Gaussian Generalized Adrien Numbers

Abstract. In this study, we define Gaussian Generalized Adrien numbers, focusing on two specific cases: Gaussian Adrien numbers and Gaussian Adrien-Lucas numbers. We then examine and present various properties of these sequences, including identities, matrix representations, recurrence relations, Binet's formulas, generating functions, exponential functions, Simson's formulas, and summation formulas.

Keywords. Gaussian Adrien numbers, Gaussian Adrien-Lucas numbers.

1. Introduction

Second-order, third-order, and fourth-order linear recurrence relations are generalized forms of sequences where each term depends on a fixed number of preceding terms. A second-order recurrence follows $a_n = A_{n-1} + B_{n-2}$, with a characteristic equation $x^2 - Ax - B = 0$. Extending this, a third-order recurrence incorporates three previous terms: $a_n = A_{n-1} + B_{n-2} + C_{n-3}$, leading to a cubic characteristic equation. Similarly, a fourth-order recurrence includes four preceding terms, $a_n = A_{n-1} + B_{n-2} + C_{n-3} + D_{n-4}$, resulting in a quartic characteristic equation. These higher-order relations naturally generalize second-order recurrence relations by increasing the number of dependencies and leading to more complex solutions, which can involve distinct, repeated, or complex roots.

In this section, we present key foundational results on Adrien numbers, which are governed by a fourth-order homogeneous recurrence relation.

The generalized Adrien sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation as

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4}, \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)},$$

for $n=1,2,3,....$ Hence, recurrence (1.1) holds for all integer n.Soykan has conducted a study on this particular sequence, for more details, see [8]

Characteristic equation of $\{W_n\}$ is

$$z^4-3z^3+z^2+1=(z^3-2z^2-z-1)(z-1)=0,$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{2}{3}+\left(\frac{61}{54}+\sqrt{\frac{29}{36}}\right)^{1/3}+\left(\frac{61}{54}-\sqrt{\frac{29}{36}}\right)^{1/3}, \\
\beta &= \frac{2}{3}+\omega\left(\frac{61}{54}+\sqrt{\frac{29}{36}}\right)^{1/3}+\omega^2\left(\frac{61}{54}-\sqrt{\frac{29}{36}}\right)^{1/3}, \\
\gamma &= \frac{2}{3}+\omega^2\left(\frac{61}{54}+\sqrt{\frac{29}{36}}\right)^{1/3}+\omega\left(\frac{61}{54}-\sqrt{\frac{29}{36}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= 3, \\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= 1, \\
\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Note also that

$$\begin{aligned}
\alpha+\beta+\gamma &= 2, \\
\alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

$$\begin{aligned}
p_1 &= W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0, \\
p_2 &= W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0, \\
p_3 &= W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0, \\
p_4 &= W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0,
\end{aligned}$$

where

$$\begin{aligned} A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}, \\ A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

For $n = 1, 2, 3....$ Hence, recurrence (1.1)is true for all integer n.

For the fourth-order reccurrance relations has been studied by many authors, for more detail see 12, 13, 14, 15, 11, 16, 9 17].

We now present Binet's formula for the generalized Adrien numbers.

Theorem 1.1. *[8] Binet formula of generalized Adrien numbers can be presented as follows:*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)\alpha^n}{4\alpha^2 + 3\alpha - 1} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)\beta^n}{4\beta^2 + 3\beta - 1} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)\gamma^n}{4\gamma^2 + 3\gamma - 1} \\ &+ \frac{W_3 - 2W_2 - W_1 - W_0}{-3}. \end{aligned}$$

Now we define two special cases of the sequence $\{W_n\}$ as follows: The Adrien sequence $\{A_n\}_{n\geq 0}$ and the Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations as:

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,\ A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8,\ n \geq 4, \tag{1.2}$$

$$B_n = 3B_{n-1} - B_{n-2} - B_{n-4}\ ,\ B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18,\ n \geq 4. \tag{1.3}$$

The sequences $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, can be extended to negative subscripts by defining,

$$\begin{aligned} A_{-n} &= -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)}, \\ B_{-n} &= -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)}. \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.2)-(1.2) hold for all integer n.Binet's formulas as follows.

Now we introduce Binet's formula of Adrien and Adrien-Lucas numbers.

Corollary 1.2. *For all integers n, Binet's formula of Adrien and Adrien-Lucas numbers are*

$$A_n = \frac{(2\alpha^2 + \alpha + 1)\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{(2\beta^2 + \beta + 1)\beta^n}{4\beta^2 + 3\beta - 1} + \frac{(2\gamma^2 + \gamma + 1)\gamma^n}{4\gamma^2 + 3\gamma - 1} - \frac{1}{3},$$

and

$$B_n = \alpha^n + \beta^n + \gamma^n + 1.$$

respectively.

LEMMA 1.3. *Suppose that* $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized Adrien sequence* $\{W_n\}$. *Then,* $\sum_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 3W_0)z + (W_2 - 3W_1 + W_0)z^2 + (W_3 - 3W_2 + W_1)z^3}{1 - 3z + z^2 + z^4}.$$

Proof. Take $r = 3, s = -1, t = 0, u = -1$ in Lemma 8. □

Next, we give some information about Gaussian sequences from literature.

