On Dual Hyperbolic Generalized Edouard Numbers

Abstract. In this research, the generalized dual hyperbolic Edouard numbers are introduced. Various special cases are explored (including dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers). Binet's formulas, generating functions and summation formulas for these numbers are presented. Moreover, along with matrices associated with these sequences.

2010 Mathematics Subject Classification. 11B39, 11B83.

Keywords. Dual hyperbolic Edouard numbers, dual hyperbolic Edouard-Lucas numbers.

1. Introduction

In mathematical and geometric contexts, a hypercomplex system refers to a framework that generalizes the principles of complex numbers. These systems possess rich algebraic structures and are frequently studied for their diverse applications in physics and engineering. Below, we provide a concise overview of the key application areas of hypercomplex number systems in these fields.

In contrast to complex numbers, hypercomplex systems provide a more sophisticated framework for representing transformations and symmetries in higher-dimensional spaces. As noted by Kantor in [15], these systems can be viewed as extensions of the real number line, offering algebraic tools tailored to multidimensional analysis. The principal types of hypercomplex number systems encompass complex numbers, hyperbolic numbers, and dual numbers. Complex numbers, defined by a real and an imaginary component, serve as the foundational structure for more advanced hypercomplex systems. Hyperbolic numbers build upon the complex number framework and are employed in diverse mathematical models, particularly those involving Lorentz transformations and spacetime geometries. Dual numbers, distinguished by the presence of a dual unit whose square is zero, are instrumental in various algebraic constructions, including automatic differentiation and kinematic analysis.

The following sections offer more detailed insights into the mathematical properties and application areas of these hypercomplex systems.

- Complex numbers are constructed by extending the real number system through the introduction of an imaginary unit, denoted as "i", which satisfies the identity $i^2 = -1$. A complex number is typically expressed in the form $z = a + bi$, where a and b are real numbers, and i represents the imaginary unit.
- Hyperbolic numbers also referred to as double numbers or split complex numbers extend the real number system by introducing a new unit element j, which satisfies the identity $j^2 = 1$ [17]. These numbers are distinct from real and complex numbers due to their unique algebraic properties. A hyperbolic number is defined as:

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

  where a and b are real numbers and j is the hyperbolic unit. This structure enables the modeling of systems with split-signature metrics and has notable applications in areas such as special relativity and signal processing.
- Dual numbers [9] expand the real number system through the incorporation of a new element ε, which satisfies the identity $\varepsilon^2 = 0$. This infinitesimal unit distinguishes dual numbers from other hypercomplex systems and makes them especially valuable in modeling instantaneous rates of change. A dual number is defined as:

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

  where a and b are real numbers, and ε is the nilpotent unit. Dual numbers are commonly used in applications such as automatic differentiation, kinematics, and perturbation analysis, due to their ability to elegantly encode infinitesimal variations.
- Among the non-commutative examples of hypercomplex number systems are quaternions [11]. Quaternions generalize complex numbers by incorporating three distinct imaginary units, typically denoted as i,j, and k. A quaternion has the form as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.These multiplication rules result in a non-commutative structure, meaning the order of multiplication affects the result.The set of quaternion numbers is formally defined as:

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- Additional hypercomplex systems include octonions and sedenions, which are discussed in [13] and [18]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions), and $\mathbb{S}$ (sedenions) are all constructed as real algebras derived from the real numbers $\mathbb{R}$ using a recursive procedure known as the Cayley–Dickson Process. This technique successively doubles the dimension of each algebra and continues beyond sedenions to produce what are collectively referred to as the 2^n-ions.The following table highlights selected publications from the literature that investigate the properties and applications of these extended number systems.

Table 1. Papers that have been published in the literature raleted to 2^n-ions.

Authors and Title of the paper↓		Papers↓
Biss, D.K., Dugger, D., Isaksen, D.C., Large annihilators in Cayley-Dickson algebras		[4]
Hamilton, W.R., Elements of Quaternions		[11]
Imaeda, K., Sedenions: algebra and analysis		[12]
Moreno, G., The zero divisors of the Cayley-Dickson algebras over the real numbers		[16]
Göcen, M., Soykan, Y., Horadam 2^k-Ions		[10]
Soykan,Y., Tribonacci and Tribonacci-Lucas Sedenions		[18]
On higher order Fibonacci hyper complex numbers		[14]

A dual hyperbolic number is a type of hypercomplex number, specifically a member of the hyperbolic number system. A dual hyperbolic number is defined as follows

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

$\mathbb{H}_{\mathbb{D}}$, the set of all dual hyperbolic numbers, are generally denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linearly independent, and the algebra $\mathbb{H}_{\mathbb{D}}$ is generated by their span, i.e. $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$

Therefore, $\{1, j, \varepsilon, \varepsilon j\}$ forms a basis for the dual hyperbolic algebra $\mathbb{H}_{\mathbb{D}}$. For more detail, see [2].

The next properties are holds for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

We claim that p and q be two dual hyperbolic numbers that $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ and then we can write the product of p and q as

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and we can write the sum dual hyperbolic numbers p and q as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. $\mathbb{H}_{\mathbb{D}}$ is not field since every dual hyperbolic numbers doesn't have an inverse. For more detail about dual hyperbolic numbers, see [2].

Next, we give some proporties about generalized Edouard numbers.

A generalized Edouard sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3};\ W_0, W_1, W_2 \quad (n \geq 3) \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be given to negative subscripts by defining

$$W_{-n} = 7W_{-(n-1)} - 7W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (1.1) is true for all integer n.

In the Table 2 we give the first few generalized Edouard numbers with positive subscript and negative subscript

Table 2. A few generalized Edouard numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$7W_0 - 7W_1 + W_2$
2	W_2	$42W_0 - 48W_1 + 7W_2$
3	$W_0 - 7W_1 + 7W_2$	$246W_0 - 287W_1 + 42W_2$
4	$7W_0 - 48W_1 + 42W_2$	$1435W_0 - 1680W_1 + 246W_2$
5	$42W_0 - 287W_1 + 246W_2$	$8365W_0 - 9799W_1 + 1435W_2$
6	$246W_0 - 1680W_1 + 1435W_2$	$48756W_0 - 57120W_1 + 8365W_2$

If we take $W_0 = 0, W_1 = 1, W_2 = 7$ then $\{E_n\}$ is the Edouard sequence, if we take $W_0 = 3, W_1 = 7, W_2 = 35$ then $\{K_n\}$ is the Edouard-Lucas sequence. In other words, Edouard sequence $\{E_n\}_{n\geq 0}$, Edouard-Lucas sequence $\{K_n\}_{n\geq 0}$ are given by the third-order recurrence relations

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \qquad E_0 = 0, E_1 = 1, E_2 = 7, \tag{1.2}$$

$$K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 7, K_2 = 35, \tag{1.3}$$

In addition that the sequences given above can be extended to negative subscripts by defining,

$$\begin{aligned} E_{-n} &= 7E_{-(n-1)} - 7E_{-(n-2)} + E_{-(n-3)}, \\ K_{-n} &= 7K_{-(n-1)} - 7K_{-(n-2)} + K_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.2)-(1.3) are true for all integer n.

