\|_{L^2}^2 + \|u_3\|_{L^2}^2), \end{aligned} \tag{3.14}$$

To bound M_3, by Young's inequality, we get

$$M_3 = -\int \theta u_3 dx \leq \frac{1}{4}\|u_3\|_{L^2}^2 + C\|\theta\|_{L^2}^2. \tag{3.15}$$

Combining the estimates (3.13)-(3.15) respectively, it follows

$$\|u_3\|_{L^2}^2 \leq 2\frac{d}{dt}\int \theta u_3 dx + C\|\theta\|_{L^2}^2 + (\|\theta\|_{H^2} + \|u\|_{H^2})(\|u\|_{H^2}^2 + \|\nabla\theta\|_{L^2}^2) \tag{3.16}$$

Next, applying $\partial_i (i = 1, 2, 3)$ to $(1.1)_1$ and dotting it with $\partial_i u_3$ in L^2, we can deduce that

$$\begin{aligned} \sum_{i=1}^{3}\|\partial_i u_3\|_{L^2}^2 &= -\sum_{i=1}^{3}\int \partial_t\partial_i\theta \cdot \partial_i u_3 dx - \sum_{i=1}^{3}\int \partial_i u_3 \partial_i(u \cdot \nabla\theta) dx - \sum_{i=1}^{3}\int \partial_i u_3 \cdot \partial_i\theta dx \\ &:= N_1 + N_2 + N_3. \end{aligned}$$

Applying the struture of equation $(1.1)_2$ and integration by parts,

$$\begin{aligned} N_1 &= \sum_{i=1}^{3}\frac{d}{dt}\int \partial_i\theta\partial_i u_3 dx + \sum_{i=1}^{3}\int \partial_i\theta\partial_i(\theta - \partial_3 P - u \cdot \nabla u_3) dx \\ &:= N_{11} + N_{12} + N_{13} + N_{14}. \end{aligned} \tag{3.17}$$

Using Hölder's inequality and Young's inequality to get

$$\begin{aligned} N_{12} + N_{14} &= \sum_{i=1}^{3}\int \partial_i\theta\partial_i(\theta - u \cdot \nabla u_3) dx \\ &\leq \sum_{i=1}^{3}\|\partial_i\theta\|_{L^2}^2 + \sum_{i=1}^{3}\|\partial_i\theta\|_{L^\infty}(\|u\|_{L^2}\|\partial_i\nabla u_3\|_{L^2} + \|\partial_i u\|_{L^2}\|\nabla u_3\|_{L^2}) \\ &\leq \|\nabla\theta\|_{L^2}^2 + C\|\theta\|_{H^3}\|u\|_{H^2}^2 \end{aligned} \tag{3.18}$$

Due to Hölder's inequality,

$$N_{13} = \sum_{i=1}^{3}\int \partial_i\theta\partial_i\partial_3 p dx \leq C\sum_{i=1}^{3}\|\partial_i\theta\|_{L^2}\|\partial_i\partial_3 p\|_{L^2}, \tag{3.19}$$

where

$$\begin{aligned} \|\partial_i\partial_3 p\|_{L^2} &\leq \|(-\Delta)^{-1}\nabla \cdot \partial_i\partial_3(u \cdot \nabla u)\|_{L^2} + \|(-\Delta)^{-1}\partial_i\partial_3^2 u_3\|_{L^2} + \|(-\Delta)^{-1}\partial_i\partial_3^2\theta\|_{L^2} \\ &:= N_{131} + N_{132} + N_{133}. \end{aligned}$$