- Horadam [6] introduced Gaussian Fibonacci numbers and defined by

$$GF_n = F_n + iF_{n-1}$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact, he defined these numbers as $GF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.

- Pethe and Horadam [7] introduced Gaussian generalized Fibonacci numbers by

$$GF_n = F_n + iF_{n-1},$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Halıcı and Öz [5] studied Gaussian Pell and Pell Lucas numbers by written , respectively,

$$GP_n = P_n + iP_{n-1},$$
$$GQ_n = Q_n + iQ_{n-1}$$

We give some Gaussian numbers with second third recurence relations.

- Yılmaz and Soykan [19] studied Gaussian Guglielmo and Guglielmo-Lucas numbers by written respectively,

$$GT_n = T_n + iT_{n-1},$$
$$GH_n = H_n + iH_{n-1}.$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$, and $H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}$, $H_0 = 3, H_1 = 3, H_2 = 3$.

- Dikmen [2] presented Gaussian Leonardo and Leonardo-Lucas numbers by written respectively,

$$
\begin{aligned}
Gl_n &= l_n + il_{n-1}, \\
GH_n &= H_n + iH_{n-1}.
\end{aligned}
$$

where $l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3$, and $H_n = 2H_{n-1} - H_{n-3}, H_0 = 3, H_1 = 2, H_2 = 4$.

- Ayrılma and Soykan [1] presented Gaussian Edouard and Edouard-Lucas numbers by written respectively,

$$
\begin{aligned}
GE_n &= E_n + iE_{n-1}, \\
GK_n &= K_n + iK_{n-1}.
\end{aligned}
$$

where $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, E_0 = 0, E_1 = 1, E_2 = 7$, and $K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}$, $K_0 = 3, K_1 = 7, K_2 = 35$.

- Soykan and Okumuş and Bilgin [18] describe Gaussian Bigollo and Bigollo-Lucas numbers by written respectively,

$$
\begin{aligned}
GB_n &= B_n + iB_{n-1}, \\
GC_n &= C_n + iC_{n-1}.
\end{aligned}
$$

where $B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, B_0 = 0, B_1 = 1, B_2 = 4$ and $C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3}$, $C_0 = 3, C_1 = 4, C_2 = 6$.

- Eren and Soykan [3] describe Gaussian Woodall and Woodall-Lucas numbers by written respectively,

$$
\begin{aligned}
GR_n &= R_n + iR_{n-1}, \\
GC_n &= C_n + iC_{n-1}.
\end{aligned}
$$

where $R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}$, $R_0 = -1, R_1 = 1, R_2 = 7,$, and $C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}$, $C_0 = 1, C_1 = 3, C_2 = 9$.

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

LEMMA 1.4. *Suppose that* $f_{GW_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Adrien sequence* $\{W_n\}$.

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{4\beta^2 + 3\beta - 1} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} + (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x}.\end{aligned}$$

Proof: Using the Binet's formula of generating Adrien numbers we get

$$\begin{aligned}\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} (\frac{p_1\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{p_2\beta^n}{4\beta^2 + 3\beta - 1} + \frac{p_3\gamma^n}{4\gamma^2 + 3\gamma - 1} + \frac{W_3 - 2W_2 - W_1 - W_0}{-3}) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{p_1\alpha^n}{4\alpha^2 + 3\alpha - 1} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{p_2\beta^n}{4\beta^2 + 3\beta - 1} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{p_3\gamma^n}{4\gamma^2 + 3\gamma - 1} \frac{x^n}{n!} \\ &+ \sum_{n=0}^{\infty} (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) \frac{x^n}{n!} \\ &= \frac{p_1}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} + \frac{p_2}{4\beta^2 + 3\beta - 1} e^{\beta x} + \frac{p_3}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} + \frac{W_3 - 2W_2 - W_1 - W_0}{-3} e^{\delta x}. \square\end{aligned}$$

The previous Lemma 1.4 gives the following results as particular examples.

Corollary 1.5. *Exponential generating function of Adrien and Adrien-Lucas numbers*

a): $\sum\limits_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{\alpha^{n+3}}{4\alpha^2 + 3\alpha - 1} + \frac{\beta^{n+3}}{4\beta^2 + 3\beta - 1} + \frac{\gamma^{n+3}}{4\gamma^2 + 3\gamma - 1} - \frac{1}{3}) \frac{x^n}{n!}$
$= \frac{\alpha^3 e^{\alpha x}}{4\alpha^2 + 3\alpha - 1} + \frac{\beta^3 e^{\beta x}}{4\alpha^2 + 3\alpha - 1} + \frac{\gamma^3 e^{\gamma x}}{4\alpha^2 + 3\alpha - 1} - \frac{e^x}{3}.$

b): $\sum\limits_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

2. Generalized Gaussian Adrien Numbers

In this section, we introduces Gaussian numbers and explores some of their key properties including Binet's formula and generating functions.