We can enumerate several essential properties of generalized Edouard numbers that are required.

Binet formula of generalized Edouard sequence can be calculated using its characteristic equation given as

$$z^3 - 7z^2 + 7z - 1 = \left(z^2 - 6z + 1\right)(z - 1) = 0,$$

where the roots of above equation are

$$\begin{aligned}\alpha &= 3+2\sqrt{2},\\ \beta &= 3-2\sqrt{2},\\ \gamma &= 1.\end{aligned}$$

Using these roots and the recurrence relation of $\{W_n\}$, we can write the Binet's formula can be written as

$$\begin{aligned}W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}-\frac{z_3\gamma^n}{4} \\ &= A_1\alpha^n+A_2\beta^n+A_3\gamma^n,\end{aligned} \tag{1.4}$$

where z_1, z_2 and z_3 are given below

$$\begin{aligned}z_1 &= W_2-(\beta+1)W_1+\beta W_0,\\ z_2 &= W_2-(\alpha+1)W_1+\alpha W_0,\\ z_3 &= W_2-6W_1+W_0,\end{aligned}$$

and

$$\begin{aligned}A_1 &= \frac{W_2-(\beta+1)W_1+\beta W_0}{(\alpha-\beta)(\alpha-\gamma)},\\ A_2 &= \frac{W_2-(\alpha+1)W_1+\alpha W_0}{(\beta-\alpha)(\beta-\gamma)},\\ A_3 &= \frac{W_2-6W_1+W_0}{(\gamma-\alpha)(\gamma-\beta)}.\end{aligned} \tag{1.5}$$

Binet's formula of Edouard, Edouard-Lucas sequences can be written as

$$\begin{aligned}E_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4},\\ K_n &= \alpha^n+\beta^n+1.\end{aligned}$$

After then we can write the generating function of generalized Edouard numbers,

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-7W_0)x+(W_2-7W_1+7W_0)x^2}{1-7x+7x^2-x^3}. \tag{1.6}$$

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

LEMMA 1. *[3, Lemma 1.4]. Suppose that* $f_{W_n}(x)=\sum\limits_{n=0}^{\infty} W_n\frac{x^n}{n!}$ *is the exponential generating function of the generalized Edouard sequence* $\{W_n\}$*. Then*

$$\sum_{n=0}^{\infty} W_n\frac{x^n}{n!} = \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)}e^{\alpha x}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)}e^{\beta x}-\frac{(W_2-6W_1+W_0)}{4}e^{x}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2. *Exponential generating function of Edouard and Edouard-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}) \frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4} e^x.$$

b):

$$\sum_{n=0}^{\infty} K_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n + 1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^x.$$

For more details about generalized Edouard numbers, see [22].

Now, we give some information on published papers related to hyperbolic and dual hyperbolic numbers in litarature.

- Cockle [8] explored hyperbolic numbers with complex coefficients, contributing to the early development of hypercomplex algebra.
- Cheng and Thompson [6] introduced dual numbers with complex coefficients, expanding the algebraic versatility of dual number systems for applications in polynomial equations and transformation theory.
- Akar at al [2] introduced the concept of dual hyperbolic numbers, combining characteristics of dual and hyperbolic systems into a unified algebraic structure.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan at al [19] introduced the concept of dual hyperbolic generalized Pell numbers, extending classical Pell sequences into the framework of hypercomplex algebra. These numbers are defined as:

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

  where $j^2 = 1$ and $\varepsilon^2 = 0$, reflecting the underlying dual hyperbolic structure.The scalar components V_n follow the recurrence relation for generalized Pell numbers:$V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$with initial values V_0, V_1 not both zero.
- Cihan at al [1] examined the structure of dual hyperbolic Fibonacci and dual hyperbolic Lucas numbers, which integrate classical sequences into a hypercomplex framework. These numbers are defined as:

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

  where $j^2 = 1$, $\varepsilon^2 = 0$.The scalar components are governed by the classical recurrence relations. Fibonacci sequence: $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, Lucas sequence: $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan at al [[20] investigated dual hyperbolic generalized Jacobsthal numbers, extending classical recurrence relations within a hypercomplex framework. These numbers are expressed as:

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $j^2 = 1$, $\varepsilon^2 = 0$ and the scalar components J_n follow the recurrence relation $J_n = J_{n-1}+2J_{n-2}$, $J_0 = a,\ J_1 = b$.

- Bród at al [5] examined the structure of dual hyperbolic generalized balancing numbers, integrating classical recurrence sequences with dual and hyperbolic number theory. These numbers are defined as:

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $j^2 = 1$, $\varepsilon^2 = 0$ and the base sequence B_n follows the recurrence $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0,\ B_1 = 1$.

Next section, we present the dual hyperbolic generalized Edouard numbers and give some properties of these numbers.

2. Dual Hyperbolic Generalized Edouard Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual hyperbolic generalized Edouard numbers then using this defination, we present generating functions and Binet's formula of dual hyperbolic generalized Edouard numbers.

We now examine dual hyperbolic generalized Edouard numbers within the algebra $\mathbb{H}_{\mathbb{D}}$. The nth such number is defined as

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{2.1}$$

with the initial values $\widehat{W}_0, \widehat{W}_1, \widehat{W}_2$. (2.1) can be written to negative subscripts by defining,

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3}. \tag{2.2}$$

so identity (2.1) holds for all integers n.

