# ON SOLUTION OF SINGULAR CAUCHY PROBLEM OF GENERALIZED EULER POISSON DARBOUX EQUATION USING EXTENDED $K$-INTEGRAL TRANSMUTATION COMPOSITION METHOD.

**Abstract**

This study solves the initial value problem for the Generalized Euler-Poisson-Darboux (GEPD) equation using Extended $k$-integral transmutation composition method (E-$k$ITCM). In E-$k$ITCM, the study intends to use the integral transform approach to determine the Extended $k$-transmutation operator. The Extended $k$-special functions are crucial in representing the solution to the GEPD equation. The use of Extended $k$-functions is due to the involvement of geralized special functions. Overall, the study will have a significant impact in the areas of wave propagation and special functions theory, which is of interest to specialists in Mathematics and Physics.

**Keywords:** Integral Transform Composition Method, Singular Cauchy Generalized Euler-Poisson-Darboux equation, Extended $k-$Beta and Extended $k-$Gamma function,Transmutation operator, Extended $k$-Transmutation Operator, Hankel Transform.

## 1. INTRODUCTION

The Euler-Poisson-Darboux (EPD) equation is a second-order partial differential equation (PDE) that plays a crucial role in various fields of applied mathematics. It arises in numerous problems involving wave propagation and is widely applicable in areas such as gas dynamics, hydrodynamics, acoustics, electromagnetism, geophysics, quantum mechanics, elasticity, and plasticity, among others..

The classical $n$-dimensional EPD equation takes the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\omega}{t}\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \quad u \in \mathbb{R}^n \times \{t > 0\}. \tag{1}$$

Where in the EPD equation (1), $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a point in $\mathbb{R}^n$, $\omega$ is a real-valued parameter, $t$ denotes the time variable, and $u(x,t)$ represents the unknown function to be determined. Equation (1) is called a Cauchy problem when subjected to the initial conditions (2).

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \quad u \in \mathbb{R}^n \times \{t = 0\}. \tag{2}$$

Equation (1) is referred to as Singular when the coefficient of the associated operator $\frac{\partial u}{\partial t} \to \infty$ as $t \to 0$. On multiplying equation (1) by $t$ on both sides and letting $t \to 0$, it degenerates to the first-order equation $\frac{\partial u}{\partial t} = 0$. With initial conditions (2), equation (1- 2) is referred to as the singular Cauchy problem of EPD type. When $\omega = 0$, equation (1) coincides with the standard wave equation in $n$-dimension.

The singular Cauchy problem (1-2) has been extensively explored in $n$-dimensional space and for general values of parameter $\omega$ since the foundational work of [1]. Variants of the equation (1) with diverse applications in fields such as gas dynamics, hydrodynamics, mechanics, elasticity, and plasticity have also been investigated in [2, 3, 4].

For the special case where $n = 1$, equation (1) first appeared in the work of [5, 6, 7, 8].

This study solves Generalized Euler-Poisson-Darboux (GEPD) type of equation defined by (3):

$$\frac{\partial^2 u}{\partial t^2} + \frac{\omega}{t}\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \left( \frac{\partial^2 u}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i} \right), \quad u \in \mathbb{R}^n \times \{t \in \mathbb{R}^+\}. \tag{3}$$

where $u = u(x,t)$, $\omega \in \mathbb{R}$, $t \in \mathbb{R}^+$, $x = (x_1, \ldots, x_n)$, $x_i > 0$, $\gamma_i > 0$, $i = 1, 2, \ldots, n$.
With initial conditions given as:

$$u(x,0) = f(x), \quad u_t(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \quad u \in \mathbb{R}^n \times \{t = 0\}. \tag{4}$$

The operator acting with respect to the time variable $t$ is the Singular Bessel operator and is defined by (5)

$$(B_\omega)_t = \frac{\partial^2}{\partial t^2} + \frac{\omega}{t}\frac{\partial}{\partial t}, \quad t > 0; \quad \omega = \text{real parameter}. \tag{5}$$

The operator acting with respect to the spatial variable $x$ is referred to as the Elliptical Singular-Bessel operator and is defined by (6)

$$(\Delta_\gamma)_x = \sum_{i=1}^{n} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i} \right), \quad \gamma_i = \text{real parameter}. \tag{6}$$

It is well established that fundamental solutions to singular Cauchy partial differential equations often involve hypergeometric functions [9], owing to their appropriate behavior near singular points. Inspired by this observation, recent developments have introduced various generalized forms of classical special functions (such as the Beta and Gamma functions), incorporating additional parameters.

These generalized forms are commonly referred to as either $k$-special functions or extended special functions. A $k$-special function generalizes classical special functions by introducing a new parameter $k$, while an extended $k$-special function represents a further extension by including additional parameters (e.g., $b$), thus offering greater flexibility and broader applicability.

Consequently, representing solutions to differential equations using these extended special functions is of significant interest, as the essential properties of the original special functions are retained in their generalizations.

Therefore, to carry out this study, a solid foundation in special functions is essential. The following subsection provides a brief introduction to several classical special functions and their known generalizations. In addition, an overview of the Hankel and integral transmutation composition method is also presented.

## 1.1 Special Functions Overview

A special function is a function, either real or complex, that depends on one or more real or complex variables and is well-defined, such that its numerical values can be tabulated. In this context the adjective *special* is used because the fundamental properties of functions are not concerned but only the properties of functions which arise in the solution of specific problems are put into considerations.

### 1.1.1 Gamma Function and Its Generalization

The integral representation of the classical Gamma function, which generalizes the factorial function, is defined as:

$$\Gamma(n) = (n-1)! = \int_0^\infty t^{n-1} e^{-t}\, dt, \qquad \Re(n) > 0, \tag{7}$$

For $k > 1$ and $\Re(n) > 0$, Gamma function satisfies the following recursion property

$$\Gamma(n+k) = n(n+1)(n+2)\ldots(n+(k-1))\Gamma(n) \tag{8}$$

[10] extended the domain of (7) to the entire complex plane by inserting a regularization factor $e^{-bt}$ in the integrand of (7) to get,

$$\Gamma_b(n) = \int_0^\infty t^{n-1} \exp\left( -t - \frac{b}{t} \right) dt, \quad \Re(b) \geq 0, \Re(n) > 0, \tag{9}$$

and for $b = 0$ eqn (9) reduces to the original Gamma function in (7).

Using a step size of $k$, the integral representation of the $k$-Gamma function $\Gamma_k$ is defined by:

$$\Gamma_k(n) = \int_0^\infty t^{n-1} e^{-\frac{t^k}{k}}\, dt, \quad k \in \mathbb{R}^+,\, \Re(n) > 0. \tag{10}$$

and for $k = 1$ eqn (10) reduces to the original Gamma function in (7). By applying the substitution $u = \frac{t^k}{k}$ in equation (10), the relationship between the $k$-Gamma function $\Gamma_k(n)$, as defined in equation (10), and the classical Gamma function $\Gamma(n)$, given in equation (7), is established and is expressed by equation (11).

$$\Gamma_k(n) = k^{(\frac{n}{k}-1)}\Gamma(\frac{n}{k}), \qquad k \in \mathbb{R}^+,\, \Re(n) > 0. \tag{11}$$

Similarly the $k$-Gamma function satisfies the recursion formula (12):

$$\Gamma_k(a+k) = a\Gamma_k(a), \qquad \Re(a) > 0 \tag{12}$$

This study defines the Extended $k$-Gamma function $\Gamma_{k;b}(n)$ as:

$$\Gamma_{k;b}(n) = \int_0^\infty t^{n-1} \exp\left( -\frac{t^k}{k} - \frac{kb^k}{t^k} \right) dt, \quad k \in \mathbb{R}^+,\, \Re(b) \geq 0,\, \Re(n) > 0. \tag{13}$$

and for $k = 1$ equation (13) simplifies to equation (9), the Extended Gamma function $\Gamma_b(n)$. On transformation by setting $u = \frac{t^k}{k}$ in equation (13), the relationship between the Extended Gamma function $\Gamma_b(n)$ in (9) and the Extended $k$-Gamma function $\Gamma_{k;b}(n)$ in (13) is achieved and is defined by (14):

$$\Gamma_{k;b}(n) = k^{(\frac{n}{k}-1)}\Gamma_b(\frac{n}{k}), \qquad k \in \mathbb{R}^+,\, \Re(b) \geq 0,\, \Re(n) > 0. \tag{14}$$

### 1.1.2 Beta Function and Its Generalization

In terms of the familiar Gamma function, the classical Beta function is defined by:

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \int_0^1 t^{n-1}(1-t)^{m-1}\, dt. \tag{15}$$

Equation (15) was later extended by the same authors [11] as

$$B_b(n,m) = \int_0^1 t^{n-1}(1-t)^{m-1} \exp\left( \frac{-b}{t(1-t)} \right) dt, \quad \Re(b) \geq 0. \tag{16}$$

[11] managed to provide Product formula relation between this Extended Gamma function $\Gamma_b(x)$, defined in (9) and Extended Beta function $B_b(n,m)$ defined in (16) given by:

$$\Gamma_b(n)\Gamma_b(m) = 2\int_0^\infty t^{2(n+m)-1} e^{-t^2} B_{\frac{b}{t^2}}(n,m)\, dt, \quad \Re(b) > 0, \Re(n) > 0, \Re(m) > 0. \tag{17}$$

When $b = 0$ is applied to Equation (17), it yields the classical relation between the Gamma and Beta function. With respect to the $k$-Gamma function (10), the $k$-Beta function $B_k(n,m)$ is defined by:

$$B_k(n,m) = \frac{\Gamma_k(n)\Gamma_k(m)}{\Gamma_k(n+m)} = \frac{1}{k}\int_0^1 t^{\frac{n}{k}-I}(1-t)^{\frac{m}{k}-I}\, dt, \quad k \in \mathbb{R}^+; \Re(n), \Re(m) > 0. \tag{18}$$

and for $k = 1$ eqn (18) reduces to the original Beta function in (15).

[12] introduced the Extended $k$-Beta function defined as:

$$B_{b,k}(n,m) = \frac{1}{k}\int_0^1 t^{\frac{n}{k}-1}(1-t)^{\frac{m}{k}-1}\exp\left(-\frac{b^k}{kt(1-t)}\right)dt, \tag{19}$$

where $k > 0$, $\Re(b) > 0$, $\Re(n) > 0$, $\Re(m) > 0$.

Given that when $b = 0$, $B_{b,k}(n,m)$ reduces to $B_k(n,m)$ in (18). If $k = 1$, $B_{b,k}(n,m)$ approaches $B_b(n,m)$ defined in (16).Furthermore, if both $k = 1$ and $b = 0$, then $B_{b;k}(n,m)$ reduces to Beta function $B(n,m)$ in equation (15).

The below integral representation of the Extended $k$-Beta function $B_{b,k}(n,m)$ , holds true

$$\int_0^\infty p^{s-1}B_{b,k}(n,m)\,dp = \Gamma_k(s)B_k(n+s,m+s), \tag{20}$$

where $\Re(s) > 0$, $\Re(n+s) > 0$, and $\Re(m+s) > 0$.