Gaussian generalized Adrien numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2, GW_3)\}_{n\geq 0}$ are defined by

$$GW_n = 3GW_{n-1} - GW_{n-2} - GW_{n-4}, \tag{2.1}$$

with the initial conditions

$$\begin{aligned} GW_0 &= W_0 + i(3W_2 - W_1 - W_3), \\ GW_1 &= W_1 + iW_0, \\ GW_2 &= W_2 + iW_1, \\ GW_3 &= W_3 + iW_2. \end{aligned}$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = -GW_{-(n-2)} + 3GW_{-(n-3)} - GW_{-(n-4)}. \tag{2.2}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (2.1) hold for all integer n. Note that for all integers n, we get

$$GW_n = W_n + iW_{n-1}, \tag{2.3}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}. \tag{2.4}$$

The first few generalized Gaussian Adrien numbers with positive subscript and negative subscript are presented in the following table.

Table 1. The first few generalized Gaussian Adrien numbers with positive subscript

n	GW_n
0	$W_0 + i(3W_2 - W_1 + -W_3)$
1	$W_1 + iW_0$
2	$W_2 + iW_1$
3	$W_3 + iW_2$
4	$3W_3 - W_2 - W_0 + iW_3$
5	$8W_3 - W_1 - 3W_2 - 3W_0 + i(3W_3 - W_2 - W_0 + iW_3)$

TABLE 2. Negative subscript

n	GW_{-n}
0	$W_3 + i(3W_2 - W_1 - W_3)$
1	$3W_2 - W_1 - W_3 + i(3W_1 - W_0 - W_2)$
2	$3W_3 + W_0 - W_2 + i(3W_0 - 3W_2 + W_3)$
3	$3W_0 - 3W_2 + W_3 + i(10W_2 - 6W_1 - 3W_3)$
4	$10W_0 - 6W_1 - 3W_3 + i(10W_1 - 6W_0 - 3W_2)$
5	$10W_1 - 6W_0 - 3W_2 + i(10W_0 + 3W_1 - 18W_2 + 6W_3)$

We can define two special cases of GW_n : $GW_n(0, 1, 3 + i, 8 + 3i) = GA_n$ is the sequence of Gaussian Adrien numbers , $GW_n : (4, 3+4i, 7+3i, 18+7i) = GB_n$ is the sequence of Gaussian Adrien-Lucas numbers.

So Gaussian Adrien numbers are defined by

$$GA_n = 3GA_{n-1} - GA_{n-2} - GA_{n-4}, \tag{2.5}$$

with the initial conditions

$$GA_0 = 0, GA_1 = 1, GA_2 = 3 + i, GA_3 = 8 + 3i.$$

Gaussian Adrien-Lucas numbers are defined by

$$GB_n = 3GB_{n-1} - GB_{n-2} - GB_{n-4}, \tag{2.6}$$

with the initial conditions

$$GB_0 = 4 + 4i, GB_1 = 3 + 4i, GB_2 = 7 + 3i, GB_3 = 18 + 7i.$$

$$GA_n = A_n + iA_{n-1},$$

$$GB_n = B_n + iB_{n-1}.$$

The first few values of Gaussian Adrien numbers, Gaussian Adrien-Lucas numbers, with positive and negative subscript are given in the Table 3.

Table 3. Special cases of Gaussian generalized Adrien numbers and Gaussian Adrien-Lucas numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8
GA_n	0	1	$3+i$	$8+3i$	$21+8i$	$54+21i$	$138+54i$	$352+138i$	$897+352i$
GA_{-n}	0	0	$-i$	-1	i	$1-3i$	-3	$6i$	$6-10i$
GB_n	4	$3+4i$	$7+3i$	$18+7i$	$43+18i$	$108+43i$	$274+108i$	$696+274i$	$1771+696i$
GB_{-n}	4	$-2i$	$-2+9i$	$9-2i$	$-2-15i$	$-15-2i$	31	$-74i$	$-74+108i$

Next, we describe the Binet's formula for the Gaussian generalized Adrien numbers.

The Binet's formula for the Gaussian generalized Adrien numbers is

$$\begin{aligned} GW_n \quad = \quad & \frac{(\alpha GW_3 - \alpha(3-\alpha)GW_2 + (-\alpha^2 + (3-1)\alpha + 1)GW_1 - GW_0)\alpha^n}{4\alpha^2 + 3\alpha - 1} \\ & + \frac{(G\beta W_3 - \beta(3-\beta)GW_2 + (-\beta^2 + (3-1)\beta + 1)GW_1 - GW_0)\beta^n}{4\beta^2 + 3\beta - 1} \\ & + \frac{(\gamma GW_3 - \gamma(3-\gamma)GW_2 + (-\gamma^2 + (3-1)\gamma + 1)GW_1 - GW_0)\gamma^n}{3\gamma - 2} \\ & + \frac{GW_3 - 2GW_2 - GW_1 - GW_0}{-3} \\ & + i(\frac{(\alpha GW_3 - \alpha 3 - \alpha)GW_2 + (-\alpha^2 + (3-1)\alpha + 1)GW_1 - GW_0)\alpha^{n-1}}{4\alpha^2 + 3\alpha - 1} \\ & + \frac{(G\beta W_3 - \beta(3-\beta)GW_2 + (-\beta^2 + (3-1)\beta + 1)GW_1 - GW_0)\beta^{n-1}}{4\beta^2 + 3\beta - 1} \\ & + \frac{(\gamma W_3 - \gamma(3-\gamma)GW_2 + (-\gamma^2 + (3-1)\gamma + 1)GW_1 - GW_0)\gamma^{n-1}}{4\gamma^2 + 3\gamma - 1} \\ & + \frac{GW_3 - 2GW_2 - GW_1 - GW_0}{-3}). \end{aligned}$$

Proof. The proof follows from (1.1) and (2.3). □

The previous Theorem gives the following results.