Using the fact of Riesz operator $\partial_i(-\Delta)^{-1/2}$ with $i = 1, 2, 3$ is bounded in $L^r, 0<r<\infty$, one find

$$\begin{aligned} N_{131} &= \|(-\Delta)^{-1} \cdot \partial_i\partial_3\partial_j(u_k\partial_k u_j)\|_{L^2} \\ &\le C\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2} \\ &\le C\|u\|_{H^3}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} N_{132} + N_{133} &= \|(-\Delta)^{-1}\partial_3^2\partial_i u_3\|_{L^2} + \|(-\Delta)^{-1}\partial_3^2\partial_i\theta\|_{L^2} \\ &\le C(\|\nabla\theta\|_{L^2} + \|\nabla u_3\|_{L^2}). \end{aligned}$$

Combining the estimates for N_{131}, N_{132} and N_{133}, we have

$$\|\partial_i\partial_3 p\|_{L^2} \le C(\|\nabla\theta\|_{L^2} + \|\nabla u_3\|_{L^2} + \|u\|_{H^3}^2). \tag{3.20}$$

Putting (3.20) into (3.19), one can get

$$N_{13} \le C\|\nabla\theta\|_{L^2}^2 + \frac{1}{4}\|\nabla u_3\|_{L^2}^2 + \|\nabla\theta\|_{L^2}\|u\|_{H^3}^2. \tag{3.21}$$

Therefore, it follows from (3.21) and (3.18),

$$N_1 \le \sum_{i=1}^{3} \frac{d}{dt}\int \partial_i\theta\partial_i u_3 dx + C\|\nabla\theta\|_{L^2}^2 + \frac{1}{4}\|\nabla u_3\|_{L^2}^2 + \|\theta\|_{H^3}\|u\|_{H^3}^2. \tag{3.22}$$

By Hölder's inequality and Young's inequality, one has

$$\begin{aligned} N_2 &= \sum_{i=1}^{3}\int \partial_i(u\cdot\nabla\theta)\partial_i u_3 dx \\ &\le C\|\nabla u_3\|_{L^\infty}(\|\nabla\theta\|_{L^2}\|\nabla u_3\|_{L^2} + \|\nabla^2\theta\|_{L^2}\|u\|_{L^2}) \\ &\le C\|u\|_{H^3}(\|\nabla\theta\|_{H^1}^2 + \|u\|_{H^1}^2), \end{aligned} \tag{3.23}$$

To bound N_3, by Young's inequality, we get

$$N_3 = -\sum_{i=1}^{3}\int \partial_i\theta\partial_i u_3 dx \le \frac{1}{4}\|\nabla u_3\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2. \tag{3.24}$$

Combining the estimates (3.22)-(3.24) respectively, it follows

$$\|\nabla u_3\|_{L^2}^2 \le 2\frac{d}{dt}\int \nabla\theta\nabla u_3 dx + C\|\nabla\theta\|_{L^2}^2 + (\|\theta\|_{H^3} + \|u\|_{H^3})(\|u\|_{H^3}^2 + \|\theta\|_{H^2}^2) \tag{3.25}$$

Similarly,

$$\|\nabla^2 u_3\|_{L^2}^2 \le 2\frac{d}{dt}\int \nabla^2\theta\nabla^2 u_3 dx + C\|\nabla^2\theta\|_{L^2}^2 + (\|\theta\|_{H^3} + \|u\|_{H^3})(\|u\|_{H^3}^2 + \|\theta\|_{H^3}^2) \tag{3.26}$$

Finally, applying $\partial_i^3 (i = 1, 2, 3)$ to $(1.1)_1$ and dotting it with $\partial_i^3 u_3$ in L^2, we can deduce that

$$\begin{aligned} \sum_{i=1}^{3}\|\partial_i^3 u_3\|_{L^2}^2 &= -\sum_{i=1}^{3}\int \partial_t\partial_i^3\theta\cdot\partial_i^3 u_3 dx - \sum_{i=1}^{3}\int \partial_i^3 u_3\partial_i^3(u\cdot\nabla\theta)dx - \sum_{i=1}^{3}\int \partial_i^3 u_3\cdot\partial_i^3\theta dx \\ &:= Q_1 + Q_2 + Q_3. \end{aligned}$$