### 1.1.3 Pochhammer's symbol and Its Generalization

EThe Pochhammer symbol(or shifted/increasing factorial) $(\alpha)_n$ related to the Gamma function $\Gamma(n)$ is defined by:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \frac{(\alpha+n-1)!}{(\alpha-1)!} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & \text{if } n\in\mathbb{N}^+,\ \alpha\in\mathbb{C}, \\ 1 & \text{if } n=0,\ \alpha\in\mathbb{C}\setminus\{0\}. \end{cases} \tag{21}$$

and for $n, k \in \mathbb{N}^+$, the $k$-Pochhammer symbol is defined as

$$(\alpha)_{nk} = \prod_{i=0}^{n-1}(\alpha+ik) = \alpha(\alpha+k)(\alpha+2k)\cdots(\alpha+(n-1)k) \tag{22}$$

The relationship between the $k$-Pochhammer symbol and the $k$-Gamma function defined in equation (10), is expressed by the following relation:

$$(\alpha)_{nk} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)} \tag{23}$$

The Extended Pochhammer symbol $(\alpha)_{n,b}$ is defined by:

$$(\alpha)_{n,b} = \begin{cases} \frac{\Gamma_b(\alpha+n)}{\Gamma(\alpha)} & \text{if } \Re(b) > 0,\ \alpha,n\in\mathbb{C}, \\ (\alpha)_n & \text{if } b=0;\ \alpha,n\in\mathbb{C}\setminus\{0\}. \end{cases} \tag{24}$$

Here, $\Gamma_b$ denotes the Extended Gamma function, as introduced in equation (9). This yields the following integral representation for the Extended Pochhammer symbol $(\alpha)_{n,b}$, which is defined by:

$$(\alpha)_{n,b} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha+n-1}\exp\left(-t-\frac{b}{t}\right)dt, \quad \Re(b)\geq 0;\ \alpha,n\in\mathbb{C}. \tag{25}$$

### 1.1.4 Hypergeometric function and Its Generalization

The classical Gauss hypergeometric function is defined as follows:

$${}_2F_1(\alpha,\beta;c;x) = \sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n}{(c)_n}\frac{x^n}{n!}, \qquad |x|<1. \qquad \text{where } (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}. \tag{26}$$

The integral expression of Gauss hypergeometric function is as follows:

$$ {}_2F_1(\alpha,\beta;c;x) = \frac{\Gamma(c)}{\Gamma(\beta)\Gamma(c-\beta)} \int_0^1 t^{b'-1}(1-t)^{c-\beta-1}(1-xt)^{-\alpha}\, dt, \quad \Re(c) > \Re(\beta) > 0, |x| < 1. \tag{27} $$

Using Extended Beta function $B_b(m,n)$ (16), the Extended Gauss hypergeometric function is defined by:

$$ {}_2F_{1;b}(\alpha,\beta;c;x) = \frac{\Gamma(c)}{\Gamma(b')\Gamma(c-\beta)} \sum_{n=0}^{\infty} (\alpha)_n B_b(\beta+n, c-\beta) \frac{x^n}{n!}, \quad |x| < 1, \Re(c) > \Re(\beta) > 0, |x| < 1. \tag{28} $$

Using (22), the $k$-Gauss hypergeometric function is defined by:

$$ {}_2F_{1;k}(\alpha,\beta;c;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_{nk}(\beta)_{nk}}{(c)_{nk}} \frac{x^n}{n!}, \quad n \in \mathbb{N}_0, k \geq 1, \Re(c) > \Re(\beta) > 0, |x| < 1. \tag{29} $$

Using equation (19), the Extended $k$-Gauss hypergeometric function is defined as follows:

$$ {}_2F_{1;b,k}(\alpha,\beta;c;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n B_{b,k}(\beta+n, c-b')}{B(\beta, c-\beta)} \frac{x^n}{n!}, \quad k \geq 1,\ b \geq 0, |x| < 1, \Re(c) > \Re(\beta) > 0. \tag{30} $$

It follows that when $b = 0$, the function ${}_2F_{1;b,k}(\alpha,\beta;c;x)$ simplifies to ${}_2F_{1;k}(\alpha,\beta;c;x)$, as given in equation (29). Similarly, if $k = 1$, the function ${}_2F_{1;b,k}(\alpha,\beta;c;x)$ reduces to ${}_2F_{1;b}(\alpha,\beta;c;x)$, which is defined in equation (28). Furthermore, when both parameters $b = 0$ and $k = 1$, ${}_2F_{1;b,k}(\alpha,\beta;c;x)$ coincides with the classical Gauss hypergeometric function ${}_2F_1(\alpha,\beta;c;x)$, as shown in equation (26).

To this extent, it is evident that the use of generalized special functions to extend hypergeometric and other special functions is essential, given their significant applications in a wide range of real-world problems in Mathematics and Physics that involve complex functions with intricate coefficients.

#### 1.1.5 The Bessel function and The Hankel transform

The Bessel function of the first kind of order $\nu$, denoted by $J_\nu(x)$, is defined by the following power series expansion centered at $x = 0$:

$$ J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu} \qquad \nu \in \mathbb{N}_0. \tag{31} $$

Using the relationship between the $k$-Gamma function $\Gamma_k(n)$ in (10) and the Gamma function $\Gamma(n)$ in (7) defined by (11), the $k-$Bessel function of the first kind of order $\nu$, denoted by $J_{\nu,k}(xk)$, is defined by (32).

$$ J_{\nu,k}(xk) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma_k((n+1)k)\,\Gamma_k((n+\nu+1)k)} \left(\frac{xk}{2}\right)^{2n+\nu} \qquad k \in \mathbb{R}^+, \nu \in \mathbb{N}_0. \tag{32} $$

Let $\mathcal{G}$ denote the set of smooth functions characterized by rapid decay on the interval $(0,\infty)$. This set is defined by:

$$ \mathcal{G} = \left\{ f \in C^\infty(0,\infty) : \sup_{x\in(0,\infty)} \left| x^\beta D^\alpha f(x) \right| < \infty,\ \forall\, \alpha, \beta \in \mathbb{Z}_+ \right\} \tag{33} $$

Given a function $f \in \mathcal{G}$, its Hankel transform of order $\nu$ is defined by:

$$ \mathcal{H}_\nu[f](x) = \hat{f}(x) = \int_0^\infty j_{\frac{\nu-1}{2}}(xt)\, f(t)\, t^\nu\, dt, \quad \nu \in \mathbb{N}_0. \tag{34} $$

where the function $j_\nu(t)$ is expressed as:

$$j_\nu(t) = \frac{2^\nu \Gamma(\nu+1) J_\nu(t)}{t^\nu} \tag{35}$$

The inverse Hankel transform takes the form:

$$\mathcal{H}_\nu^{-1}[\hat{f}](x) = f(x) = 2^{1-\nu}\Gamma^{-2}\left(\frac{\nu+1}{2}\right)\int_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,\hat{f}(\xi)\,\xi^\nu\,d\xi \tag{36}$$

To this extent, it has been outlined clearly that the general idea of using the generalized special functions in generalizing hypergeometric functions and other special functions cannot be neglected as they have important applications in considerable range of real-world problems in areas of Mathematics and Physics that require more complex functions of complicated coefficients.

#### 1.1.6 Overview of the Integral Transmutation Composition Method.

Consider a pair of arbitrary operators $A$ and $B$, along with the associated Fourier transforms $F_A$ and $F_B$, which are invertible and operate according to the expressions:

$$F_A A = g(t) F_A, \quad F_B B = g(t) F_B, \tag{37}$$

where $t$ is a dual variable and $g$ represents an arbitrary function with appropriate properties.

The core idea of ITCM is to derive a transmutation operator $T$ or $S$ using the following formulas:

$$S = F_B^{-1}\frac{1}{w(t)}F_A, \quad T = F_A^{-1} w(t) F_B \tag{38}$$

where $w(t)$ is an arbitrary function. If $S$ and $T$ are the required transmutation operators interconnecting $A$ and $B$, they must satisfy:

$$SA = BS, \quad TB = AT. \tag{39}$$

## 2. METHODOLOGY

In this section, the Singular Cauchy GEPD equation (3 - 4) is reduced into two independent variables, with a pair of Singular Bessel operators.

### 2.1 The GEPD equation in Two independent variables.

The Singular Cauchy GEPD equation is expressed as:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\omega}{t}\frac{\partial u}{\partial t} - \sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i}\right) = 0, \quad u \in \mathbb{R}^n \times \{t > 0\}. \tag{40}$$

where $u = u(x, t; \omega)$, $\omega \in \mathbb{R}$, $t > 0$, $x = (x_1, \ldots, x_n)$, $\gamma_i > 0$, $x_i > 0$, $i = 1, 2, \ldots, n$.

With initial conditions given as:

$$u(x, 0) = f(x), \quad u_t(x, 0) = \frac{\partial u(x, 0)}{\partial t} = 0, \quad u \in \mathbb{R}^n \times \{t = 0\}. \tag{41}$$

This study deals with the subset of the Euclidean space defined by:

$$\mathbb{R}_+^{n+1} = \{(t, \mathbf{x}) = (t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}, t > 0, x_1 > 0, \ldots, x_n > 0\}.$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $r = |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^n x_i^2}$, and let $\Omega$ be a finite or infinite open set in $\mathbb{R}^{n+1}$, symmetric with respect to each hyperplane $t = 0$, $x_i = 0$, for $i = 1, \ldots, n$. A multi-index parameter $\gamma = (\gamma_1, \ldots, \gamma_n)$ is defined as a collection of positive real parameters, where $\gamma_i > 0$ for each $i = 1, \ldots, n$, and the sum is given by $|\gamma| = \gamma_1 + \cdots + \gamma_n$.

Now transforming equation (40) for the function $u(x, t; \omega, \gamma)$ into an equation involving only two independent variables $t$(time) and $r$(the radial coordinate). We have

$$r = |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}, \tag{42}$$

This implies that,

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i} = \frac{u_r x_i}{r}. \tag{43}$$

and using quotient rule on (43) we obtain:

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{u_r x_i}{r}\right) = \frac{u_r}{r} + \frac{x_i^2 u_{rr}}{r^2} - \frac{u_r x_i^2}{r^3}. \tag{44}$$

Therefore, $\sum_{i=1}^n \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i}\right)$ in equation (40) reduces to

$$\sum_{i=1}^n \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i}\right) = \sum_{i=1}^n \left(\frac{u_r}{r} + \frac{x_i^2 u_{rr}}{r^2} - \frac{u_r x_i^2}{r^3} + \frac{\gamma_i}{x_i}\frac{u_r x_i}{r}\right). \tag{45}$$

Therefore, equations (40-41) reduce to:

$$\frac{\partial^2 u}{\partial r^2} + \frac{n + |\gamma| - 1}{r}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial t^2} - \frac{\omega}{t}\frac{\partial u}{\partial t} = 0. \tag{46}$$

Initial conditions given as:

$$u(r, 0) = f(r), \quad u_t(r, 0) = \frac{\partial u(r, 0)}{\partial t} = 0. \tag{47}$$

Setting $\omega = 2N$ and $n + |\gamma| - 1 = 2M$ whereby $M, N \in \mathbb{R}$, then equation (46) reduces to:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2M}{r}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial t^2} - \frac{2N}{t}\frac{\partial u}{\partial t} = 0. \tag{48}$$

Initial conditions given as:

$$u(r, 0) = f(r), \quad u_t(r, 0) = \frac{\partial u(r, 0)}{\partial t} = 0. \tag{49}$$

Equation (48) is a second-order linear hyperbolic partial differential equation in two independent variables $t$ and $r$ only with two pairs of singular Bessel operators (50):

$$(B_\omega)_t u = (B_\tau)_r u, \qquad \omega = 2N, \tau = n + |\gamma| - 1 = 2M; M, N \in \mathbb{R}. \tag{50}$$

with initial conditions

$$u(r, 0; \omega, \tau) = f(r); \quad u_t(r, 0) = 0. \tag{51}$$

The next subsection will explore the derivation of integral transmutation operator useful in representation of solution to (50 - 51).