Corollary 2.1. *For all integers n, we have following identities,*

(a): $GA_n = \frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3} + i(\frac{(2\alpha^2+\alpha+1)\alpha^{n-1}}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^{n-1}}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^{n-1}}{4\gamma^2+3\gamma-1} - \frac{1}{3}).$

(b): $GB_n = \alpha^n + \beta^n + \gamma^n + 1 + i(\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + 1).$

The next Theorem presents the generating function of Gaussian generalized Adrien numbers.

Theorem 2.2. *Let $f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n$ donate the generating function of Gaussian generalized Adrien numbers is given as follows:*

$$\begin{aligned} f_{GW_n}(z) &= \sum_{n=0}^{\infty} GW_n x^n \\ &= \frac{GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + GW_0)x^2 + (GW_3 - 3GW_2 + GW_1)x^3}{1 - 3x + x^2 + x^4}. \end{aligned} \tag{2.7}$$

Proof. Using the definition of Gaussian Adrien numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain $(1 + x^2 - 3x^3 + x^4)f_{GW_n}(x)$

$$\begin{aligned} (1 - 3x + x^2 + x^4)f_{GW_n}(x) &= \sum_{n=0}^{\infty} GW_n x^n - 3x\sum_{n=0}^{\infty} GW_n x^n + x^2\sum_{n=0}^{\infty} GW_n x^n + x^4\sum_{n=0}^{\infty} GW_n x^n, \\ &= \sum_{n=0}^{\infty} GW_n x^n - 3\sum_{n=0}^{\infty} GW_n x^{n+1} + \sum_{n=0}^{\infty} GW_n x^{n+2} + \sum_{n=0}^{\infty} GW_n x^{n+4}, \\ &= \sum_{n=0}^{\infty} GW_n x^n - 3\sum_{n=1}^{\infty} GW_{(n-1)} x^n + \sum_{n=2}^{\infty} GW_{(n-2)} x^n + \sum_{n=4}^{\infty} GW_{(n-4)} x^n, \\ &= (GW_0 + GW_1x + GW_2x^2 + GW_3x^3) - 3(GW_0x + GW_1x^2 + GW_2x^3) \\ &\quad +3(GW_0x^2 + GW_1x^3) + \sum_{n=4}^{\infty}(GW_n - 3GW_{n-1} + GW_{n-2} + GW_{n-4})x^n, \\ &= GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + 3GW_0)x^2 \\ &\quad +(GW_3 - 3GW_2 + GW_1)x^3. \end{aligned}$$

and modifying above equation, we get (2.2). □

Corollary 2.3. *For all integers n, we have following identities:*

(a): $f_{GA_n}(x) = \sum_{n=0}^{\infty} GA_n x^n = \frac{ix^2 + x}{x^4 + 2x^3 + 3x^2 - 7x + 1},$

(b): $f_{GB_n}(x) = \sum_{n=0}^{\infty} GB_n x^n = \frac{2ix^3 + (2-9i)x^2 - (9-4i)x + 4}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

Theorem (2.2) gives the following results as special cases,

$(1-3x+x^2+x^4)f_{GA_n}(x) = GA_0 + (GA_1 - 3GA_0)x + (GA_2 - 3GA_1 + 3GA_0)x^2 + (GA_3 - 3GA_2 + GA_1)x^3$

$= ix^2 + x, (1 - 3x + x^2 + x^4),$

$f_{GB_n}(x) = GB_0 + (GB_1 - 3GB_0)x + (GB_2 - 3GB_1 + GB_0)x^2 + (GB_3 - 3GB_2 + GB_1)x^3$

$= 2ix^3 + (2 - 9i)\,x^2 - (9 - 4i)\,x + 4.$

Lemma 2.4. *Suppose that* $f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ *is the exponential Gaussian generating function of the generalized Adrien sequence* $\{GW_n\}$.