Now, we define some special cases of dual hyperbolic generalized Edouard numbers. The nth dual hyperbolic Edouard numbers, the nth dual hyperbolic Edouard-Lucas numbers, respectively, are given as the nth dual hyperbolic Edouard numbers is given $\widehat{E}_n = E_n + jE_{n+1} + \varepsilon E_{n+2} + j\varepsilon E_{n+3}$, with the initial

values

$$\begin{aligned}
\widehat{E}_0 &= E_0 + jE_1 + \varepsilon E_2 + j\varepsilon E_3, \\
\widehat{E}_1 &= E_1 + jE_2 + \varepsilon E_3 + j\varepsilon E_4, \\
\widehat{E}_2 &= E_2 + jE_3 + \varepsilon E_4 + j\varepsilon E_5,
\end{aligned}$$

the nth dual hyperbolic Edouard-Lucas numbers is given $\widehat{K}_n = K_n + jK_{n+1} + \varepsilon K_{n+2} + j\varepsilon K_{n+3}$ with the initial values

$$\begin{aligned}\widehat{K}_0 &= K_0 + jK_1 + \varepsilon K_2 + j\varepsilon K_3,\\ \widehat{K}_1 &= K_1 + jK_2 + \varepsilon K_3 + j\varepsilon K_4,\\ \widehat{K}_2 &= K_2 + jK_3 + \varepsilon K_4 + j\varepsilon K_5.\end{aligned}$$

Note that, for dual hyperbolic Edouard numbers (by using $W_n = E_n$, $E_0 = 0$, $E_1 = 1$, $E_2 = 7$) we get

$$\begin{aligned}\widehat{E}_0 &= j + 7\varepsilon + 42j\varepsilon,\\ \widehat{E}_1 &= 1 + 7j + 42\varepsilon + 246j\varepsilon,\\ \widehat{E}_2 &= 7 + 42j + 246\varepsilon + 1435j\varepsilon,\end{aligned}$$

for dual hyperbolic Edouard-Lucas numbers (by using $W_n = K_n$, $K_0 = 3$, $K_1 = 7$, $K_2 = 35$) we obtain

$$\begin{aligned}\widehat{K}_0 &= 3 + 7j + 35\varepsilon + 199j\varepsilon,\\ \widehat{K}_1 &= 7 + 35j + 199\varepsilon + 1155j\varepsilon,\\ \widehat{K}_2 &= 35 + 199j + 1155\varepsilon + 6727j\varepsilon.\end{aligned}$$

So, using (2.1), we can write the following identity for non negative integers n,

$$\widehat{W}_n = 7\widehat{W}_{n-1} - 7\widehat{W}_{n-2} + \widehat{W}_{n-3}, \tag{2.3}$$

and the sequence $\{\widehat{W}_n\}_{n\geq 0}$ can be given as

$$\widehat{W}_{-n} = 7\widehat{W}_{-(n-1)} - 7\widehat{W}_{-(n-2)} + \widehat{W}_{-(n-3)},$$

for $n = 1, 2, 3, ...$ by using (2.2). As a result., recurrence (2.3) holds for all integer n.

Table 3 presents the initial values of the dual hyperbolic generalized Edouard numbers$\widehat{W}_n$, showcasing terms with both positive and negative subscripts for a comprehensive view of the sequence's symmetric structure.

Table 3. A few dual hyperbolic generalized Edouard numbers

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$7\widehat{W}_0 - 7\widehat{W}_1 + \widehat{W}_2$
2	$\widehat{W}_2$	$42\widehat{W}_0 - 48\widehat{W}_1 + 7\widehat{W}_2$
3	$\widehat{W}_0 - 7\widehat{W}_1 + 7\widehat{W}_2$	$246\widehat{W}_0 - 287\widehat{W}_1 + 42\widehat{W}_2$
4	$7\widehat{W}_0 - 48\widehat{W}_1 + 42\widehat{W}_2$	$1435\widehat{W}_0 - 1680\widehat{W}_1 + 246\widehat{W}_2$
5	$42\widehat{W}_0 - 287\widehat{W}_1 + 246\widehat{W}_2$	$8365\widehat{W}_0 - 9799\widehat{W}_1 + 1435\widehat{W}_2$
6	$246\widehat{W}_0 - 1680\widehat{W}_1 + 1435\widehat{W}_2$	$48756\widehat{W}_0 - 57120\widehat{W}_1 + 8365\widehat{W}_2$

Note that

$$\begin{aligned} \widehat{W}_0 &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3, \\ \widehat{W}_1 &= W_1 + jW_2 + \varepsilon W_3 + j\varepsilon W_4, \\ \widehat{W}_2 &= W_2 + jW_3 + \varepsilon W_4 + j\varepsilon W_5. \end{aligned}$$

A few dual hyperbolic Edouard numbers, dual hyperbolic Edouard-Lucas numbers with positive subscript and negative subscript are given in the following Table 4, Table 5.

Table 4. Dual hyperbolic Edouard numbers

n	$\widehat{E}_n$	$\widehat{E}_{-n}$
0	$j + 7\varepsilon + 42j\varepsilon$	
1	$1 + 7j + 42\varepsilon + 246j\varepsilon$	$\varepsilon + 7j\varepsilon$
2	$7 + 42j + 246\varepsilon + 1435j\varepsilon$	$1 + j\varepsilon$
3	$42 + 246j + 1435\varepsilon + 8365j\varepsilon$	$7 + j$
4	$246 + 1435j + 8365\varepsilon + 48756j\varepsilon$	$42 + 7j + \varepsilon$
5	$1435 + 8365j + 48756\varepsilon + 284172j\varepsilon$	$246 + 42j + 7\varepsilon + j\varepsilon$

Table 5. Dual hyperbolic Edouard-Lucas numbers

n	$\widehat{K}_n$	$\widehat{K}_{-n}$
0	$3 + 7j + 35\varepsilon + 199j\varepsilon$	
1	$7 + 35j + 199\varepsilon + 1155j\varepsilon$	$7 + 3j + 7\varepsilon + 35j\varepsilon$
2	$35 + 199j + 1155\varepsilon + 6727j\varepsilon$	$35 + 7j + 3\varepsilon + 7j\varepsilon$
3	$199 + 1155j + 6727\varepsilon + 39203j\varepsilon$	$199 + 35j + 7\varepsilon + 3j\varepsilon$
4	$1155 + 6727j + 39203\varepsilon + 228487j\varepsilon$	$1155 + 199j + 35\varepsilon + 7j\varepsilon$
5	$6727 + 39203j + 228487\varepsilon + 1331715j\varepsilon$	$6727 + 1155j + 199\varepsilon + 35j\varepsilon$