## 2.2 The Integral Transmutation Operator.

This section applies the ITCM to construct an integral transmutation operator $T$ that transmutes the singular Bessel operator $(B_\omega)_t$, as defined in (5), into the corresponding operator $(B_\tau)_r$ with a different real parameter $\tau$.

To perform this transformation, ITCM is utilized in conjunction with the Hankel transform described in (34). The objective is to determine the transmutation operator $T^{(\phi)}_{\omega,\tau}$, defined as

$$T^{(\phi)}_{\omega,\tau} = \mathcal{H}^{-1}_\tau \left[\phi(t)\, \mathcal{H}_\omega\right], \tag{52}$$

which satisfies the intertwining relation (53):

$$T^{(\phi)}_{\omega,\tau} B_\omega = B_\tau T^{(\phi)}_{\omega,\tau}, \tag{53}$$

where $\mathcal{H}_\omega$ and $\mathcal{H}^{-1}_\tau$ denote the Hankel transform and its inverse respectively, as defined in (34) and (36).

Assuming the function $\phi(t)$ takes the form $\phi(t) = Ct^\beta$, with $C \in \mathbb{R}$ independent of $t$, the operator is denoted by $T^{(\beta)}_{\omega,\tau} := T^{(\phi)}_{\omega,\tau}$. The constant $C$ is conveniently chosen so that the operator $T^{(\beta)}_{\omega,\tau}$ satisfies $T^{(\beta)}_{\omega,\tau} 1 = 1$. Under this assumptions, the below theorem is obtained.

**Theorem 2.1.** *With $\mathcal{G}$ defined by (33), let $u \in \mathcal{G}$ for which the composition*

$$T^{(\beta)}_{\omega;\tau} u(x) = \mathcal{H}^{-1}_\tau \big[\phi(t)\mathcal{H}_\omega[u](t)\big](x) \tag{54}$$

*is well-defined. Then Integral Transmutation Operator $T^{(\beta)}_{\omega,\tau}$ satisfying (53) admits the following integral representation:*

$$\begin{aligned} T^{(\beta)}_{\omega,\tau} u(x) = C\frac{2^{\beta+1}\Gamma(\frac{\tau+\beta+1}{2})}{\Gamma\left(\frac{\tau+1}{2}\right)} &\Bigg[\frac{x^{-(\beta+\tau+1)}}{\Gamma(-\frac{\beta}{2})\Gamma(\frac{\omega+1}{2})} \int_0^x u(y) y^{\omega} {}_2F_1(\frac{\tau+\beta+1}{2}, \frac{\beta}{2}+1; \frac{\omega+1}{2}; \frac{y^2}{x^2})\, dy \\ &+ \frac{1}{\Gamma(\frac{\omega-\tau-\beta}{2})\Gamma(\frac{\tau+1}{2})} \int_x^\infty u(y) y^{\omega-\tau-\beta-1} {}_2F_1(\frac{\tau+\beta+1}{2}, \frac{\tau-\omega+\beta}{2}+1; \frac{\tau+1}{2}; \frac{x^2}{y^2})\, dy\Bigg], \end{aligned}$$

*for $x, y, \Re(\tau+\beta+1) > 0, \Re(\beta + \frac{\tau-\omega}{2}) < 0, (\omega-\tau-\beta) \in \mathbb{N}$.* (55)

*Here, ${}_2F_1$ denotes the Gauss hypergeometric function (26).*

*Proof.* The proof of theorem (2.1) follows directly by substituting $\phi(t) = Ct^\beta$ into the operator definition (54) and applying known Bessel integral identities appearing as entry in [13, p.691, eqn **6.574** 1]. □

**Remark 2.1.** *In (55), it can be clearly seen that if certain conditions on the parameters are met, parts of the integral transmutation operator vanishes:*

1. *If $\beta \geq 0$, then the first part of (55) vanishes.*
2. *If $(\omega - \tau - \beta) \leq 0$ , then the second part of (55) vanishes.*

## 3. RESULTS AND DISCUSSION

This section first develops the Extended $k$-Integral Transmutation operator, proves key theorems simplifying the Extended $k$-Integral Transmutation operator under different parameter conditions. Provides explicit solution to the Singular GEPD equation, verify the compatibility of the solution with respect to initial conditions and furthermore gives two relations for the Extended $k$-Integral Transmutation operator to satisfy the transmutation operator property.

### 3.1 The Extended $k-$Integral Transmutation Operator $T^{(\beta)}_{\omega,\tau;b;k}u(x)$.

**Theorem 3.1** (Extended $k$-Integral Transmutation Operator $T^{(\beta)}_{\omega,\tau;b;k}u(x)$). *With $\mathcal{G}$ defined by (33), let $u \in \mathcal{G}$; $x, y, (\beta+\tau+1), (\tau-\omega+\beta), (\omega-\tau-\beta) \in \mathbb{R}^+$; $\omega, \tau \in \mathbb{R}$; $n \in \mathbb{N}_0$; $\Re(\beta+\frac{\tau-\omega}{2}) < 0; \omega-\beta > 1; b \geq 0; k \geq 1$ and $\beta \notin \mathbb{N}_0$, then the Extended $k$-Integral Transmutation Operator $\left(T^{(\beta)}_{\omega,\tau;b;k}u\right)(x)$satisfying :*

$$T^{(\beta)}_{\omega,\tau;b;k}B_\omega = B_\tau T^{(\beta)}_{\omega,\tau;b;k} \tag{56}$$

*admits the integral expression:*

$$\begin{aligned} &T^{(\beta)}_{\omega,\tau;b;k}u(x) \\ &= C\frac{2^{\beta+1}\Gamma(\frac{\tau+\beta+1}{2})}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\beta+\tau+1)}}{\Gamma(-\frac{\beta}{2})\Gamma(\frac{\omega+1}{2})}\int_0^x u(y)y^\omega \sum_{n=0}^{\infty}\frac{\left(\frac{\tau+\beta+1}{2}\right)_n B_{b;k}\left(\frac{\beta+n+2}{2},\frac{\omega-\beta-1}{2}\right)}{B\left(\frac{\beta+2}{2},\frac{\omega-\beta-1}{2}\right)n!}\left(\frac{y^2}{x^2}\right)^n dy\right. \\ &\quad \left. + \frac{1}{\Gamma(\frac{\omega-\tau-\beta}{2})\Gamma(\frac{\tau+1}{2})}\int_x^\infty u(y)y^{\omega-\tau-\beta-1}\sum_{n=0}^{\infty}\frac{\left(\frac{\tau+\beta+1}{2}\right)_n B_{b;k}\left(\frac{\tau-\omega+\beta+2+2n}{2},\frac{\omega-\beta-1}{2}\right)}{B\left(\frac{\tau-\omega+\beta+2}{2},\frac{\omega-\beta-1}{2}\right)n!}\left(\frac{x^2}{y^2}\right)^n dy\right], \end{aligned}$$

*whereby $B$ and $B_{b;k}$ are defined by equations* (15) *and* (19) *respectively*; $\Re(\beta+\dfrac{\tau-\omega}{2}) < 0;$

$$x, y, b, k, (\tau+\beta+1), (\tau-\omega+\beta), (\omega-\tau-\beta) \in \Re^+; \omega-\beta > 1; \omega, \tau \in \Re; n \in \mathbb{N}_0; \beta \notin \mathbb{N}_0. \tag{57}$$

*Proof.* The proof follows by substituting $\phi(t) = Ct^\beta$ into the operator definition (54) and applying known Bessel integral identities appearing as entry in [13, p.691, eqn **6.574** 1].

We get:

$$\begin{aligned} \left(T^{(\beta)}_{\omega,\tau}u\right)(x) &= C\frac{2^{\beta+1}\Gamma(\frac{\tau+\beta+1}{2})}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\beta+\tau+1)}}{\Gamma(-\frac{\beta}{2})\Gamma(\frac{\omega+1}{2})}\int_0^x u(y)y^\omega {}_2F_1(\frac{\tau+\beta+1}{2},\frac{\beta}{2}+1;\frac{\omega+1}{2};\frac{y^2}{x^2})\,dy\right. \\ &\quad \left. + \frac{1}{\Gamma(\frac{\omega-\tau-\beta}{2})\Gamma(\frac{\tau+1}{2})}\int_x^\infty u(y)y^{\omega-\tau-\beta-1}{}_2F_1(\frac{\tau+\beta+1}{2},\frac{\tau-\omega+\beta}{2}+1;\frac{\tau+1}{2};\frac{x^2}{y^2})\,dy\right], \end{aligned}$$

for $x, y, \Re(\tau+\beta+1) > 0, \Re(\beta+\dfrac{\tau-\omega}{2}) < 0, (\omega-\tau-\beta) \in \mathbb{N}.$ (58)

With the Extended $k-$Gauss hypergeometric function defined by (30), then (58) becomes:

$$\begin{aligned} &T^{(\beta)}_{\omega,\tau;b;k}u(x) \\ &= C\frac{2^{\beta+1}\Gamma(\frac{\tau+\beta+1}{2})}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\beta+\tau+1)}}{\Gamma(-\frac{\beta}{2})\Gamma(\frac{\omega+1}{2})}\int_0^x u(y)y^\omega \sum_{n=0}^{\infty}\frac{\left(\frac{\tau+\beta+1}{2}\right)_n B_{b;k}\left(\frac{\beta+n+2}{2},\frac{\omega-\beta-1}{2}\right)}{B\left(\frac{\beta+2}{2},\frac{\omega-\beta-1}{2}\right)n!}\left(\frac{y^2}{x^2}\right)^n dy\right. \\ &\quad \left. + \frac{1}{\Gamma(\frac{\omega-\tau-\beta}{2})\Gamma(\frac{\tau+1}{2})}\int_x^\infty u(y)y^{\omega-\tau-\beta-1}\sum_{n=0}^{\infty}\frac{\left(\frac{\tau+\beta+1}{2}\right)_n B_{b;k}\left(\frac{\tau-\omega+\beta+2+2n}{2},\frac{\omega-\beta-1}{2}\right)}{B\left(\frac{\tau-\omega+\beta+2}{2},\frac{\omega-\beta-1}{2}\right)n!}\left(\frac{x^2}{y^2}\right)^n dy\right], \end{aligned}$$

where $B$ and $B_{b;k}$ are defined by equation (15) and (19) respectively; $\Re\left(\beta+\dfrac{\tau-\omega}{2}\right) < 0;$

$$x, y, b, k, (\tau+\beta+1), (\tau-\omega+\beta), (\omega-\tau-\beta) \in \mathbb{R}^+; \omega-\beta > 1; \omega, \tau \in \mathbb{R}; n \in \mathbb{N}_0; \beta \notin \mathbb{N}_0. \tag{59}$$

This concludes the proof of theorem (3.1). □

The following useful Extended $k-$Integral Transmutation operators arises as a special case of transmutation operator (57) when certain parameter conditions are met.