Then $\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ is given by

$$
\begin{aligned}
\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} \\
&+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{4\beta^2 + 3\beta - 1} e^{\beta x} \\
&+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} \\
&+ (\frac{W_3 - 2W_2 - W_1 - W_0}{-3})e^{x}. \\
&+ i(\frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{\alpha(4\alpha^2 + 3\alpha - 1)} e^{\alpha x} \\
&+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{\beta(4\beta^2 + 3\beta - 1)} e^{\beta x} \\
&+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{\gamma(4\gamma^2 + 3\gamma - 1)} e^{\gamma x} \\
&+ (\frac{W_3 - 2W_2 - W_1 - W_0}{-3})e^{x})
\end{aligned}
$$

Proof. The proof follows from the Binet's formula of GW_n and $GW_n = W_n + iW_{n-1}$ Lemma(1.4).

$$\begin{aligned}
\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty}(W_n + iW_{n-1})\frac{x^n}{n!} = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} iW_{n-1}\frac{x^n}{n!} \\
& \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - GW_0)}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} \\
& + \frac{(GW_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{4\beta^2 + 3\beta - 1} e^{\beta x} \\
& + \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} \\
& + (\frac{W_3 - 2W_2 - W_1 - GW_0}{-3}) e^{x}. \\
& + i(\frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{\alpha(4\alpha^2 + 3\alpha - 1)} e^{\alpha x} \\
& + \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{\beta(4\beta^2 + 3\beta - 1)} e^{\beta x} \\
& + \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - GW_0)}{\gamma(4\gamma^2 + 3\gamma - 1)} e^{\gamma x} + (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x})
\end{aligned}$$

The previous Lemma 2.4 gives the following results as particular examples.

Corollary 2.5. *Exponential Gaussian generating function of Adrien and Adrien-Lucas numbers*

a): $\sum\limits_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{\alpha^{n+3}}{4\alpha^2+3\alpha-1} + \frac{\beta^{n+3}}{4\beta^2+3\beta-1} + \frac{\gamma^{n+3}}{4\gamma^2+3\gamma-1} - \frac{1}{3} + i(\frac{\alpha^{n+2}}{4\alpha^2+3\alpha-1} + \frac{\beta^{n+2}}{4\beta^2+3\beta-1} + \frac{\gamma^{n+2}}{4\gamma^2+3\gamma-1} - \frac{1}{3}))\frac{x^n}{n!}$
$= \frac{\alpha^3 e^{\alpha x}}{4\alpha^2+3\alpha-1} + \frac{\beta^3 e^{\beta x}}{4\beta^2+3\beta-1} + \frac{\gamma^3 e^{\gamma x}}{4\gamma^2+3\gamma-1} - \frac{e^x}{3} + i(\frac{\alpha^2 e^{\alpha x}}{4\alpha^2+3\alpha-1} + \frac{\beta^2 e^{\beta x}}{4\beta^2+3\beta-1} + \frac{\gamma^2 e^{\gamma x}}{4\gamma^2+3\gamma-1} - \frac{e^x}{3}).$

b): $\sum\limits_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1 + i(\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + 1))\frac{x^n}{n!}$
$= e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x + i(\frac{1}{\alpha}e^{\alpha x} + \frac{1}{\beta}e^{\beta x} + \frac{1}{\gamma}e^{\gamma x} + e^x).$

3. Obtaining Binet Formula From Generating Function

We next find Binet formula generalized Gaussian Adrien number $\{GW_n\}$ by the use of generating function for GW_n.

Theorem 3.1. *(Binet formula of generalized Gaussian Adrien numbers)*

$$GW_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{3.1}$$

where

$$\begin{aligned}
q_1 &= GW_0\alpha^3 + (GW_1 - 3GW_0)\alpha^2 + (GW_2 + GW_1 + GW_0)\alpha + (GW_3 + GW_2 + GW_1),\\
q_2 &= GW_0\beta^3 + (GW_1 - 3GW_0)\beta^2 + (GW_2 + GW_1 + GW_0)\beta + (GW_3 + GW_2 + GW_1),\\
q_3 &= GW_0\gamma^3 + (GW_1 - 3GW_0)\gamma^2 + (GW_2 + GW_1 + GW_0)\gamma + (GW_3 + GW_2 + GW_1),\\
q_4 &= GW_0\delta^3 + (GW_1 - 3GW_0)\delta^2 + (GW_2 + GW_1 + GW_0)\delta + (GW_3 + GW_2 + GW_1).
\end{aligned}$$

Proof. Let

$$h(x) = 1 - 3x + x^2 + x^4.$$

Then for some α, β, γ and δ we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x),$$

i.e.,

$$1 - 3x + x^2 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x), \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{3}{x} + \frac{1}{x^2} + \frac{1}{x^4} = 0.$$

This implies $x^4 - 3x^3 + x^2 + u = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + GW_0)x^2 + (GW_3 - 3GW_2 + GW_1)x^3}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}.$$

Then we write

$$\begin{aligned}
&\frac{GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + GW_0)x^2 + (GW_3 - 3GW_2 + GW_1)x^3}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}\\
=\ &\frac{B_1}{(1 - \alpha x)} + \frac{B_2}{(1 - \beta x)} + \frac{B_3}{(1 - \gamma x)} + \frac{B_4}{(1 - \delta x)}.
\end{aligned} \tag{3.3}$$

So

$$\begin{aligned}
&GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + GW_0)x^2 + (GW_3 - 3GW_2 + GW_1)x^3\\
=\ &B_1(1 - \beta x)(1 - \gamma x)(1 - \delta x) + B_2(1 - \alpha x)(1 - \gamma x)(1 - \delta x)\\
&+B_3(1 - \alpha x)(1 - \beta x)(1 - \delta x) + B_3(1 - \alpha x)(1 - \beta x)(1 - \gamma x).
\end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $GW_0 + (GW_1 - 3GW_0)\frac{1}{\alpha} + (GW_2 - 3GW_1 + GW_0)\frac{1}{\alpha^2} + (GW_3 - 3GW_2 + GW_1)\frac{1}{\alpha^3}$ $= B_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})(1 - \frac{\delta}{\alpha})$.