Now, we will give some expressions that we will use in the rest of the paper and then we define Binet's formula for the dual hyperbolic generalized Edouard numbers.

$$\widehat{\alpha} = 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \tag{2.4}$$

$$\widehat{\beta} = 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3, \tag{2.5}$$

$$\widehat{\gamma} = 1 + j + \varepsilon + j\varepsilon. \tag{2.6}$$

Note that using above equalities we can write the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+\alpha^2+2j\alpha+2\varepsilon(\alpha^4+\alpha^2)+4j\varepsilon\alpha^3,\\
\widehat{\beta}^2 &= 1+\beta^2+2j\beta+2\varepsilon(\beta^2+\beta^4)+4j\varepsilon\beta^3,\\
\widehat{\gamma}^2 &= 3+2j+4\varepsilon+4j\varepsilon,\\
\widehat{\alpha}\widehat{\beta} &= 1+\alpha\beta+j(\beta+\alpha)+\varepsilon(\beta^2+\alpha^2+\alpha^3\beta+\alpha\beta^3)+\varepsilon j(\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3),\\
\widehat{\alpha}\widehat{\gamma} &= 1+\alpha+j(1+\alpha)+\varepsilon(1+\alpha+\alpha^2+\alpha^3)+j\varepsilon(1+\alpha+\alpha^2+\alpha^3),\\
\widehat{\beta}\widehat{\gamma} &= 1+\alpha+j(1+\alpha)+\varepsilon(1+\alpha+\alpha^2+\alpha^3)+j\varepsilon(1+\alpha+\alpha^2+\alpha^3).
\end{aligned}$$

THEOREM 3. *(Binet's Formula) Let n be any integer then the Binet's formula of dual hyperbolic generalized Edouard number is*

$$\widehat{W}_n=\widehat{\alpha}A_1\alpha^n+\widehat{\beta}A_2\beta^n+\widehat{\gamma}A_3 \tag{2.7}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ are given as (2.4)-(2.5)-(2.6).

Proof. Using Binet's formula of the generalized Edouard numbers given below

$$W_n=A_1\alpha^n+A_2\beta^n+A_3$$

where A_1, A_2, A_2 are given (1.5) we get

$$\begin{aligned}
\widehat{W}_n &= W_n+jW_{n+1}+\varepsilon W_{n+2}+j\varepsilon W_{n+3},\\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n+(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1})j+(A_1\alpha^{n+2}+A_2\beta^{n+2}+A_3\gamma^{n+2})\varepsilon\\
&\quad +(A_1\alpha^{n+3}+A_2\beta^{n+3}+A_3)j\varepsilon.\\
&= \widehat{\alpha}A_1\alpha^n+\widehat{\beta}A_2\beta^n+\widehat{\gamma}A_3.
\end{aligned}$$

This proves (2.7). □

In particular, for any integer n, the Binet's Formula of nth dual hyperbolic Edouard number, Edouard-Lucas numbers, respectively, provided by

$$\begin{aligned}
\widehat{E}_n &= \frac{\widehat{\alpha}\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)}+\frac{\widehat{\beta}\beta^{n+1}}{(\beta-\alpha)(\beta-1)}-\frac{\widehat{\gamma}}{4},\\
\widehat{K}_n &= \widehat{\alpha}\alpha^n+\widehat{\beta}\beta^n+\widehat{\gamma},
\end{aligned}$$

In the following Theorem, we now derive the generating function for the sequence of dual hyperbolic generalized Edouard numbers, providing a compact analytical representation of their structure and recursive behavior.

THEOREM 4. *The generating function for the dual hyperbolic generalized Edouard numbers is*

$$f_{\widehat{W}_n}(x)=\frac{\widehat{W}_0+(\widehat{W}_1-7\widehat{W}_0)x+(\widehat{W}_2-7\widehat{W}_1+7\widehat{W}_0)x^2}{(1-7x+7x^2-x^3)}. \tag{2.8}$$

Proof. We assume that $f_{\widehat{W}_n}(x)$ is the generating function of the dual hyperbolic generalized Edouard numbers and then we can write

$$f_{\widehat{W}_n}(x)=\sum_{n=0}^{\infty}\widehat{W}_n x^n.$$

Then, in light of the definition of the dual hyperbolic generalized Edouard numbers, and substracting $7xg(x)$ and $-7x^2g(x)$ from $x^3g(x)$, we get

$$\begin{aligned}
(1-7x+7x^2-x^3)f_{G\widehat{W}_n}(x) &= \sum_{n=0}^{\infty}\widehat{W}_n x^n-7x\sum_{n=0}^{\infty}\widehat{W}_n x^n+7x^2\sum_{n=0}^{\infty}\widehat{W}_n x^n-x^3\sum_{n=0}^{\infty}\widehat{W}_n x^n,\\
&= \sum_{n=0}^{\infty}\widehat{W}_n x^n-7\sum_{n=0}^{\infty}\widehat{W}_n x^{n+1}+7\sum_{n=0}^{\infty}\widehat{W}_n x^{n+2}-\sum_{n=0}^{\infty}\widehat{W}_n x^{n+3},\\
&= \sum_{n=0}^{\infty}\widehat{W}_n x^n-7\sum_{n=1}^{\infty}\widehat{W}_{n-1} x^{n}+7\sum_{n=2}^{\infty}\widehat{W}_{n-2} x^{n}-\sum_{n=3}^{\infty}\widehat{W}_{n-3} x^{n},\\
&= (\widehat{W}_0+\widehat{W}_1x+\widehat{W}_2x^2)-7(\widehat{W}_0x+\widehat{W}_1x^2)+7GW_0x^2\\
&\quad+\sum_{n=3}^{\infty}(\widehat{W}_n-7\widehat{W}_{n-1}+7\widehat{W}_{n-2}-\widehat{W}_{n-3})x^n,\\
&= \widehat{W}_0+\widehat{W}_1x+\widehat{W}_2x^2-7\widehat{W}_0x-7\widehat{W}_1x^2+7\widehat{W}_0x^2,\\
&= \widehat{W}_0+(\widehat{W}_1-7\widehat{W}_0)x+(\widehat{W}_2-7\widehat{W}_1+7\widehat{W}_0)x^2.
\end{aligned}$$