**Corollary 3.2.** *Taking into account remark (2.1) No.2, letting* $u \in L^2(0,\infty)$*; with* $\beta = -\tau$*;* $\omega = 0$*; and* $1 < \tau < 2$*, the following useful Extended "k-descent "Integral Transmutation operator is obtained:*

$$T^{(-\tau)}_{0,\tau;b;k} u(x) = \frac{2x^{1-\tau}}{B_{b;k}\left(\frac{\tau}{2},\frac{1}{2}\right)} \int_0^x u(y)\left(x^2-y^2\right)^{\frac{\tau}{2}-1}\, dy,$$

*where* $x, y \in \mathbb{R}^+; 1 < \tau < 2; b \geq 0; k \geq 1; B_{b;k}$ *is defined by equation* (19). (60)

*such that*

$$T^{(-\tau)}_{0,\tau;b;k} 1 = 1. \tag{61}$$

*Proof.* Putting $\beta = -\tau; \omega = 0$ into (57) and taking into account remark (2.1) No.2, given that $1 < \tau < 2$, we have:

$$\begin{aligned}
&T^{(-\tau)}_{0,\tau;b;k} u(x)\\
&= C\frac{2^{1-\tau}\sqrt{\pi}}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-1}}{\Gamma(\frac{\tau}{2})\sqrt{\pi}}\int_0^x u(y)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{B\left(\frac{2-\tau}{2},\frac{\tau-1}{2}\right)n!}\left(\frac{y^2}{x^2}\right)^n dy\right],\\
&= C\frac{2^{1-\tau}}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\tau}{2}\right)}\left[\frac{1}{x}\int_0^x u(y)\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}\right)B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2-\tau}{2}\right)\Gamma\left(\frac{\tau-1}{2}\right)n!}\left(\frac{y^2}{x^2}\right)^n dy\right],\\
&= \frac{C\cdot 2^{1-\tau}}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\tau}{2}\right)}\left[\frac{1}{x}\int_0^x u(y)\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2-\tau}{2}+n\right)\Gamma\left(\frac{1+2n}{2}\right)B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2-\tau}{2}\right)\Gamma\left(\frac{\tau-1}{2}\right)\Gamma\left(\frac{2-\tau}{2}+n\right)\Gamma\left(\frac{1+2n}{2}\right)n!}\left(\frac{y^2}{x^2}\right)^n dy\right],\\
&= C\frac{2^{1-\tau}}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\tau}{2}\right)}\left[\frac{1}{x}\int_0^x u(y)\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2-\tau}{2}+n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2-\tau}{2}\right)\Gamma\left(\frac{1+2n}{2}\right)n!}\left(\frac{y^2}{x^2}\right)^n\frac{B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{B\left(\frac{\tau-1}{2},\frac{2-\tau}{2}+n\right)} dy\right],\\
&= C\frac{2^{1-\tau}}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\tau}{2}\right)}\left[\frac{1}{x}\int_0^x u(y)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{2-\tau}{2}\right)_n}{\left(\frac{1}{2}\right)_n n!}\left(\frac{y^2}{x^2}\right)^n\frac{B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{B\left(\frac{\tau-1}{2},\frac{2-\tau}{2}+n\right)} dy\right],\\
&= C\frac{2^{1-\tau}B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\tau}{2}\right)B\left(\frac{\tau-1}{2},\frac{2-\tau}{2}+n\right)}\left[\frac{1}{x}\int_0^x u(y)\,{}_2F_1\left(\frac{1}{2},\frac{2-\tau}{2};\frac{1}{2};\frac{y^2}{x^2}\right)dy\right],
\end{aligned}$$

whereby $x, y \in \mathbb{R}^+; b \geq 0; k \geq 1; \quad 1 < \tau < 2; \quad n \in \mathbb{N}_0; B$ and $B_{b;k}$ are defined by equations (15) and (19) respectively. (62)

Using the Gauss hypergeometric function (26), a clear observation is seen that in (62), $\alpha = c$, and therefore using binomial theorem for integers $n \geq 0$, $\beta \geq 1$, the identity below holds true:

$${}_2F_1(\alpha,\beta;a;t) = \sum_{n=0}^{\infty}\frac{(\beta)_n(t)^n}{n!} = \sum_{n=0}^{\infty}\binom{n+\beta-1}{n}(t)^n = (1-t)^{-\beta}. \tag{63}$$

Therefore using (63), the Extended $k-$Integral Transmutation operator (62) reduces to:

$$\begin{aligned}
T^{(-\tau)}_{0,\tau;b;k} u(x) &= C\frac{2^{1-\tau}B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\tau}{2}\right)B\left(\frac{\tau-1}{2},\frac{2-\tau}{2}+n\right)}\left[\frac{1}{x}\int_0^x u(y)\left(1-\frac{y^2}{x^2}\right)^{\frac{\tau}{2}-1}dy\right],\\
&= C\frac{2^{1-\tau}B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\tau}{2}\right)B\left(\frac{\tau-1}{2},\frac{2-\tau}{2}+n\right)}\left[\frac{1}{x}\int_0^x u(y)x^{2-\tau}\left(x^2-y^2\right)^{\frac{\tau}{2}-1}dy\right],
\end{aligned}$$

$$= C\frac{2^{1-\tau}B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\tau}{2}\right)B\left(\frac{\tau-1}{2},\frac{2-\tau}{2}+n\right)}\left[x^{1-\tau}\int_0^x u(y)\left(x^2-y^2\right)^{\frac{\tau}{2}-1}\,dy\right],$$

whereby $x, y, \in \mathbb{R}^+; b \geq 0; k \geq 1; 1 < \tau < 2; n \in \mathbb{N}_0; B$ and $B_{b;k}$ are defined by equations $(15)$ and $(19)$ respectively. (64)

Using the transformations $y = xt$ and $t^2 = q$, it can be clearly seen that:

$$x^{1-\tau}\int_0^x \left(x^2-y^2\right)^{\frac{\tau}{2}-1}\,dy = \int_0^1 \left(1-t^2\right)^{\frac{\tau}{2}-1}\,dt = \frac{1}{2}\int_0^1 q^{\frac{1}{2}-1}\left(1-q\right)^{\frac{\tau}{2}-1}\,dq. = \frac{\sqrt{\pi}\Gamma(\frac{\tau}{2})}{2\Gamma\left(\frac{\tau+1}{2}\right)} \tag{65}$$

and therefore into equation (64), choosing

$$C = \frac{2^{\tau}\Gamma^2\left(\frac{\tau+1}{2}\right)B\left(\frac{\tau-1}{2},\frac{2-\tau+2n}{2}\right)}{\sqrt{\pi}B_{b;k}\left(\frac{n+2-\tau}{2},\frac{\tau-1}{2}\right)} \tag{66}$$

guarantees that condition (61) is satisfied. Therefore, for $C$ defined by (66), equation (64) reduces to:

$$T^{(-\tau)}_{0,\tau;b;k}u(x) = \frac{2x^{1-\tau}}{B_{b;k}\left(\frac{\tau}{2},\frac{1}{2}\right)}\int_0^x u(y)\left(x^2-y^2\right)^{\frac{\tau}{2}-1}\,dy,$$

where $x, y \in \mathbb{R}^+; 1 < \tau < 2; b \geq 0; k \geq 1; B_{b;k}$ is defined by equation $(19)$. (67)

This completes Corollary (3.2). □

**Corollary 3.3.** *The following Extended "k-descent "integral transmutation operator is obtained for* $u \in L^2(0,\infty)$*;* $\beta = \omega - \tau$*;* $\tau \in (1,\infty)$ *and* $\omega \in (\tau-2,\tau)$

$$T^{(\omega-\tau)}_{\omega,\tau;b;k}u(x) = 2B_{b;k}\left(\frac{\tau-\omega}{2},\frac{\omega+1}{2}\right)x^{1-\tau}\int_0^x u(y)y^{\omega}\left(x^2-y^2\right)^{\frac{\tau-\omega-2}{2}}\,dy,$$

*where* $B_{b;k}$ *is defined by equation* $(19); x, y, (\omega+1) \in \Re^+; \Re(\dfrac{\tau-\omega}{2}) > 0;$
$\tau \in (1,\infty); \tau, \omega \in \Re; \omega \in (\tau-2,\tau); b \geq 0; k \geq 1.$ (68)

*satisfying*

$$T^{(\omega-\tau)}_{\omega,\tau;b;k}B_{\omega} = B_{\tau}T^{(\omega-\tau)}_{\omega,\tau;b;k}. \tag{69}$$

$$T^{(\omega-\tau)}_{\omega,\tau;b;k}1 = 1. \tag{70}$$

*Proof.* The proof of (68) is directly obtained by taking into account remark (2.1) No.2 and replacing $\beta = \omega - \tau$ into (57), we have:

$$T^{(\omega-\tau)}_{\omega,\tau;b;k}u(x)$$
$$= C\frac{2^{\omega-\tau+1}\Gamma(\frac{\omega+1}{2})}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\omega+1)}}{\Gamma(\frac{\tau-\omega}{2})\Gamma(\frac{\omega+1}{2})}\int_0^x u(y)y^{\omega}\sum_{n=0}^{\infty}\frac{\left(\frac{\omega+1}{2}\right)_n B_{b;k}\left(\frac{\omega-\tau+n+2}{2},\frac{\tau-1}{2}\right)}{B\left(\frac{\omega-\tau+2}{2},\frac{\tau-1}{2}\right)n!}\left(\frac{y^2}{x^2}\right)^n dy\right],$$
$$= C\frac{2^{\omega-\tau+1}\Gamma(\frac{\omega+1}{2})}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\omega+1)}}{\Gamma(\frac{\tau-\omega}{2})\Gamma(\frac{\omega+1}{2})}\int_0^x u(y)y^{\omega}\sum_{n=0}^{\infty}\frac{\left(\frac{\omega+1}{2}\right)_n B_{b;k}\left(\frac{\omega-\tau+n+2}{2},\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{\omega-\tau+2}{2}\right)\Gamma\left(\frac{\tau-1}{2}\right)\left(\Gamma\left(\frac{\omega+1}{2}\right)\right)^{-1}n!}\left(\frac{y^2}{x^2}\right)^n dy\right],$$
$$= C\frac{2^{\omega-\tau+1}}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\omega+1)}}{\Gamma(\frac{\tau-\omega}{2})}\int_0^x u(y)y^{\omega}\sum_{n=0}^{\infty}\frac{\left(\frac{\omega+1}{2}\right)_n\left(\frac{\omega-\tau+2}{2}\right)_n}{\left(\frac{\omega+1}{2}\right)_n n!}\left(\frac{y^2}{x^2}\right)^n\frac{B_{b;k}\left(\frac{\omega-\tau+n+2}{2},\frac{\tau-1}{2}\right)}{B\left(\frac{\omega-\tau+2+2n}{2},\frac{\tau-1}{2}\right)}dy\right],$$

$$= C\frac{2^{\omega-\tau+1}}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\omega+1)}}{\Gamma(\frac{\tau-\omega}{2})}\int_0^x u(y)y^{\omega}{}_2F_1\left(\frac{\omega+1}{2},\frac{\omega-\tau+2}{2};\frac{\omega+1}{2};\frac{y^2}{x^2}\right)\frac{B_{b;k}\left(\frac{\omega-\tau+n+2}{2},\frac{\tau-1}{2}\right)}{B\left(\frac{\omega-\tau+2+2n}{2},\frac{\tau-1}{2}\right)}\,dy\right],$$

whereby $B$ and $B_{b;k}$ are defined by (15) and (19) respectively; $x, y, (\omega+1) \in \Re^+$;
$\tau \in (1,\infty); \omega \in (\tau-2,\tau); \omega,\tau \in \Re; \tau-\omega > 0; b \geq 0; k \geq 1; n \in \mathbb{N}_0.$ (71)