This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3(GW_0 + (GW_1 - 3GW_0)\frac{1}{\alpha} + (GW_2 - 3GW_1 + GW_0)\frac{1}{\alpha^2} + (GW_3 - 6GW_2 + GW_1)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{GW_0\alpha^3 + (GW_1 - GW_0)\alpha^2 + (GW_2 - 3GW_1 + GW_0)\alpha + (GW_3 - 3GW_2 + GW_1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{GW_0\beta^3 + (GW_1 - 3GW_0)\beta^2 + (GW_2 - 3GW_1 + GW_0)\beta + (GW_3 - 3GW_2 + GW_1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{GW_0\gamma^3 + (GW_1 - 3GW_0)\gamma^2 + (GW_2 - 3GW_1 + GW_0)\gamma + (GW_3 - 3GW_2 + GW_1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{GW_0\delta^3 + (GW_1 - 3GW_0)\delta^2 + (GW_2 - 3GW_1 + GW_0)\delta + (GW_3 - 3GW_2 + GW_1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (3.3)can be written as

$$\sum_{n=}^{\infty} GW_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} GW_n x^n &= B_1\sum_{n=0}^{\infty}\alpha^n x^n + B_2\sum_{n=0}^{\infty}\beta^n x^n + B_3\sum_{n=0}^{\infty}\gamma^n x^n + B_4\sum_{n=0}^{\infty}\delta^n x^n \\ &= \sum_{n=0}^{\infty}(B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n)x^n. \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$GW = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

and then we get (3.1). □

4. Some Identities About Recurrence Relations of Gaussian Generalized Adrien Numbers

In this section, we present some identities on Gaussian Adrien, GaussianAdrien-Lucas.

THEOREM 4.1. *The following equations hold for all integer n*

$$\begin{aligned} GA_n &= \frac{34}{261}GB_{n+3} - \frac{50}{261}GB_{n+2} - \frac{22}{261}GB_{n+1} - \frac{49}{261}GB_n, \qquad (4.1) \\ GB_n &= -GA_{n+3} + 3GA_{n+2} + GA_{n+1} - 4GA_n. \end{aligned}$$

Proof. To proof identity (4.1), we can write

$$GA_n = aGB_{n+3} + bGB_{n+2} + cGB_{n+1} + dGB_n.$$

Solving the system of equations

$$\begin{aligned} GA_0 &= aGB_3 + bGB_2 + cGB_1 + dGB_0, \\ GA_1 &= aGB_4 + bGB_3 + cGB_2 + dGB_1, \\ GA_2 &= aGB_5 + bGB_4 + cGB_3 + dGB_2, \\ GA_3 &= aGB_6 + bGB_5 + cGB_4 + dGB_3. \end{aligned}$$

we get $a = \frac{34}{261}, b = -\frac{50}{261}, c = -\frac{22}{261}, d = -\frac{49}{261}$.The other identities can be found similarly. □

Lemma 4.2. *([4])Let's assume that* $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}.$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}.$$

respectively.

The generating functions of the even and odd-indexed Gaussian generalized Adrien sequences are provided by the following theorem.

Theorem 4.3. *The generating functions of the sequence* GW_{2n} *and* GW_{2n+1} *are provided by*

$$f_{GW_{2n}}(x) = \frac{x^3(GW_2 - 3GW_1 + GW_0) + x^2(3GW_3 - 8GW_2 + 2GW_0) + x(GW_2 - 7GW_0) + GW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}, \quad (4.2)$$

$$f_{GW_{2n+1}}(x) = \frac{x^3(GW_3 - 3GW_2 + GW_1) + x^2\left(GW_3 - 3GW_2 + 2GW_1 - 3GW_0\right) + x\left(GW_3 - 7GW_1\right) + GW_1}{x^4 + 2x^3 + 3x^2 - 7x + 1}. \quad (4.3)$$

Proof. We only proof (4.2). From Theorem (2.2) we can obtain following identities:

$$f_{GW_n}(\sqrt{x}) = \frac{(GW_2 - 3GW_1 + GW_0)\sqrt{x^3} + \left(3GW_3 - 8GW_2 + 2GW_0\right)\sqrt{x^2} + \left(GW_2 - 7GW_0\right)\sqrt{x} + GW_0}{x^2 + 2\sqrt{x^3} + 3x - 7\sqrt{x} + 1},$$

$$f_{GW_n}(-\sqrt{x}) = -\frac{(GW_3 - 3GW_2 + GW_1)\sqrt{x^3} + \left(GW_3 - 3GW_2 + 2GW_1 - 3GW_0\right)\sqrt{x^2} + \left(GW_3 + 7GW_1\right)\sqrt{x} - GW_1}{x^2 + 2\sqrt{x^3} + 3x - 7\sqrt{x} + 1}.$$

Thus, the result follows from Lemma (4.2) can be proved . The other identity can be found similarly. □

From Theorem (4.3), we get the following Corollary.