Applying the structure of equation $(1.1)_2$ and integration by parts,

$$\begin{aligned} Q_1 &= \sum_{i=1}^{3} \frac{d}{dt}\int \partial_i^3\theta\partial_i^3 u_3 dx + \sum_{i=1}^{3}\int \partial_i^3\theta\partial_i^3(\theta - \partial_3 P - u\cdot\nabla u_3)dx \\ &:= Q_{11} + Q_{12} + Q_{13} + Q_{14}. \end{aligned} \tag{3.27}$$

Using Hölder's inequality and Young's inequality to get

$$\begin{aligned} Q_{12} + Q_{14} &= \sum_{i=1}^{3}\int \partial_i^3\theta\partial_i^3(\theta - u\cdot\nabla u_3)dx \\ &= \sum_{i=1}^{3}\int \partial_i^3\theta\partial_i^3\theta - \sum_{i=1}^{3}\int \partial_i^3\theta(\partial_i^3 u\cdot\nabla u_3 + 3\partial_i^2 u\cdot\nabla\partial_i u_3 + 3\partial_i u\cdot\nabla\partial_i^2 u_3 + u\cdot\nabla\partial_i^3 u_3)dx \\ &\le \sum_{i=1}^{3}\|\partial_i^3\theta\|_{L^2}^2 + \sum_{i=1}^{3}\|\partial_i^3\theta\|_{L^2}(\|\nabla u_3\|_{L^2}\|\partial_i^3 u\|_{L^2} + \|\partial_i^2 u\|_{L^4}\|\nabla\partial_i u_3\|_{L^4}) \\ &\quad - \sum_{i=1}^{3}\int \partial_i^3\theta u\cdot\nabla\partial_i^3 u_3 dx \\ &\le C\|\nabla^3\theta\|_{L^2}^2 + C\|\theta\|_{H^3}\|u\|_{H^3}^2 - \sum_{i=1}^{3}\int \partial_i^3\theta u\cdot\nabla\partial_i^3 u_3 dx \end{aligned} \tag{3.28}$$

Due to Hölder's inequality,

$$Q_{13} = \sum_{i=1}^{3}\int \partial_i^3\theta\partial_i^3\partial_3 p dx \le C\sum_{i=1}^{3}\|\partial_i^3\theta\|_{L^2}\|\partial_i^3\partial_3 p\|_{L^2}, \tag{3.29}$$

where

$$\begin{aligned} \|\partial_i^3\partial_3 p\|_{L^2} &\le \|(-\Delta)^{-1}\nabla\cdot\partial_i^3\partial_3(u\cdot\nabla u)\|_{L^2} + \|(-\Delta)^{-1}\partial_i^3\partial_3^2 u_3\|_{L^2} + \|(-\Delta)^{-1}\partial_i^3\partial_3^2\theta\|_{L^2} \\ &:= Q_{131} + Q_{132} + Q_{133}. \end{aligned}$$

Using the fact of Riesz operator $\partial_i(-\Delta)^{-1/2}$ with $i = 1,2$ is bounded in $L^r, 0<r<\infty$, one find

$$\begin{aligned} Q_{131} &= \|(-\Delta)^{-1}\partial_i^3\partial_1\partial_j(u_k\partial_k u_j)\|_{L^2} \\ &= \|\partial_i^2(\partial_j u_k\partial_k u_j)\|_{L^2} \\ &= \|\partial_i^2\partial_j u_k\partial_k u_j\|_{L^2} + \|\partial_i\partial_j u_k\partial_k\partial_i u_j\|_{L^2} + \|\partial_j u_k\partial_k\partial_i^2 u_j\|_{L^2} \\ &\le C(\|\nabla u\|_{L^\infty}\|\nabla u\|_{H^2} + \|\nabla^2 u\|_{L^4}^2) \\ &\le C\|u\|_{H^3}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} Q_{132} + Q_{133} &= \|(-\Delta)^{-1}\partial_3^2\partial_i^3 u_3\|_{L^2} + \|(-\Delta)^{-1}\partial_3^2\partial_i^3\theta\|_{L^2} \\ &\le C(\|\nabla^3\theta\|_{L^2} + \|\nabla^3 u_3\|_{L^2}). \end{aligned}$$