Using the identity (63), (71) reduces to:

$$T^{(\omega-\tau)}_{\omega,\tau;b;k}u(x) = C\frac{2^{\omega-\tau+1}}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\omega+1)}}{\Gamma(\frac{\tau-\omega}{2})}\int_0^x u(y)y^{\omega}\left(1-\frac{y^2}{x^2}\right)^{\frac{\tau-\omega-2}{2}}\frac{B_{b;k}\left(\frac{\omega-\tau+n+2}{2},\frac{\tau-1}{2}\right)}{B\left(\frac{\omega-\tau+2+2n}{2},\frac{\tau-1}{2}\right)}\,dy\right],$$

$$= C\frac{2^{\omega-\tau+1}}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{x^{-(\omega+1)}}{\Gamma(\frac{\tau-\omega}{2})}\int_0^x u(y)y^{\omega}x^{\omega-\tau+2}\left(x^2-y^2\right)^{\frac{\tau-\omega-2}{2}}\frac{B_{b;k}\left(\frac{\omega-\tau+n+2}{2},\frac{\tau-1}{2}\right)}{B\left(\frac{\omega-\tau+2+2n}{2},\frac{\tau-1}{2}\right)}\,dy\right],$$

$$= C\frac{2^{\omega-\tau+1}B_{b;k}\left(\frac{\omega-\tau+n+2}{2},\frac{\tau-1}{2}\right)}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma(\frac{\tau-\omega}{2})B\left(\frac{\omega-\tau+2+2n}{2},\frac{\tau-1}{2}\right)}x^{1-\tau}\int_0^x u(y)y^{\omega}\left(x^2-y^2\right)^{\frac{\tau-\omega-2}{2}}\,dy,$$

whereby $B$ and $B_{b;k}$ are defined by (15) and (19) respectively; $x, y, (\omega+1) \in \Re^+$;
$\Re(\frac{\tau-\omega}{2}) > 0; \tau \in (1,\infty); \omega \in (\tau-2,\tau); \omega,\tau \in \Re; b \geq 0; k \geq 1; n \in \mathbb{N}_0.$ (72)

It's evident from (72), using the transformations $y = xt$ and $t^2 = q$, we get:

$$x^{1-\tau}\int_0^x (y)^{\omega}\left(x^2-y^2\right)^{\frac{\tau-\omega}{2}-1}\,dy = \int_0^1 t^{\omega}\left(1-t^2\right)^{\frac{\tau-\omega}{2}-1}\,dt = \frac{1}{2}\int_0^1 q^{\frac{\omega+1}{2}-1}(1-q)^{\frac{\tau-\omega}{2}-1}\,dq.$$

$$= \frac{\Gamma\left(\frac{\omega+1}{2}\right)\Gamma(\frac{\tau-\omega}{2})}{2\Gamma\left(\frac{\tau+1}{2}\right)}. \quad (73)$$

and therefore into (72), choosing

$$C = \frac{\Gamma^2\left(\frac{\tau+1}{2}\right)B\left(\frac{\omega-\tau+2+2n}{2},\frac{\tau-1}{2}\right)}{2^{\omega-\tau}\Gamma\left(\frac{\omega+1}{2}\right)B_{b;k}\left(\frac{\omega-\tau+n+2}{2},\frac{\tau-1}{2}\right)} \quad (74)$$

guarantees that condition (70) is satisfied. Therefore, for $C$ defined by (74), equation (72) reduces to:

$$T^{(\omega-\tau)}_{\omega,\tau;b;k}u(x) = 2B_{b;k}\left(\frac{\tau-\omega}{2},\frac{\omega+1}{2}\right)x^{1-\tau}\int_0^x u(y)y^{\omega}\left(x^2-y^2\right)^{\frac{\tau-\omega-2}{2}}\,dy,$$

where $B_{b;k}$ is defined by equation (19); $x, y, (\omega+1) \in \Re^+; \Re(\frac{\tau-\omega}{2}) > 0; \omega,\tau \in \Re$;
$\tau \in (1,\infty); \omega \in (\tau-2,\tau); b \geq 0; k \geq 1.$ (75)

This completes Corollary (3.3). □

**Corollary 3.4.** *Let $u \in L^1_f$ given that $f(y) = |y|^{\omega-\tau}$, i.e., $\int_{\mathbb{R}}|u(y)|f(y)\,dy < \infty$. With $\beta = 0, \omega \in (1,3)$, and $-1 < \tau < \omega$, we obtain following Extended "$k-descent$" integral transmutation operator:*

$$T^{(0)}_{\omega,\tau;b;k}u(x) = \frac{2B_{b;k}\left(\frac{\omega-\tau}{2},\frac{\tau-2\omega+3}{2}\right)}{\Gamma^2_{b;k}\left(\frac{\omega-\tau}{2}\right)}x^{\tau-\omega}\int_x^{\infty} u(y)y^{2\omega-\tau-2}\left(y^2-x^2\right)^{\frac{1-\omega}{2}}\,dy,$$

*where $\Gamma_{b;k}$ and $B_{b;k}$ are defined by equation (13) and (19) respectively*; $x, y, (\omega-\tau) \in \Re^+$;
$\Re(\frac{\tau-\omega}{2}) < 0; \omega \in (1,3);$ *and* $-1 < \tau < \omega; \omega,\tau \in \Re; b \geq 0; k \geq 1.$ (76)

*satisfying*

$$T^{(0)}_{\omega,\tau;b;k}1 = x^{\omega-\tau}. \quad (77)$$

*Proof.* The proof of (76) follows directly by replacing $\beta = 0$ into (57) and taking into account remark (2.1) No.1. Therefore we have:

$$T^{(0)}_{\omega,\tau;b;k}u(x)$$
$$= C\frac{2^1\Gamma(\frac{\tau+1}{2})}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{1}{\Gamma(\frac{\omega-\tau}{2})\Gamma(\frac{\tau+1}{2})}\int_x^\infty u(y)y^{\omega-\tau-1}\sum_{n=0}^{\infty}\frac{\left(\frac{\tau+1}{2}\right)_n B_{b;k}\left(\frac{\tau-\omega+2+2n}{2},\frac{\omega-1}{2}\right)}{B\left(\frac{\tau-\omega+2}{2},\frac{\omega-1}{2}\right)n!}\left(\frac{x^2}{y^2}\right)^n dy\right],$$
$$= C\frac{2}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{1}{\Gamma(\frac{\omega-\tau}{2})}\int_x^\infty u(y)y^{\omega-\tau-1}\sum_{n=0}^{\infty}\frac{\left(\frac{\tau+1}{2}\right)_n\left(\frac{\omega-1}{2}\right)_n}{\left(\frac{\tau+1}{2}\right)_n n!}\left(\frac{x^2}{y^2}\right)^n\frac{B_{b;k}\left(\frac{\tau-\omega+2+2n}{2},\frac{\omega-1}{2}\right)}{B\left(\frac{\omega-1+2n}{2},\frac{\tau-\omega+2}{2}\right)}dy\right],$$
$$= C\frac{2}{\Gamma\left(\frac{\tau+1}{2}\right)}\left[\frac{1}{\Gamma(\frac{\omega-\tau}{2})}\int_x^\infty u(y)y^{\omega-\tau-1}\left(1-\frac{x^2}{y^2}\right)^{\frac{1-\omega}{2}}\frac{B_{b;k}\left(\frac{\tau-\omega+2+2n}{2},\frac{\omega-1}{2}\right)}{B\left(\frac{\omega-1+2n}{2},\frac{\tau-\omega+2}{2}\right)}dy\right],$$
$$= C\frac{2B_{b;k}\left(\frac{\tau-\omega+2+2n}{2},\frac{\omega-1}{2}\right)}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma(\frac{\omega-\tau}{2})B\left(\frac{\omega-1+2n}{2},\frac{\tau-\omega+2}{2}\right)}\int_x^\infty u(y)y^{2\omega-\tau-2}\left(y^2-x^2\right)^{\frac{1-\omega}{2}}dy,$$

where $B$ and $B_{b;k}$ are defined by (15) and (19) respectively; $\Re(\frac{\tau-\omega}{2})<0; n\in\mathbb{N}_0;$
$x, y, (\tau+1), (\tau-\omega), (\omega-\tau)\in\Re^+; \omega\in(1,3);$ and $-1<\tau<\omega; \omega,\tau\in\Re; b\geq 0; k\geq 1.$ (78)

Using the transformations $y=\frac{x}{t}$ and $t^2=q$, it's clear that:

$$\int_x^\infty y^{2\omega-\tau-2}\left(y^2-x^2\right)^{\frac{1-\omega}{2}}dy = \int_1^0 (xt^{-1})^{2\omega-\tau-2}\left(x^2t^{-2}-x^2\right)^{\frac{1-\omega}{2}}\left(-xt^{-2}\right)dt.$$
$$= x^{\omega-\tau}\int_0^1 t^{\tau-\omega-1}\left(1-t^2\right)^{\frac{1-\omega}{2}}dt. = \frac{x^{\omega-\tau}}{2}\int_0^1 q^{\frac{\tau-\omega}{2}-1}(1-q)^{\frac{3-\omega}{2}-1}dq.$$
$$= \frac{x^{\omega-\tau}}{2}\int_0^1 q^{\frac{\omega-\tau}{2}-1}(1-q)^{\frac{\omega-\tau}{2}-1}dq. = \frac{x^{\omega-\tau}\Gamma\left(\frac{\omega-\tau}{2}\right)\Gamma\left(\frac{3-\omega}{2}\right)}{2\Gamma\left(\frac{\tau-2\omega+3}{2}\right)}. \quad (79)$$

and therefore into (78), choosing

$$C = \frac{\Gamma\left(\frac{\tau-2\omega+3}{2}\right)\Gamma\left(\frac{\tau+1}{2}\right)B\left(\frac{\omega-1+2n}{2},\frac{\tau-\omega+2}{2}\right)}{\Gamma\left(\frac{3-\omega}{2}\right)B_{b;k}\left(\frac{\tau-\omega+2+2n}{2},\frac{\omega-1}{2}\right)} \quad (80)$$

will satisfy condition (77). Thus, from (79) and for $C$ defined by (80), equation (78) reduces to:

$$T^{(0)}_{\omega,\tau;b;k}u(x) = \frac{2B_{b;k}\left(\frac{\omega-\tau}{2},\frac{\tau-2\omega+3}{2}\right)}{\Gamma^2_{b;k}\left(\frac{\omega-\tau}{2}\right)}x^{\tau-\omega}\int_x^\infty u(y)y^{2\omega-\tau-2}\left(y^2-x^2\right)^{\frac{1-\omega}{2}}dy,$$

where $\Gamma_{b;k}$ and $B_{b;k}$ are defined by (13) and (19) respectively; $x, y, (\omega-\tau)\in\Re^+;$
$\Re(\frac{\tau-\omega}{2})<0; \omega\in(1,3);$ and $-1<\tau<\omega; \omega,\tau\in\Re; b\geq 0; k\geq 1.$ (81)

This concludes the proof of Corollary (3.4). □

**Theorem 3.5.** *Into the operator definition (54), letting $\phi(t)=j_{\frac{\omega-1}{2}}(zt)$ and the order $\tau=\omega$, we obtain the following useful transmutation operator:*

$$_{x,z}T_{\omega;b,k}u(x) = \frac{2^{\omega-2}(xz)^{1-\omega}}{B_{b;k}\left(\frac{\omega}{2},\frac{1}{2}\right)}\int_{|x-z|}^{x+z}u(y)y\Delta^{\omega-2}dy,$$

*where* $\Delta = \frac{1}{4}\sqrt{[z^2-(x-y)^2][(x+y)^2-z^2]}, \omega, x, y, z>0; b\geq 0; k\geq 1.$ (82)

*satisfying*

$$ {}_{x,z}T_{\omega;b,k}\left(B_\omega\right)_x = \left(B_\omega\right)_z \, {}_{x,z}T_{\omega;b,k}. \tag{83}$$

*Proof.* The proof of (82) follows by substituting $\phi(t) = j_{\frac{\omega-1}{2}}(zt)$ into the operator definition (54) and applying explicit Bessel integral appearing as entry in [13, p.694, eqn **6.578** 9].