Corollary 4.4.

$$\begin{aligned}
f_{GA_{2n}}(x) &= \frac{ix^3+ix^2+(3+i)\,x}{x^4+2x^3+3x^2-7x+1},\\
f_{GA_{2n+1}}(x) &= \frac{x^2+(1+3i)\,x+1}{x^4+2x^3+3x^2-7x+1},\\
f_{GB_{2n}}(x) &= \frac{(2-9i)\,x^3+(6-3i)\,x^2-(21-3i)\,x+4}{x^4+2x^3+3x^2-7x+1},\\
f_{GB_{2n+1}}(x) &= \frac{2ix^3-(9-6i)\,x^2-(3+21i)\,x+(3+4i)}{x^4+2x^3+3x^2-7x+1}.
\end{aligned}$$

From Corollary (4.4) we can obtain the following corollary which presents the identities on Gaussian Adrien sequences.

Corollary 4.5. **(a):** $(3+i)GB_{2n-2}+iGB_{2n-4}+iGB_{2n-6}=4GA_{2n}-(21-3i)GA_{2n-2}$ $+(6-3i)GA_{2n-4}+(2-9i)GA_{2n-6}$,

(b): $GB_{2n}+(1+3i)\,GB_{2n-2}+GB_{2n-4}=4GA_{2n+1}-(21-3i)GA_{2n-1}+(6-3i)GA_{2n-3}$ $+(2-9i)GA_{2n-5}$,

(c): $(3+4i)GB_{2n}-(3+21i)GB_{2n-2}-(9-6i)GB_{2n-4}+(2i)GB_{2n-6}=4GB_{2n+1}-(21-3i)GB_{2n-1}$ $+(6-3i)GB_{2n-3}+(2-9i)GB_{2n-5}$,

d): $GB_{2n+1}+(1+3i)GB_{2n-1}+GB_{2n-3}=(3+4i)GA_{2n+1}-(3+21i)GA_{2n-1}-(9-6i)GA_{2n-3}$ $+(2i)GA_{2n-5}$,

(e): $(3+i)GB_{2n-1}+iGB_{2n-3}+iGB_{2n-5}=(3+4i)GA_{2n}-(3+21i)GA_{2n-2}-(9-6i)GA_{2n-4}$ $+(2i)GA_{2n-6}$,

(f): $GA_{2n}+(1+3i)GA_{2n-2}+GA_{2n-4}=(3+i)GA_{2n-1}+iGA_{2n-3}+iGA_{2n-5}$.

Proof. From Corollary (4.4) we obtain

$$(ix^3+ix^2+(3+i)\,x)f_{GA_{2n}}(x)=((2-9i)\,x^3+(6-3i)\,x^2-(21-3i)x+4)f_{GB_{2n}}(x).$$

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ix^3+ix^2+(3+i)\,x\sum_{n=0}^{\infty}GB_{2n}x^n,\\
&= (3+i)\,x\sum_{n=0}^{\infty}GB_{2n}x^n+ix^2\sum_{n=0}^{\infty}GB_{2n}x^n+ix^3\sum_{n=0}^{\infty}GB_{2n}x^n,\\
&= (3+i)\sum_{n=0}^{\infty}GB_{2n}x^{n+1}+i\sum_{n=0}^{\infty}GB_{2n}x^{n+2}+i\sum_{n=0}^{\infty}GB_{2n}x^{n+3},\\
&= (3+i)\sum_{n=1}^{\infty}GB_{2n-2}x^n+i\sum_{n=2}^{\infty}GB_{2n-4}x^n+i\sum_{n=3}^{\infty}GB_{2n-6}x^n,\\
&= (12+4i)x+(18+20i)x^2+\sum_{n=3}^{\infty}((3+i)\,GB_{2n-2}+iGB_{2n-4}+iGB_{2n-6})x^n.
\end{aligned}$$

whereas the RHS (right hand side) is equal to

$$\begin{aligned}
RHS &= ((2-9i)\,x^3+(6-3i)\,x^2-(21-3i)x+4)\sum_{n=0}^{\infty}GA_{2n}x^n \\
&= 4\sum_{n=0}^{\infty}GA_{2n}x^n-(21-3i)x\sum_{n=0}^{\infty}GA_{2n}x^n+(6-3i)\,x^2\sum_{n=0}^{\infty}GA_{2n}x^n+(2-9i)\,x^3\sum_{n=0}^{\infty}GA_{2n}x^n \\
&= 4\sum_{n=0}^{\infty}GA_{2n}x^n-(21-3i)\sum_{n=0}^{\infty}GA_{2n}x^{n+1}+(6-3i)\sum_{n=0}^{)\infty}GA_{2n}x^{n+2}+(2-9i)\sum_{n=0}^{\infty}GA_{2n}x^{n+3} \\
&= 4\sum_{n=0}^{\infty}GA_{2n}x^n-(21-3i)\sum_{n=1}^{\infty}GA_{2n-2}x^n+(6-3i)\sum_{n=2}^{\infty}GA_{2n-4}x^n+(2-9i)\sum_{n=3}^{\infty}GA_{2n-6}x^n \\
&\quad (12+4i)x+(18+20i)x^2+\sum_{n=3}^{\infty}(4GA_{2n}-(21-3i)GA_{2n-2}+(6-3i)GA_{2n-4}+(2-9i)GA_{2n-6})x^n
\end{aligned}$$