Note that , using the recurrence relation $\widehat{W}_n=7\widehat{W}_{n-1}-7\widehat{W}_{n-2}+\widehat{W}_{n-3}$ and rearranging above equation, the (2.8) has been obtained. □

Now we can write the generating functions of the dual hyperbolic Edouard, Edouard-Lucas numbers as

$$\begin{aligned}
f_{\widehat{E}_n}(x) &= \frac{(j+7\varepsilon+42j\varepsilon)+(1-48j\varepsilon-7\varepsilon)x+(\varepsilon+7j\varepsilon)x^2}{(1-7x+7x^2-x^3)},\\
f_{\widehat{K}_n}(x) &= \frac{(3+7j+35\varepsilon+199j\varepsilon)+(-14-14j-46\varepsilon-238j\varepsilon)\,x+(7+3j+7\varepsilon+35j\varepsilon)\,x^2}{(1-7x+7x^2-x^3)},
\end{aligned}$$

respectively. □

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}$ of the sequence $\widehat{W}_n$.

Lemma 5. *Suppose that* $f_{\widehat{W}_n}(x)=\sum\limits_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}$ *is the exponential generating function of the dual hyperbolic generalized Edouard sequence* $\{\widehat{W}_n\}$.

Then $\sum\limits_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}W_n\frac{x^n}{n!}+j\sum_{n=0}^{\infty}W_{n+1}\frac{x^n}{n!}+\varepsilon\sum_{n=0}^{\infty}W_{n+2}\frac{x^n}{n!}+j\varepsilon\sum_{n=0}^{\infty}W_{n+3}\frac{x^n}{n!} \\
&= \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)}e^{\alpha x}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)}e^{\beta x}-\frac{(W_2-6W_1+W_0)}{4}e^{x} \\
&\quad +j(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha}{(\alpha-\beta)(\alpha-1)}e^{\alpha x}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta}{(\beta-\alpha)(\beta-1)}e^{\beta x}-\frac{(W_2-6W_1+W_0)}{4}e^{x}) \\
&\quad +\varepsilon(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^2}{(\alpha-\beta)(\alpha-1)}e^{\alpha x}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^2}{(\beta-\alpha)(\beta-1)}e^{\beta x}-\frac{(W_2-6W_1+W_0)}{4}e^{x}) \\
&\quad +j\varepsilon(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^3}{(\alpha-\beta)(\alpha-1)}e^{\alpha x}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^3}{(\beta-\alpha)(\beta-1)}e^{\beta x}-\frac{(W_2-6W_1+W_0)}{4}e^{x}).
\end{aligned}$$

Proof: Note that we have

$$\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}=\sum_{n=0}^{\infty}(W_n+jW_{n+1}+\varepsilon W_{n+2}+j\varepsilon W_{n+3})\frac{x^n}{n!}.$$

Then using the Binet's formula of dual hiperbolic generalized Edouard numbers or exponential generating function of the generalized Edouard sequence we get the required identy.

The previous Lemma gives the following results as particular examples.

Corollary 6. *Exponential generating function of dual hiperbolic Edouard and dual hyperbolic Edouard-Lucas numbers are*

a):

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{E}_n\frac{x^n}{n!} &= \frac{\alpha e^{\alpha x}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta e^{\beta x}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}e^{x}+j(\frac{\alpha^2 e^{\alpha x}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^2 e^{\beta x}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}e^{x}) \\
&\quad +\varepsilon(\frac{\alpha^3 e^{\alpha x}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^3 e^{\beta x}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}e^{x})+j\varepsilon(\frac{\alpha^4 e^{\alpha x}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^4 e^{\beta x}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}e^{x}).
\end{aligned}$$

b):

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{K}_n\frac{x^n}{n!} &= e^{\alpha x}+e^{\beta x}+e^{x}+j(\alpha e^{\alpha x}+\beta e^{\beta x}+e^{x}). \\
&\quad +\varepsilon(\alpha^2 e^{\alpha x}+\beta^2 e^{\beta x}+e^{x})+j\varepsilon(\alpha^3 e^{\alpha x}+\beta^3 e^{\beta x}+e^{x}).
\end{aligned}$$

3. Obtaining Binet Formula From Generating Function

Next ,by using generating function $f_{\widehat{W}_n}(x)$, we investigate Binet formula of $\{\widehat{W}_n\}$.

Theorem 7. *(Binet formula of dual hyperbolic generalized Edouard numbers)*

$$\widehat{W}_n=\widehat{\alpha}A_1\alpha^n+\widehat{\beta}A_2\beta^n+\widehat{\gamma}A_3 \tag{3.1}$$

Proof. Using the $\sum_{n=0}^{\infty}\widehat{W}_n x^n$ we can write

$$\sum_{n=0}^{\infty}\widehat{W}_n x^n=\frac{\widehat{W}_0+(\widehat{W}_1-7\widehat{W}_0)x+(\widehat{W}_2-7\widehat{W}_1+7\widehat{W}_0)x^2}{(1-7x+7x^2-x^3)}=\frac{d_1}{(1-\alpha x)}+\frac{d_2}{(1-\beta x)}+\frac{d_3}{(1-x)}, \tag{3.2}$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{W}_n x^n &= \frac{d_1}{(1-\alpha x)}+\frac{d_2}{(1-\beta x)}+\frac{d_3}{(1-x)},\\ &= \frac{d_1(1-x)(1-\beta x)+d_2\,(1-\alpha x)\,(1-x)+d_3\,(1-\alpha x)\,(1-\beta x)}{(x^2-6x+1)\,(1-x)},\end{aligned}$$

thus, we obtain

$$\widehat{W}_0+(\widehat{W}_1-7\widehat{W}_0)x+(\widehat{W}_2-7\widehat{W}_1+7\widehat{W}_0)x^2=d_1+d_2+d_3+(-d_2-\alpha d_2-\beta d_1-\alpha d_3-\beta d_3)x+(\alpha d_2+\beta d_1+\alpha\beta d_3)x^2.$$

By equating the coefficients of corresponding powers of x in the above equation, we get

$$\begin{aligned}\widehat{W}_0 &= d_1+d_2+d_3,\\ \widehat{W}_1-7\widehat{W}_0 &= -d_2-\alpha d_2-\beta d_1-\alpha d_3-\beta d_3,\\ \widehat{W}_2-7\widehat{W}_1+7\widehat{W}_0 &= \alpha d_2+\beta d_1+\alpha\beta d_3.\end{aligned} \tag{3.3}$$