Therefore we have:

$$\begin{aligned}
T^{(\phi)}_{\omega,\omega}u(x) &= \mathcal{H}_\omega^{-1}\left[j_{\frac{\omega-1}{2}}(zt)\mathcal{H}_\omega[u](t)\right](x) = {}_{x,z}T_\omega u(x)\\
&= \frac{2^{1-\omega}}{\Gamma^2\left(\frac{\omega+1}{2}\right)}\int_0^\infty j_{\frac{\omega-1}{2}}(xt)\,t^\omega\,dt\cdot j_{\frac{\omega-1}{2}}(zt)\int_0^\infty j_{\frac{\omega-1}{2}}(yt)u(y)y^\omega\,dy.\\
&= \frac{2^{1-\omega}}{\Gamma^2\left(\frac{\omega+1}{2}\right)}\int_0^\infty u(y)y^\omega\,dy\int_0^\infty j_{\frac{\omega-1}{2}}(xt)j_{\frac{\omega-1}{2}}(yt)j_{\frac{\omega-1}{2}}(zt)t^\omega\,dt, \qquad \text{recall (35).}\\
&= \frac{2^{\frac{\omega-1}{2}}\Gamma\left(\frac{\omega+1}{2}\right)}{(xz)^{\frac{\omega-1}{2}}}\int_0^\infty u(y)y^{\frac{1+\omega}{2}}\,dy\int_0^\infty t^{1-\frac{\omega-1}{2}}J_{\frac{\omega-1}{2}}(xt)J_{\frac{\omega-1}{2}}(yt)J_{\frac{\omega-1}{2}}(zt)\,dt.
\end{aligned}\tag{84}$$

Using Bessel integral formula appearing as entry in [13, p.694, eqn **6.578** 9], we have:

$$\int_0^\infty J_{\frac{\omega-1}{2}}(xt)J_{\frac{\omega-1}{2}}(zt)J_{\frac{\omega-1}{2}}(yt)t^{1-\frac{\omega-1}{2}}\,dt = \begin{cases} 0, & \text{if } 0<y\le|x-z| \text{ or } y\ge x+z,\\[4pt] \dfrac{2^{\frac{\omega-3}{2}}\Delta^{\omega-2}}{(xyz)^{\frac{\omega-1}{2}}\Gamma\left(\frac{\omega}{2}\right)\Gamma\left(\frac{1}{2}\right)}, & \text{if } |x-z|<y<x+z,\end{cases}$$

where $\quad \Delta = \dfrac{1}{4}\sqrt{[z^2-(x-y)^2][(x+y)^2-z^2]}, x>0, \quad y>0, \quad z>0, \quad \Re\left(\dfrac{\omega-1}{2}\right) > -\dfrac{1}{2},$

$\Delta > 0$ corresponds to the area of a triangle whose sides are $x, y,$ and $z$. (85)

Therefore using the above (85) results, (84) becomes:

$$\begin{aligned}
T^{(\phi)}_{\omega,\omega}u(x) &= \mathcal{H}_\omega^{-1}\left[j_{\frac{\omega-1}{2}}(zt)\mathcal{H}_\omega[u](t)\right](x) = {}_{x,z}T_\omega u(x)\\
&= \frac{2^{\omega-2}\Gamma\left(\frac{\omega+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\omega}{2}\right)(xz)^{\omega-1}}\int_{|x-z|}^{x+z}u(y)y\Delta^{\omega-2}\,dy,\\
&= \frac{2^{\omega-2}\Gamma\left(\frac{\omega+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\omega}{2}\right)\Gamma\left(\frac{1}{2}\right)(xz)^{\omega-1}}\int_{|x-z|}^{x+z}u(y)y\Delta^{\omega-2}\,dy,\\
&= \frac{2^{\omega-2}(xz)^{1-\omega}}{B\left(\frac{\omega}{2},\frac{1}{2}\right)}\int_{|x-z|}^{x+z}u(y)y\Delta^{\omega-2}\,dy, \quad \omega,x,y,z>0.
\end{aligned}\tag{86}$$

With the Extended $k-$Beta function defined by (19), then Extended $k-$ form of the operator (86) becomes:

$$ {}_{x,z}T_{\omega;b,k}u(x) = \frac{2^{\omega-2}(xz)^{1-\omega}}{B_{b;k}\left(\frac{\omega}{2},\frac{1}{2}\right)}\int_{|x-z|}^{x+z}u(y)y\Delta^{\omega-2}\,dy, \quad \omega,x,y,z>0; b\ge 0; k\ge 1. \tag{87}$$

This completes theorem (3.5). □

To this extent, it is evident that ITCM provides a powerful and effective approach for constructing Extended $k-$transmutation operators. Therefore, it plays a crucial role in deriving connection formulas and explicit integral representations of solutions for a wide range models described by differential equations, particularly those involving Bessel operators.

### 3.2 Application of the Extended $k-$integral transmutation operator to derive generalized solution for the singular Cauchy GEPD equation.

By utilizing the Extended $k-$integral transmutation operators (60), (68) and (82) generated through ITCM, generalized solutions to hyperbolic singular differential equations with singular Bessel operators can now be explicitly represented. Into radial coordinate system, the initial value GEPD equation (3-4) is reduced into (48-49) with a pair of Singular Bessel operators (50 - 51): Therefore ITCM permit us to utilize the Extended $k-$integral transmutation operators (68), (76) and (82) to represent solution to the Cauchy problem (50 - 51).

Thus, for $\tau \in (1, \infty)$ and $\omega \in (\tau - 2, \tau)$, a solution of the initial value GEPD equation (50 - 51) is arrived at by using (68) and (82) takes the form:

$$u\left(r,t;\tau,\omega;b,k\right) = 2B_{b;k}\left(\frac{\tau-\omega}{2},\frac{\omega+1}{2}\right)t^{1-\tau}\int_0^t u(r,y)y^{\omega}\left(t^2-y^2\right)^{\frac{\tau-\omega-2}{2}}\,{}_{r,y}T_{\omega;b,k}f(r)\,dy,$$

where $B_{b;k}$ is defined by equation (19); $t, y, (\omega+1) \in \Re^+; \Re(\frac{\tau-\omega}{2}) > 0; \omega, \tau \in \Re;$

$\tau \in (1, \infty); \omega \in (\tau - 2, \tau); b \geq 0; k \geq 1; {}_{r,y}T_{\omega;b,k}f(r)$ is defined by $(89)$. (88)

$${}_{r,y}T_{\omega;b,k}f(r) = \frac{2^{\omega-2}\left(ry\right)^{1-\omega}}{B_{b;k}\left(\frac{\omega}{2},\frac{1}{2}\right)}\int_{|r-y|}^{r+y} f(z)z\Delta^{\omega-2}\,dz,$$

where $\Delta = \frac{1}{4}\sqrt{[y^2-(r-z)^2][(r+z)^2-y^2]}, \omega, r, y, z > 0; b \geq 0; k \geq 1.$ (89)

and for $\omega \in (1, 3)$ and $-1 < \tau < \omega$, a solution of the Singular Cauchy GEPD equation (50 - 51) is arrived at by using (76) and (82) taking the form:

$$u\left(r,t;\tau,\omega;b,k\right) = \frac{2B_{b;k}\left(\frac{\omega-\tau}{2},\frac{\tau-2\omega+3}{2}\right)}{\Gamma^2_{b;k}\left(\frac{\omega-\tau}{2}\right)}t^{\tau-\omega}\int_t^{\infty} u(r,y)y^{2\omega-\tau-2}\left(y^2-t^2\right)^{\frac{1-\omega}{2}}\,{}_{r,y}T_{\omega;b,k}f(r)\,dy,$$

where $\Gamma_{b;k}$ and $B_{b;k}$ are defined by equation $(13)$ and $(19)$ respectively; $t, y, (\omega - \tau) \in \Re^+;$

$\Re(\frac{\tau-\omega}{2}) < 0; \omega \in (1, 3); -1 < \tau < \omega; \omega, \tau \in \Re; b \geq 0; k \geq 1; {}_{r,y}T_{\omega;b,k}f(r)$ is defined by $(89)$. (90)

The next objective is to confirm that expressions (88) and (90), which represent solutions to the singular GEPD equation (50), indeed satisfy the initial conditions specified in (51). Additionally, we aim to verify that the Extended $k$-integral transmutation operator (57) fulfills the transmutation property given in (91).

$$T^{(\beta)}_{\omega,\tau;b,k}\left(B_{\omega}\right)_t u(r,t) = \left(B_{\tau}\right)_r T^{(\beta)}_{\omega,\tau;b,k}u(r,t) \tag{91}$$

where $(B_\omega)_t$ and $(B_\tau)_r$ are the Singular Bessel operators defined in equation $(50)$.

**Theorem 3.6.** *Define $u\left(r,t;\tau,\omega;b,k\right)$ by equations (88) and (90) as the solution to the GEPD equation (50) with respect to initial conditions (51). Then:*

*(i)* $u \in C^2((0,\infty)\times[0,\infty))$,

*(ii)* $(B_\tau)_r u - (B_\omega)_t u = 0$ *in* $(0,\infty)\times[0,\infty)$,

*(iii)* $\lim_{(r,t)\to(r_0,0)} u\left(r,t;\tau,\omega;b,k\right) = f(r_0), \quad \lim_{(r,t)\to(r_0,0)} u_t\left(r,t;\tau,\omega;b,k\right) = 0,$ *for each* $r_0 > 0$.

*Proof.* $(i)$ and $(ii)$ is straightforward. Therefore, if $u\left(r,t;\tau,\omega;b,k\right)$ is given by (88), then using the transformation $y = tp$ implies that:

$$u\left(r,t;\tau,\omega;b,k\right)\Big|_{t=0} = 2B_{b;k}\left(\frac{\tau-\omega}{2},\frac{\omega+1}{2}\right)t^{1-\tau}\int_0^t u(r,y)y^{\omega}\left(t^2-y^2\right)^{\frac{\tau-\omega-2}{2}}\,{}_{r,y}T_{\omega;b,k}f(r)\,dy\Big|_{t=0}$$

$$= 2B_{b;k}\left(\frac{\tau-\omega}{2},\frac{\omega+1}{2}\right)\int_0^1 u(r,tp)\Big|_{t=0} p^{\omega}\left(1-p^2\right)^{\frac{\tau-\omega-2}{2}}\ {}_{r,0}T_{\omega;b,k}f(r)\,dp = f(r) \tag{92}$$

and

$$\frac{\partial}{\partial t}\Big(u\left(r,t;\tau,\omega;b,k\right)\Big)\Big|_{t=0} = u_t\left(r,t;\tau,\omega;b,k\right)\Big|_{t=0}$$
$$= 2B_{b;k}\left(\frac{\tau-\omega}{2},\frac{\omega+1}{2}\right)\int_0^1 u_t(r,tp)\Big|_{t=0} p^{\omega}\left(1-p^2\right)^{\frac{\tau-\omega-2}{2}}\ {}_{r,0}T_{\omega;b,k}f(r)\,dp = 0 \tag{93}$$