Combining the estimates for Q_{131}, Q_{132} and Q_{133}, we have

$$\|\partial_i^3\partial_3 p\|_{L^2} \le C(\|\nabla^3\theta\|_{L^2} + \|\nabla^3 u_3\|_{L^2} + \|u\|_{H^3}^2). \tag{3.30}$$

Putting (3.30) into (3.29), one can get

$$Q_{13} \le C\|\nabla^3\theta\|_{L^2}^2 + \frac{1}{4}\|\nabla^3 u_3\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}\|u\|_{H^3}^2. \tag{3.31}$$

Therefore, it follows from (3.27), (3.28) and (3.31),

$$Q_1 \le \sum_{i=1}^{3}\frac{d}{dt}\int \partial_i^3\theta\partial_i^3 u_3 dx + C\|\nabla^3\theta\|_{L^2}^2 + \frac{1}{4}\|\nabla^3 u_3\|_{L^2}^2 + \|\theta\|_{H^3}\|u\|_{H^3}^2 - \sum_{i=1}^{3}\int \partial_i^3\theta u\cdot\nabla\partial_i^3 u_3 dx.$$

By Hölder's inequality and Young's inequality, one has

$$\begin{aligned}
Q_2 &= -\sum_{i=1}^{3}\int \partial_i^3(u\cdot\nabla\theta)\partial_i^3 u_3 dx \\
&= \sum_{i=1}^{3}\int \partial_i^3 u\cdot\nabla\theta\partial_i^3 u_3 dx - \sum_{i=1}^{3}\int \partial_i^2 u\cdot\nabla\partial_i\theta\partial_i^3 u_3 dx \\
&\quad - \sum_{i=1}^{3}\int \partial_i u\cdot\partial_i^2\nabla\theta\partial_i^3 u_3 dx - \sum_{i=1}^{3}\int u\cdot\nabla\partial_i^3\theta\partial_i^3 u_3 dx \\
&\le C(\|\nabla\theta\|_{L^\infty}\|\nabla^3 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^4}\|\nabla^2 u\|_{L^4}\|\nabla^3 u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}\|\nabla^3 u\|_{L^2}) \\
&\quad - \sum_{i=1}^{3}\int u\cdot\nabla\partial_i^3\theta\partial_i^3 u_3 dx \\
&\le C(\|u\|_{H^3} + \|\theta\|_{H^3})(\|\theta\|_{H^3}^2 + \|u\|_{H^3}^2) - \sum_{i=1}^{3}\int u\cdot\nabla\partial_i^3\theta\partial_i^3 u_3 dx,
\end{aligned}$$

where

$$-\sum_{i=1}^{3}\int u\cdot\nabla\partial_i^3 u_3\partial_i^3\theta dx - \sum_{i=1}^{3}\int u\cdot\nabla\partial_i^3\theta\partial_i^3 u_3 dx = 0$$

Therefore

$$Q_1 + Q_2 \le \sum_{i=1}^{3}\frac{d}{dt}\int \partial_i^3\theta\partial_i^3 u_3 dx + C\|\nabla^3\theta\|_{L^2}^2 + \frac{1}{4}\|\nabla^3 u_3\|_{L^2}^2 + C(\|u\|_{H^3} + \|\theta\|_{H^3})(\|\theta\|_{H^3}^2 + \|u\|_{H^3}^2). \tag{3.32}$$

To bound Q_3, by Young's inequality, we get

$$Q_3 = -\sum_{i=1}^{3}\int \partial_i^3\theta\partial_i^3 u_3 dx \le \frac{1}{4}\|\nabla^3 u_3\|_{L^2}^2 + C\|\nabla^3\theta\|_{L^2}^2. \tag{3.33}$$