If we solve (3.3) we obtain

$$\begin{aligned}d_1 &= \frac{\widehat{W}_0\alpha^2+(\widehat{W}_1-7\widehat{W}_0)\alpha+(\widehat{W}_2-7\widehat{W}_1+7\widehat{W}_0)}{(\alpha-\beta)(\alpha-\gamma)},\\ d_2 &= \frac{\widehat{W}_0\beta^2+(\widehat{W}_1-7\widehat{W}_0)\beta+(\widehat{W}_2-7\widehat{W}_1+7\widehat{W}_0)}{(\beta-\alpha)(\beta-\gamma)},\\ d_3 &= \frac{\widehat{W}_0\gamma^2+(\widehat{W}_1-7\widehat{W}_0)\gamma+(\widehat{W}_2-7\widehat{W}_1+7\widehat{W}_0)}{(\gamma-\alpha)(\gamma-\beta)},\end{aligned}$$

Thus (3.2) can be given as

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{W}_n x^n &= d_1\sum_{n=0}^{\infty}\alpha^n x^n+d_2\sum_{n=0}^{\infty}\beta^n x^n+d_3\sum_{n=0}^{\infty}x^n,\\ &= \sum_{n=0}^{\infty}(d_1\alpha^n+d_2\beta^n+d_3)x^n,\\ &= \sum_{n=0}^{\infty}\left(\frac{\widehat{W}_2-(\beta+1)\widehat{W}_1+\beta\widehat{W}_0}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n+\frac{\widehat{W}_2-(\alpha+1)\widehat{W}_1+\alpha\widehat{W}_0}{(\beta-\alpha)(\beta-\gamma)}\beta^n+\frac{\widehat{W}_2-6\widehat{W}_1+\widehat{W}_0}{(\gamma-\alpha)(\gamma-\beta)}\right)x^n.\end{aligned}$$

Hence, we get

$$\widehat{W}_n=\widehat{\alpha}A_1\alpha^n+\widehat{\beta}A_2\beta^n+\widehat{\gamma}A_3.\ \square$$

4. Some Identities

We now introduce distinctive identities pertaining to the sequence $\{\widehat{W}_n\}$ of dual hyperbolic generalized Edouard numbers. The forthcoming theorem establishes a Simpson type formula within this framework, characterizing the structural relationships among consecutive terms of the sequence.

THEOREM 8. *(Simpson's formula for dual hyperbolic generalized Edouard numbers) For all integers n we have,*

$$\begin{vmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{vmatrix} = \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. To proof the above theorem, we can use mathematical induction. First we assume that $n \geqslant 0$. For $n = 0$ identity (4.1) is true. Let (4.1) is true for $n = k$. Consequently, the identity can be stated as follows

$$\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix} = \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}.$$

For $n = k + 1$, and using above equality, we can write

$$\begin{aligned} \begin{vmatrix} \widehat{W}_{k+3} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} &= \begin{vmatrix} 7\widehat{W}_{k+2} - 7\widehat{W}_{k+1} + \widehat{W}_k & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ 7\widehat{W}_{k+1} - 7\widehat{W}_k + \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_k \\ 7\widehat{W}_k - 7\widehat{W}_{k-1} + \widehat{W}_{k-2} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &= 7\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+1} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_k & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} - 7\begin{vmatrix} \widehat{W}_{k+1} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_k & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k-1} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &\quad + \begin{vmatrix} \widehat{W}_k & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k-2} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &= \begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix}. \end{aligned}$$

Note that, for the case $n < 0$ the proof can be done similarly. Thus, the proof is completed. □

From Theorem 4.1 we get following corollary.

COROLLARY 9.

(a): $$\begin{vmatrix} \widehat{E}_{n+2} & \widehat{E}_{n+1} & \widehat{E}_n \\ \widehat{E}_{n+1} & \widehat{E}_n & \widehat{E}_{n-1} \\ \widehat{E}_n & \widehat{E}_{n-1} & \widehat{E}_{n-2} \end{vmatrix} = -8j - 280\varepsilon - 280j\varepsilon - 8.$$

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(b): $\begin{vmatrix} \widehat{K}_{n+2} & \widehat{K}_{n+1} & \widehat{K}_{n} \\ \widehat{K}_{n+1} & \widehat{K}_{n} & \widehat{K}_{n-1} \\ \widehat{K}_{n} & \widehat{K}_{n-1} & \widehat{K}_{n-2} \end{vmatrix} = 4096j + 143\,360\varepsilon + 143\,360j\varepsilon + 4096.$

THEOREM 10. *Suppose that n and m be positive integers, E_n is Edouard numbers, the following equality is valid:*

$$\widehat{W}_{m+n} = E_{m-1}\widehat{W}_{n+2} + (E_{m-3} - 7E_{m-2})\widehat{W}_{n+1} + E_{m-2}\widehat{W}_{n}. \tag{4.2}$$

Proof. First for the proof, we assume that $m \geq 0$ .The identity (10) can be proved by mathematical induction on m. Taking $m = 0$, we get

$$\widehat{W}_{n} = E_{-1}\widehat{W}_{n+2} + (E_{-3} - 7E_{-2})\widehat{W}_{n+1} + E_{-2}\widehat{W}_{n}$$

which is true by seeing that $E_{-1} = 0, E_{-2} = 1, E_{-3} = 7$. We assume that the identity (4.2) holds for $m = k$. Then for $m = k + 1$, we get

$$\begin{aligned}
\widehat{W}_{(k+1)+n} &= 7\widehat{W}_{n+k} - \widehat{7W}_{n+k-1} + \widehat{W}_{n+k-2} \\
&= 7(E_{k-1}\widehat{W}_{n+2} + (E_{k-3} - 7E_{k-2})\widehat{W}_{n+1} + E_{k-2}\widehat{W}_{n}) \\
&\quad -7(E_{k-2}\widehat{W}_{n+2} + (E_{k-4} - 3E_{k-3})\widehat{W}_{n+1} + E_{k-3}\widehat{W}_{n}) \\
&\quad +(E_{k-3}\widehat{W}_{n+2} + (E_{k-5} - 3E_{k-4})\widehat{W}_{n+1} + E_{k-4}\widehat{W}_{n}) \\
&= (7E_{k-1} - 7E_{k-2} + E_{k-3})\widehat{W}_{n+2} + ((7E_{k-3} - 7E_{k-4} + E_{k-5}) \\
&\quad -7(7E_{k-2} - 7E_{k-3} + E_{k-4}))\widehat{W}_{n+1} + (7E_{k-2} - 7E_{k-3} + E_{k-4})\widehat{W}_{n} \\
&= E_{k}\widehat{W}_{n+2} + (E_{k-2} - 7E_{k-1})\widehat{W}_{n+1} + E_{k-1}\widehat{W}_{n} \\
&= E_{(k+1)-1}\widehat{W}_{n+2} + (E_{(k+1)-3} - 7E_{(k+1)-2})\widehat{W}_{n+1} + E_{(k+1)-2}\widehat{W}_{n}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (10). Note that, for the other cases the proof can be done similarly. □