Similarly if $u\left(r,t;\tau,\omega;b,k\right)$ is given by (90), then using the substitution $y=\frac{t}{p}$ implies that:

$$u\left(r,t;\tau,\omega;b,k\right)\Big|_{t=0} = \frac{2B_{b;k}\left(\frac{\omega-\tau}{2},\frac{\tau-2\omega+3}{2}\right)}{\Gamma^2_{b;k}\left(\frac{\omega-\tau}{2}\right)}t^{\tau-\omega}\int_t^{\infty} u(r,y)y^{2\omega-\tau-2}\left(y^2-t^2\right)^{\frac{1-\omega}{2}}\ {}_{r,y}T_{\omega;b,k}f(r)\,dy.\Big|_{t=0}$$
$$= \frac{2B_{b;k}\left(\frac{\omega-\tau}{2},\frac{\tau-2\omega+3}{2}\right)}{\Gamma^2_{b;k}\left(\frac{\omega-\tau}{2}\right)}t^{\tau-\omega}\int_1^{0} u(r,y)y^{2\omega-\tau-2}\left(y^2-t^2\right)^{\frac{1-\omega}{2}}\ {}_{r,tp^{-1}}T_{\omega;b,k}f(r)\,(-tp^{-2})dp.\Big|_{t=0}$$
$$= \frac{2B_{b;k}\left(\frac{\omega-\tau}{2},\frac{\tau-2\omega+3}{2}\right)}{\Gamma^2_{b;k}\left(\frac{\omega-\tau}{2}\right)}\ {}_{r,0}T_{\omega;b,k}f(r)\int_0^1 u(r,tp^{-1})\Big|_{t=0} p^{\tau-\omega-1}\left(1-p^2\right)^{\frac{1-\omega}{2}}\,dp = f(r) \tag{94}$$

and

$$\frac{\partial}{\partial t}\Big(u\left(r,t;\tau,\omega;b,k\right)\Big)\Big|_{t=0} = u_t\left(r,t;\tau,\omega;b,k\right)\Big|_{t=0}$$
$$= \frac{2B_{b;k}\left(\frac{\omega-\tau}{2},\frac{\tau-2\omega+3}{2}\right)}{\Gamma^2_{b;k}\left(\frac{\omega-\tau}{2}\right)}\ {}_{r,0}T_{\omega;b,k}f(r)\int_0^1 u_t(r,tp^{-1})\Big|_{t=0} p^{\tau-\omega-1}\left(1-p^2\right)^{\frac{1-\omega}{2}}\,dp = 0 \tag{95}$$

Equations (92), (93), (94), and (95) clearly confirm the validity of the function $u(r,t;\tau,\omega;b,k)$, as defined in (88) and (90), with respect to the initial conditions given in (51). This completes theorem 3.6. □

**Theorem 3.7.** *Let* $u(r,t)\in L^2(0,\infty)$ *be twice continuously differentiable on* $(0,\infty)$.

*Assume* $P\left(r,t;\omega,\tau,\beta;b;k\right)$, $Q\left(r,t;\omega,\tau,\beta;b;k\right)\in C^2([0,\infty))$ *defined by equations (96) and (97) respectively.*

$$P\left(r,t;\omega,\tau,\beta;b;k\right) = C\frac{2^{\beta+1}\Gamma\left(\frac{\tau+\beta+1}{2}\right)r^{-(\beta+\tau+1)}}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(\frac{\omega+1}{2}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{\tau+\beta+1}{2}\right)_n B_{b;k}\left(\frac{\beta+n+2}{2},\frac{\omega-\beta-1}{2}\right)}{B\left(\frac{\beta+2}{2},\frac{\omega-\beta-1}{2}\right)n!}\left(\frac{t^2}{r^2}\right)^n. \tag{96}$$

$$Q\left(r,t;\omega,\tau,\beta;b;k\right) = \frac{C\cdot 2^{\beta+1}\Gamma\left(\frac{\tau+\beta+1}{2}\right)t^{-(\beta+\tau+1)}}{\Gamma\left(\frac{\tau+1}{2}\right)\Gamma\left(\frac{\omega-\tau-\beta}{2}\right)\Gamma\left(\frac{\tau+1}{2}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{\tau+\beta+1}{2}\right)_n B_{b;k}\left(\frac{\tau-\omega+\beta+2+2n}{2},\frac{\omega-\beta-1}{2}\right)}{B\left(\frac{\tau-\omega+\beta+2}{2},\frac{\omega-\beta-1}{2}\right)n!}\left(\frac{r^2}{t^2}\right)^n. \tag{97}$$

*and the following boundary limits hold:*

$$\left.\begin{array}{ll}\textit{(i)}\lim\limits_{t\to 0}\left(P\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\right), & \textit{(iii)}\quad\lim\limits_{t\to\infty}\left(Q\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\right),\\ \textit{(ii)}\lim\limits_{t\to 0}\left(P_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\right), & \textit{(iv)}\quad\lim\limits_{t\to\infty}\left(Q_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\right).\end{array}\right\} \tag{98}$$

*Using (96) and (97), the Extended $k$-Integral Transmutation Operator (57) reduces to:*

$$T^{(\beta)}_{\omega,\tau;b;k}u(r,t) = \int_0^r P\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^{\omega}dt + \int_r^{\infty} Q\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^{\omega}dt$$
$$\Re(\beta+\frac{\tau-\omega}{2})<0; r,t,b,k,(\tau+\beta+1),(\tau-\omega+\beta),(\omega-\tau-\beta)\in\Re^{+}; \omega-\beta>1; \omega,\tau\in\Re;$$
$$n\in\mathbb{N}_0; \beta\notin\mathbb{N}_0. \tag{99}$$

*satisfying*

$$T^{(\beta)}_{\omega,\tau;b,k}\left(B_{\omega}\right)_t u(r,t) = \left(B_{\tau}\right)_r T^{(\beta)}_{\omega,\tau;b,k}u(r,t) \tag{100}$$

*where $(B_{\omega})_t$ and $(B_{\tau})_r$ are the Singular Bessel operators defined in equation (50) also satisfying the following relations*

$$\left.\begin{array}{ll} (i) & (B_{\omega})_t P\left(r,t;\omega,\tau,\beta;b;k\right) = (B_{\tau})_r P\left(r,t;\omega,\tau,\beta;b;k\right), \\ (ii) & (B_{\omega})_t Q\left(r,t;\omega,\tau,\beta;b;k\right) = (B_{\tau})_r Q\left(r,t;\omega,\tau,\beta;b;k\right). \end{array}\right\} \tag{101}$$

*and*

$$\begin{aligned}&\big(Q_r\left(r,r;\omega,\tau,\beta;b,k\right)-P_r\left(r,r;\omega,\tau,\beta;b,k\right)\big)r^{\omega}u(r,r)r^{\omega}u(r,r)\\ &\quad=(\tau+\omega)r^{\omega-1}u(r,r)\big(P\left(r,r;\omega,\tau,\beta;b;k\right)-Q\left(r,r;\omega,\tau,\beta;b;k\right)\big)\\ &\qquad+2r^{\omega}u(r,r)\big(P_r\left(r,t;\omega,\tau,\beta;b;k\right)-Q_r\left(r,t;\omega,\tau,\beta;b;k\right)\big).\end{aligned} \tag{102}$$

*Proof.* From (100), we have:

$$\begin{aligned}T^{(\beta)}_{\omega,\tau;b,k}\left(B_{\omega}\right)_t u(r,t) &= \int_0^r P\left(r,t;\omega,\tau,\beta;b;k\right)\left(B_{\omega}\right)_t u(r,t)t^{\omega}dt\\ &\quad+\int_r^{\infty} Q\left(r,t;\omega,\tau,\beta;b;k\right)\left(B_{\omega}\right)_t u(r,t)t^{\omega}dt.\end{aligned} \tag{103}$$

Replacing the Singular Bessel operator with its form given by $(B_{\omega})_t = \frac{\partial}{\partial t}+\frac{\omega}{t}\frac{\partial^2}{\partial t^2} = \frac{1}{t^{\omega}}\frac{d}{dt}t^{\omega}\frac{d}{dt}$ we have

$$\begin{aligned}&\int_0^r P\left(r,t;\omega,\tau,\beta;b,k\right)\left(B_{\omega}\right)_t u(r,t)\,t^{\omega}\,dt\\ &=\int_0^r P\left(r,t;\omega,\tau,\beta;b;k\right)\frac{d}{dt}t^{\omega}\frac{d}{dt}u(r,t)dt \text{ (integrate by parts) } \Longrightarrow U=P, dV=\frac{d}{dt}t^{\omega}\frac{d}{dt}u(r,t)dt\\ &=P\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big|_{t=0}^{r}-\int_0^r P_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\,dt \text{ (again by parts)}\\ &=P\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big|_{t=0}^{r}-t^{\omega}P_t\left(r,t;\omega,\tau,\beta;b,k\right)u(r,t)\Big|_{t=0}^{r}\\ &\quad+\int_0^r\bigg((B_{\omega})_t P\left(r,t;\omega,\tau,\beta;b;k\right)\bigg)t^{\omega}u(r,t)\,dt.\\ &=P\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big|_{t=r}-\lim_{t\to0}\bigg(P\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\bigg)\\ &\quad-P_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\Big|_{t=r}+\lim_{t\to0}\bigg(P_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\bigg)\\ &\quad+\int_0^r\bigg((B_{\omega})_t P\left(r,t;\omega,\tau,\beta;b;k\right)\bigg)t^{\omega}u(r,t)\,dt.\end{aligned} \tag{104}$$

Similarly

$$
\begin{aligned}
&\int_r^{\infty} Q\left(r,t;\omega,\tau,\beta;b,k\right)\,(B_\omega)_t u(r,t)\,t^{\omega}\,dt\\
&=\int_r^{\infty} Q\left(r,t;\omega,\tau,\beta;b,k\right)\,\frac{d}{dt}t^{\omega}\frac{d}{dt}u(r,t)\,dt\\
&=Q\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big|_{t=r}^{\infty}-\int_r^{\infty} Q_t\left(r,t;\omega,\tau,\beta;b,k\right)\,t^{\omega}\frac{d}{dt}u(r,t)\,dt\\
&=Q\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big|_{t=r}^{\infty}-Q_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\Big|_{t=r}^{\infty}\\
&\qquad+\int_r^{\infty}\Big((B_\omega)_t Q\left(r,t;\omega,\tau,\beta;b;k\right)\Big)t^{\omega}u(r,t)\,dt.\\
&=\lim_{t\to\infty}\Big(Q\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big)-Q\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big|_{t=r}\\
&\quad-\lim_{t\to\infty}\Big(Q_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\Big)+Q_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\Big|_{t=r}\\
&\qquad+\int_r^{\infty}\Big((B_\omega)_t Q\left(r,t;\omega,\tau,\beta;b;k\right)\Big)t^{\omega}u(r,t)\,dt. \qquad (105)
\end{aligned}
$$