Combining the estimates (3.32)and(3.33) respectively, it follows

$$\|\nabla^3 u_3\|_{L^2}^2 \le \sum_{i=1}^{3}\frac{d}{dt}\int \partial_i^3\theta\partial_i^3 u_3 dx + C\|\nabla^3\theta\|_{L^2}^2 + C(\|u\|_{H^3} + \|\theta\|_{H^3})(\|\theta\|_{H^3}^2 + \|u\|_{H^3}^2). \tag{3.34}$$

Combining (3.16),(3.25), (3.26), with (3.34) and integrating it over $[0,t]$ yields

$$\begin{aligned}
\int_0^t \|u_3\|_{H^3}^2 &\le C\|(u_0,\theta_0)\|_{H^3}^2 + C\|(u,\theta)(t)\|_{H^3}^2 + \int_0^t \|\theta\|_{H^3}^2 d\tau \\
&\quad + C\sup_{0\le\tau\le t}\|(u,\theta)\|_{H^3}\int_0^t(\|u\|_{H^3}^2 + \|\theta\|_{H^3}^2)d\tau \\
&\le C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t).
\end{aligned}$$

which implies (1.5). □

3.3 Proof of Theorem 1.1

Now, we use bootstrapping argument to prove Theorem 1.1. From above subsection, we have deduced that

$$\mathcal{E}_1(t) \leq C\mathcal{E}_1(0) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t), \tag{3.35}$$

$$\mathcal{E}_2(t) \leq C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t). \tag{3.36}$$

For any $t>0$, adding (3.36) to (3.35) by the appropriate constant obtains,

$$\mathcal{E}(t) \leq C\mathcal{E}(0) + C\mathcal{E}^{3/2}(t), \tag{3.37}$$

where $\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t)$, and $C>0$ is a pure constant. Let

$$\|(u_0, b_0)\|_{H^3}^2 \leq \frac{1}{16C^3}.$$

The bootstrapping argument starts with the ansatz that

$$\mathcal{E}(t) \leq \frac{1}{4C^2}.$$

It follows from (3.37) that

$$\mathcal{E}(t) \leq C\mathcal{E}(0) + C\mathcal{E}^{1/2}(t)\mathcal{E}(t) \leq C\mathcal{E}(0) + C\frac{1}{2C}\mathcal{E}(t) = C\mathcal{E}(0) + \frac{1}{2}\mathcal{E}(t),$$

then,

$$\mathcal{E}(t) \leq 2C\mathcal{E}(0).$$

The bootstrapping argument then implies that, for any $t \geq 0$,

$$\mathcal{E}(t) \leq \frac{1}{8C^2}.$$

Therefore, we finish the proof of Theorem 1.1.

4 The proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. First of all, we establish the decay results in (1.6). Due to $\nabla \cdot u = 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla u, \nabla\theta)\|_{L^2}^2 + \|\nabla u_h\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 \\
&\quad = \int \nabla\theta \cdot \nabla\theta \cdot \nabla u dx - \int \nabla u \cdot \nabla\theta \cdot \nabla\theta dx \\
&\qquad + \int \nabla\theta \cdot \nabla u \cdot \nabla\theta dx \\
&\quad \leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}^2 \\
&\quad \leq C\|u\|_{H^3}\|\nabla\theta\|_{L^2}^2.
\end{aligned} \tag{4.1}$$

For any $0 \leq s \leq t<\infty$, integrating (4.1) in time, by the upper bound in (1.2), that is

$$\|(\nabla u, \nabla\theta)(t)\|_{L^2}^2 \leq C\|(\nabla u, \nabla\theta)(s)\|_{L^2}^2. \tag{4.2}$$

The two conditions provided by (4.2) and (1.2), we can get (1.6) according to Lemma 2.2.

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