5. Linear Sums

In this section, we provide summation formulas for hyperbolic generalized Edouard numbers covering positive subscripts.

PROPOSITION 11. *For the generalized Edouard numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_k = \frac{1}{4}(-(n+3)W_n + (n+2)(7W_{n+1} - W_{n+2}) - (n+1)W_{n+1} + 2W_2 - 13W_1 + 7W_0)$.

(b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{32}(-(n+3)W_{2n} + (n+2)(-7W_{2n+2} + 48W_{2n+1} - 7W_{2n}) - (n+1)W_{2n+2} + 15W_2 - 96W_1 + 49W_0)$.

(c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{32}(-(n+3)W_{2n+1} + (n+2)(-W_{2n+2} + 42W_{2n+1} - 7W_{2n}) - (n+1)(7W_{2n+2} - 7W_{2n+1} + W_{2n}) + 9W_2 - 56W_1 + 15W_0)$.

Proof. It is given in Soykan [23, Theorem 3.3]. □

Next, we present the formulas which give the summation of the dual hyperbolic generalized Edouard numbers.

THEOREM 12. *For $n \geq 0$ then the following sum formulas holds for dual hyperbolic generalized Edouard numbers.*

(a): $\sum_{k=0}^{n} \widehat{W}_k = \frac{1}{4}(-(n+3)\widehat{W}_n + (n+2)(7\widehat{W}_{n+1} - \widehat{W}_{n+2}) - (n+1)\widehat{W}_{n+1} + 2\widehat{W}_2 - 13\widehat{W}_1 + 7\widehat{W}_0)$.

(b): $\sum_{k=0}^{n} \widehat{W}_{2k} = \frac{1}{32}(-(n+3)\widehat{W}_{2n} + (n+2)(-7\widehat{W}_{2n+2} + 48\widehat{W}_{2n+1} - 7\widehat{W}_{2n}) - (n+1)\widehat{W}_{2n+2} + 15\widehat{W}_2 - 96\widehat{W}_1 + 49\widehat{W}_0)$.

(c): $\sum_{k=0}^{n} \widehat{W}_{2k+1} = \frac{1}{32}(-(n+3)\widehat{W}_{2n+1} + (n+2)(-\widehat{W}_{2n+2} + 42\widehat{W}_{2n+1} - 7\widehat{W}_{2n}) - (n+1)(7\widehat{W}_{2n+2} - 7\widehat{W}_{2n+1} + \widehat{W}_{2n}) + 9\widehat{W}_2 - 56\widehat{W}_1 + 15\widehat{W}_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n} \widehat{W}_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1} + \varepsilon \sum_{k=0}^{n} W_{k+2} + j\varepsilon \sum_{k=0}^{n} W_{k+3}$$

and using Proposition 11 the proof is easily attainable.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n} \widehat{W}_{2k} = \sum_{k=0}^{n} W_{2k} + j\sum_{k=0}^{n} W_{2k+1} + \varepsilon \sum_{k=0}^{n} W_{2k+2} + j\varepsilon \sum_{k=0}^{n} W_{2k+3}$$

and using Proposition 11 the proof is easily attainable.

(c): Note that using (2.1), we get

$$\sum_{k=0}^{n} \widehat{W}_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + j\sum_{k=0}^{n} W_{2k+2} + \varepsilon \sum_{k=0}^{n} W_{2k+3} + j\varepsilon \sum_{k=0}^{n} W_{2k+4}$$

and using Proposition 11 the proof is easily attainable. □

As a particular case of the Theorem 12 (a), we present following corollary.

COROLLARY 13.

(a): $\sum_{k=0}^{n} \widehat{E}_k = \frac{1}{4}(-(n+3)\widehat{E}_n + (n+2)(7\widehat{E}_{n+1} - \widehat{E}_{n+2}) - (n+1)\widehat{E}_{n+1} + 1 - 34j\varepsilon - 5\varepsilon)$.

(b): $\sum_{k=0}^{n} \widehat{K}_k = \frac{1}{4}(-(n+3)\widehat{K}_n + (n+2)(7\widehat{K}_{n+1} - \widehat{K}_{n+2}) - (n+1)\widehat{K}_{n+1} - 8j - 32\varepsilon - 168j\varepsilon)$.

As a particular case of the Theorem 12 (b), we present following corollary.

COROLLARY 14.

(a): $\sum_{k=0}^{n} \widehat{E}_{2k} = \frac{1}{32}(-(n+3)\widehat{E}_{2n} + (n+2)(-7\widehat{E}_{2n+2} + 48\widehat{E}_{2n+1} - 7\widehat{E}_{2n}) - (n+1)\widehat{E}_{2n+2} + 7j + \varepsilon - 33j\varepsilon + 9)$.

(b): $\sum_{k=0}^{n} \widehat{K}_{2k} = \frac{1}{32}(-(n+3)\widehat{K}_{2n} + (n+2)(-7\widehat{K}_{2n+2} + 48\widehat{K}_{2n+1} - 7\widehat{K}_{2n}) - (n+1)\widehat{K}_{2n+2} - 32j - 64\varepsilon - 224j\varepsilon)$.

As a particular case of the Theorem 12 (c), we present following corollary.

Corollary 15.

(a): $\sum_{k=0}^{n}\widehat{E}_{2k+1} = \frac{1}{32}(-(n+3)\widehat{E}_{2n+1}+(n+2)(-\widehat{E}_{2n+2}+42\widehat{E}_{2n+1}-7\widehat{E}_{2n})-(n+1)(7\widehat{E}_{2n+2}-7\widehat{E}_{2n+1}+\widehat{E}_{2n})+j-33\varepsilon-231j\varepsilon+7)$.