Considering (98), equation (103) becomes

$$
\begin{aligned}
&T^{(\beta)}_{\omega,\tau;b,k}\,(B_\omega)_t u(r,t)=P\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big|_{t=r}-\lim_{t\to 0}\Big(P\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big)\\
&\quad-P_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\Big|_{t=r}+\lim_{t\to 0}\Big(P_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\Big)\\
&\qquad+\int_0^{r}\Big((B_\omega)_t P\left(r,t;\omega,\tau,\beta;b;k\right)\Big)t^{\omega}u(r,t)\,dt.+\lim_{t\to\infty}\Big(Q\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big)\\
&\quad-Q\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}\frac{d}{dt}u(r,t)\Big|_{t=r}-\lim_{t\to\infty}\Big(Q_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\Big)\\
&\qquad+Q_t\left(r,t;\omega,\tau,\beta;b,k\right)t^{\omega}u(r,t)\Big|_{t=r}+\int_r^{\infty}\Big((B_\omega)_t Q\left(r,t;\omega,\tau,\beta;b;k\right)\Big)t^{\omega}u(r,t)\,dt\\
&=P\left(r,r;\omega,\tau,\beta;b,k\right)r^{\omega}u_r(r,r)-P_r\left(r,r;\omega,\tau,\beta;b,k\right)r^{\omega}u(r,r)\\
&\qquad+\int_0^{r}\Big((B_\omega)_t P\left(r,t;\omega,\tau,\beta;b;k\right)\Big)t^{\omega}u(r,t)\,dt+\int_r^{\infty}\Big((B_\omega)_t Q\left(r,t;\omega,\tau,\beta;b;k\right)\Big)t^{\omega}u(r,t)\,dt\\
&\qquad+Q_r\left(r,r;\omega,\tau,\beta;b,k\right)r^{\omega}u(r,r)-Q\left(r,r;\omega,\tau,\beta;b,k\right)r^{\omega}u_r(r,r). \qquad (106)
\end{aligned}
$$

Additionally from (100), we have

$$
\begin{aligned}
&(B_\tau)_r\,T^{(\beta)}_{\omega,\tau;b,k}u(r,t)=\frac{1}{r^{\tau}}\frac{d}{dr}r^{\tau}\frac{d}{dr}T^{(\beta)}_{\omega,\tau;b,k}u(r,t)\\
&=\frac{1}{r^{\tau}}\frac{d}{dr}r^{\tau}\frac{d}{dr}\int_0^{r}P\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^{\omega}dt+\frac{1}{r^{\tau}}\frac{d}{dr}r^{\tau}\frac{d}{dr}\int_r^{\infty}Q\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^{\omega}dt. \qquad (107)
\end{aligned}
$$

Using differentiation of a definite integral with respect to a parameter (the Leibniz's integral rule) appearing as entry in [13, p.22, eqn **0.410**]. Therefore, we have

$$
\frac{1}{r^{\tau}}\frac{d}{dr}r^{\tau}\frac{d}{dr}\int_0^{r}P\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^{\omega}dt=\frac{1}{r^{\tau}}\frac{d}{dr}\left(r^{\tau}\frac{d}{dr}\int_0^{r}P\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^{\omega}dt\right)
$$

$$
\begin{aligned}
&= \frac{1}{r^\tau}\frac{d}{dr}\bigg( r^\tau P\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega \bigg|_{t=r} + r^\tau \int_0^r P_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega dt \bigg) \\
&= \frac{1}{r^\tau}\bigg( \frac{d}{dr}\big( r^{\tau+\omega} P\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r)\big) + \frac{d}{dr}\big( r^\tau \int_0^r P_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega dt\big) \bigg) \\
&= (\tau+\omega) r^{\omega-1} P\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) + r^\omega P_r\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) \\
&\quad + r^\omega P\left(r,r;\omega,\tau,\beta;b;k\right)u_r(r,r) + P_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega \bigg|_{t=r} \\
&\quad + \int_0^r \frac{1}{r^\tau}\bigg( \tau r^{\tau-1} P_r\left(r,t;\omega,\tau,\beta;b;k\right) + r^\tau P_{rr}\left(r,t;\omega,\tau,\beta;b;k\right) \bigg) u(r,t)t^\omega dt \\
&= (\tau+\omega) r^{\omega-1} P\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) + r^\omega P_r\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) \\
&\quad + r^\omega P\left(r,r;\omega,\tau,\beta;b;k\right)u_r(r,r) + P_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega \bigg|_{t=r} \\
&\quad + \int_0^r \big( (B_\tau)_r P\left(r,t;\omega,\tau,\beta;b;k\right) \big) u(r,t)t^\omega dt \qquad (108)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{r^\tau}\frac{d}{dr} r^\tau \frac{d}{dr}\int_r^\infty Q\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega dt = \frac{1}{r^\tau}\frac{d}{dr}\bigg( r^\tau \frac{d}{dr}\int_r^\infty Q\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega dt \bigg) \\
&= \frac{1}{r^\tau}\frac{d}{dr}\bigg( - r^\tau Q\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega \bigg|_{t=r} + r^\tau \int_r^\infty Q_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega dt \bigg) \\
&= \frac{1}{r^\tau}\bigg( - \frac{d}{dr}\big( r^{\tau+\omega} Q\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r)\big) + \frac{d}{dr}\big( r^\tau \int_r^\infty Q_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega dt\big) \bigg) \\
&= -(\tau+\omega) r^{\omega-1} Q\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) - r^\omega Q_r\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) \\
&\quad - r^\omega Q\left(r,r;\omega,\tau,\beta;b;k\right)u_r(r,r) - Q_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega \bigg|_{t=r} \\
&\quad + \int_r^\infty \frac{1}{r^\tau}\bigg( \tau r^{\tau-1} Q_r\left(r,t;\omega,\tau,\beta;b;k\right) + r^\tau Q_{rr}\left(r,t;\omega,\tau,\beta;b;k\right) \bigg) u(r,t)t^\omega dt \\
&= -(\tau+\omega) r^{\omega-1} Q\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) - r^\omega Q_r\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) \\
&\quad - r^\omega Q\left(r,r;\omega,\tau,\beta;b;k\right)u_r(r,r) - Q_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega \bigg|_{t=r} \\
&\quad + \int_r^\infty \big( (B_\tau)_r Q\left(r,t;\omega,\tau,\beta;b;k\right) \big) u(r,t)t^\omega dt \qquad (109)
\end{aligned}
$$

and therefore equation (107) becomes

$$
\begin{aligned}
&(B_\tau)_r T^{(\beta)}_{\omega,\tau;b,k} u(r,t) = (\tau+\omega) r^{\omega-1} P\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) + r^\omega P_r\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) \\
&\quad + r^\omega P\left(r,r;\omega,\tau,\beta;b;k\right)u_r(r,r) + P_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega \bigg|_{t=r} \\
&\quad + \int_0^r \big( (B_\tau)_r P\left(r,t;\omega,\tau,\beta;b;k\right) \big) u(r,t)t^\omega dt - (\tau+\omega) r^{\omega-1} Q\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) \\
&\quad - r^\omega Q_r\left(r,r;\omega,\tau,\beta;b;k\right)u(r,r) - r^\omega Q\left(r,r;\omega,\tau,\beta;b;k\right)u_r(r,r) \\
&\quad - Q_r\left(r,t;\omega,\tau,\beta;b;k\right)u(r,t)t^\omega \bigg|_{t=r} + \int_r^\infty \big( (B_\tau)_r Q\left(r,t;\omega,\tau,\beta;b;k\right) \big) u(r,t)t^\omega dt \\
&= (\tau+\omega) r^{\omega-1} u(r,r)\big( P\left(r,r;\omega,\tau,\beta;b;k\right) - Q\left(r,r;\omega,\tau,\beta;b;k\right) \big)
\end{aligned}
$$

$$
\begin{aligned}
&+2r^{\omega}u(r,r)\big(P_r\left(r,t;\omega,\tau,\beta;b;k\right)-Q_r\left(r,t;\omega,\tau,\beta;b;k\right)\big)\\
&+r^{\omega}u_r(r,r)\big(P\left(r,r;\omega,\tau,\beta;b;k\right)-Q\left(r,r;\omega,\tau,\beta;b;k\right)\big)\\
&+\int_0^r\big((B_\tau)_rP\left(r,t;\omega,\tau,\beta;b;k\right)\big)u(r,t)t^{\omega}dt+\int_r^{\infty}\big((B_\tau)_rQ\left(r,t;\omega,\tau,\beta;b;k\right)\big)u(r,t)t^{\omega}dt.
\end{aligned}
\tag{110}
$$

From (100), equating (106) and (110) we get

$$
\begin{aligned}
&-P_r\left(r,r;\omega,\tau,\beta;b,k\right)r^{\omega}u(r,r)+Q_r\left(r,r;\omega,\tau,\beta;b,k\right)r^{\omega}u(r,r)\\
&+\int_0^r\big((B_\omega)_tP\left(r,t;\omega,\tau,\beta;b;k\right)\big)t^{\omega}u(r,t)\,dt+\int_r^{\infty}\big((B_\omega)_tQ\left(r,t;\omega,\tau,\beta;b;k\right)\big)t^{\omega}u(r,t)\,dt=\\
&(\tau+\omega)r^{\omega-1}u(r,r)\big(P\left(r,r;\omega,\tau,\beta;b;k\right)-Q\left(r,r;\omega,\tau,\beta;b;k\right)\big)\\
&+2r^{\omega}u(r,r)\big(P_r\left(r,t;\omega,\tau,\beta;b;k\right)-Q_r\left(r,t;\omega,\tau,\beta;b;k\right)\big)\\
&+\int_0^r\big((B_\tau)_rP\left(r,t;\omega,\tau,\beta;b;k\right)\big)u(r,t)t^{\omega}dt+\int_r^{\infty}\big((B_\tau)_rQ\left(r,t;\omega,\tau,\beta;b;k\right)\big)u(r,t)t^{\omega}dt.
\end{aligned}
\tag{111}
$$

Comparing the respective terms on both sides of (111), we derive the following two relations.

$$
\left.
\begin{aligned}
&\text{(i)}\quad (B_\omega)_tP\left(r,t;\omega,\tau,\beta;b;k\right)=(B_\tau)_rP\left(r,t;\omega,\tau,\beta;b;k\right),\\
&\text{(ii)}\quad (B_\omega)_tQ\left(r,t;\omega,\tau,\beta;b;k\right)=(B_\tau)_rQ\left(r,t;\omega,\tau,\beta;b;k\right).
\end{aligned}
\right\}
\tag{112}
$$

$$
\begin{aligned}
&\big(Q_r\left(r,r;\omega,\tau,\beta;b,k\right)-P_r\left(r,r;\omega,\tau,\beta;b,k\right)\big)r^{\omega}u(r,r)r^{\omega}u(r,r)\\
&\quad=(\tau+\omega)r^{\omega-1}u(r,r)\big(P\left(r,r;\omega,\tau,\beta;b;k\right)-Q\left(r,r;\omega,\tau,\beta;b;k\right)\big)\\
&\qquad+2r^{\omega}u(r,r)\big(P_r\left(r,t;\omega,\tau,\beta;b;k\right)-Q_r\left(r,t;\omega,\tau,\beta;b;k\right)\big).
\end{aligned}
\tag{113}
$$

This completes the proof of theorem (3.7). □

## 4. CONCLUSION AND RECOMMENDATIONS

This study applied the Extended $k-$Riemann approach in formulating solution to the Singular initial value GEPD equation, yielding explicit solutions incorporating the Appell hypergeometric function of the third kind. By incorporating the Extended $k$-Beta function, the Extended $k$-Riemann function was defined with key properties and differentiability established, revealing smoothness properties except at singularities point $t = 0$.

These findings create new opportunities for exploring and solving partial differential equations in physics, mechanics, and mathematics.

Upcoming research efforts will be devoted to:

- Examining the convergence property of the derived series.
- Developing efficient numerical algorithms for implementing the Extended $k$-Riemann method.
- Applying the developed framework to problems in mathematical physics, such as quantum mechanics, fluid dynamics, and signal processing, where singular operators commonly arise.

The results are anticipated to play a pivotal role to the development of partial differential equation theory and offer practical applications in multiple fields of science and technology.

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