(b): $\sum_{k=0}^{n}\widehat{K}_{2k+1} = \frac{1}{32}(-(n+3)\widehat{K}_{2n+1}+(n+2)(-\widehat{K}_{2n+2}+42\widehat{K}_{2n+1}-7\widehat{K}_{2n})-(n+1)(7\widehat{K}_{2n+2}-7\widehat{K}_{2n+1}+\widehat{K}_{2n})-64j-224\varepsilon-1152j\varepsilon-32)$.

6. Matrices related with Dual Hyperbolic Generalized Edouard Numbers

In this section, using dual hyperbolic Edouard numbers, we give some matrices related to dual hyperbolic Edouard numbers.

We consider the triangular sequence $\{E_n\}$ defined by the third-order recurrence relation as follows

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}$$

with the initial conditions

$$E_0 = 0,\ E = 1, E_2 = 7.$$

We present the square matrix A of order 3 as

$$A = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

under the condition that $\det A = 1$. Then, we give the following Lemma.

Lemma 16. *For any integers n the following identity can be written*

$$\begin{pmatrix} \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_{n} \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}. \tag{6.1}$$

Proof. First, we prove the assertion for the case $n \geq 0$. Lemma 16 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{0} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}$$

which is true. We assume that (6.1) is true for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_{k} \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{aligned}
\begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} &= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} \\
&= \begin{pmatrix} 7\widehat{W}_{k+2} - 7\widehat{W}_{k+1} + \widehat{W}_k \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}.
\end{aligned}$$

For the other case $n < 0$ the proof is easily attainable. Consequently, using mathematical induction on n, the proof is completed.

Note that, see [21],

$$A^n = \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix}.$$

THEOREM 17. *If we define the matrices* $N_{\widehat{W}}$ *and* $S_{\widehat{W}}$ *as follow*

$$\begin{aligned}
N_{\widehat{W}} &= \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\
S_{\widehat{W}} &= \begin{pmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{pmatrix}.
\end{aligned}$$

then the following identity is true:

$$A^n N_{\widehat{W}} = S_{\widehat{W}}.$$

Proof. For the proof, we can use the following identities

$$
\begin{aligned}
A^n N_{\widehat{W}} &= \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + T_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix} \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
a_{11} &= \widehat{W}_2 E_{n+1} + \widehat{W}_1 (E_{n-1} - 7E_n) + \widehat{W}_0 E_n, \\
a_{12} &= \widehat{W}_1 E_{n+1} + \widehat{W}_0 (E_{n-1} - 7E_n) + \widehat{W}_{-1} E_n, \\
a_{13} &= \widehat{W}_0 E_{n+1} + \widehat{W}_{-1} (E_{n-1} - 7E_n) + \widehat{W}_{-2} E_n, \\
a_{21} &= \widehat{W}_2 E_n + \widehat{W}_1 (E_{n-2} - 7E_{n-1}) + \widehat{W}_0 E_{n-1}, \\
a_{22} &= \widehat{W}_1 E_n + \widehat{W}_0 (E_{n-2} - 7E_{n-1}) + \widehat{W}_{-1} E_{n-1}, \\
a_{23} &= \widehat{W}_0 E_n + \widehat{W}_{-1} (E_{n-2} - 7E_{n-1}) + \widehat{W}_{-2} E_{n-1}, \\
a_{31} &= \widehat{W}_2 E_{n-1} + \widehat{W}_1 (E_{n-3} - 7E_{n-2}) + \widehat{W}_0 E_{n-2}, \\
a_{32} &= \widehat{W}_1 E_{n-1} + \widehat{W}_0 (E_{n-3} - 7E_{n-2}) + \widehat{W}_{-1} E_{n-2}, \\
a_{33} &= \widehat{W}_0 E_{n-1} + \widehat{W}_{-1} (E_{n-3} - 7E_{n-2}) + \widehat{W}_{-2} E_{n-2}.
\end{aligned}
$$

Using the Theorem 10, the proof is done. □

From Theorem 17, we can write the following corollary.

Corollary 18.

(a): *We suppose that the matrices $N_{\widehat{E}}$ and $S_{\widehat{E}}$ are defined as following*

$$
N_{\widehat{E}} = \begin{pmatrix} \widehat{E}_2 & \widehat{E}_1 & \widehat{E}_0 \\ \widehat{E}_1 & \widehat{E}_0 & \widehat{E}_{-1} \\ \widehat{E}_0 & \widehat{E}_{-1} & \widehat{E}_{-2} \end{pmatrix},
$$

$$
S_{\widehat{E}} = \begin{pmatrix} \widehat{E}_{n+2} & \widehat{E}_{n+1} & \widehat{E}_n \\ \widehat{E}_{n+1} & \widehat{E}_n & \widehat{E}_{n-1} \\ \widehat{E}_n & \widehat{E}_{n-1} & \widehat{E}_{n-2} \end{pmatrix},
$$

so that the following identity is true for A^n, $N_{\widehat{E}}$, $S_{\widehat{E}}$,

$$
A^n N_{\widehat{E}} = S_{\widehat{E}},
$$

(b): *We suppose that the matrices* $N_{\widehat{K}}$ *and* $S_{\widehat{K}}$ *are defined as following*

$$N_{\widehat{K}} = \begin{pmatrix} \widehat{K}_2 & \widehat{K}_1 & \widehat{K}_0 \\ \widehat{K}_1 & \widehat{K}_0 & \widehat{K}_{-1} \\ \widehat{K}_0 & \widehat{K}_{-1} & \widehat{K}_{-2} \end{pmatrix},$$

$$S_{\widehat{K}} = \begin{pmatrix} \widehat{K}_{n+2} & \widehat{K}_{n+1} & \widehat{K}_n \\ \widehat{K}_{n+1} & \widehat{K}_n & \widehat{K}_{n-1} \\ \widehat{K}_n & \widehat{K}_{n-1} & \widehat{K}_{n-2} \end{pmatrix},$$

so that the following identity is true for A^n, $N_{\widehat{K}}$, $S_{\widehat{K}}$,

$$A^n N_{\widehat{K}} = S_{\widehat{K}}.$$

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