Comparing the coefficients and the proof of the first identity (a) is done. We can show other identity similarly. □

We can get an identitiy related to Gaussian Genaralized Adrien numbers given below.

Theorem 4.6. *For all integers m, n the following identities hold:*

$$GW_{m+n}=A_{m-2}GW_{n+3}+(-A_{m-3}-A_{m-5})GW_{n+2}+(-A_{m-4})GW_{n+1}-A_{m-3}GW_n.$$

Proof. First we assume that $m, n\geq 0$ then (4.6) can be proved by mathematical induction on m. If $m=0$ we get

$$GW_n=A_{-2}GW_{n+3}+(-A_{-3}-A_{-5})GW_{n+2}+(-A_{-4})GW_{n+1}-A_{-3}GW_n.$$

which is true since $A_{-2}=0$, $A_{-3}=-1$, $A_{-4}=0$, $A_{-5}=1$. Assume that the equality holds for $m\leq k$. For $m=k+1$, we get

$$\begin{aligned}
GW_{k+1+n} &= 3GW_{n+k}-GW_{n+k-1}-GW_{n+k-3}, \\
&= 3A_{k-2}GW_{n+3}+(-A_{k-3}-A_{k-5})GW_{n+2} \\
&\quad +3(-A_{k-4})GW_{n+1}-A_{k-3}GW_n \\
&\quad -(A_{k-3}GW_{n+3}+(-A_{k-4}-A_{k-6})GW_{n+2}+(-A_{-5})GW_{n+1}-A_{k-4}GW_n) \\
&\quad -A_{k-5}GW_{n+3}+(-A_{k-6}-A_{k-8})GW_{n+2}+(-A_{k-6})GW_{n+1}-A_{k-6}GW_n.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem (4.6).

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $GW_n=GA_n$ or $GW_n=GB_n$ in above Theorem, respectively, we get:

Corollary 4.7.

$$\begin{aligned}
GA_{m+n} &= A_{m-2}GA_{n+3}+(-A_{m-3}-A_{m-5})GA_{n+2}+(-A_{m-4})GA_{n+1}-A_{m-3}GA_n, \\
GB_{m+n} &= A_{m-2}GB_{n+3}+(-A_{m-3}-A_{m-5})GB_{n+2}+(-A_{m-4})GK_{n+1}-A_{m-3}GB_n.
\end{aligned}$$

5. Simson's Formula

In this section, we present Simson's formula of generalized Gaussian Adrien numbers. This is a special case of [10,Theorem 4.1].

THEOREM 5.1. *For all integers n, we can write the following equality:*

$$\begin{vmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1} & GW_{n-2} & GW_{n-3} \end{vmatrix} = \begin{vmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{vmatrix}$$

$$= (GW_0 + GW_1 + 2GW_2 - GW_3)(-GW_3^3 + 5GW_2^3 + GW_1^3 + GW_0^3 - (GW_0 + 3GW_1 - 7GW_2)GW_3^2$$
$$+(3GW_0 - 4GW_1 - 14GW_3)GW_2^2 + (2GW_0 + GW_2 - 6GW_3)GW_1^2 - (GW_1 + 2GW_3)GW_0^2$$
$$+13GW_1GW_2GW_3 + GW_0GW_2GW_3 + 5GW_0GW_1GW_3 - 7GW_0GW_1GW_2).$$

Proof. Using Theorem (2) it can be proved by using induction use [10,Theorem 4.1]

From the Theorem (5.1) we get the following Corollary.

COROLLARY 5.2. *For all integers n, the Simson's formulas of Adrien and Adrien Lucas numbers are given as respectively.*

(a): $$\begin{vmatrix} GA_{n+3} & GA_{n+2} & GA_{n+1} & GA_n \\ GA_{n+2} & GA_{n+1} & GA_n & GA_{n-1} \\ GA_{n+1} & GA_n & GA_{n-1} & GA_{n-2} \\ GA_n & GA_{n-1} & GA_{n-2} & GA_{n-3} \end{vmatrix} = 1 - 3i.$$

(b): $$\begin{vmatrix} GB_{n+3} & GB_{n+2} & GB_{n+1} & GB_n \\ GB_{n+2} & GB_{n+1} & GB_n & GB_{n-1} \\ GB_{n+1} & GB_n & GB_{n-1} & GB_{n-2} \\ GB_n & GB_{n-1} & GSB_{n-2} & GB_{n-3} \end{vmatrix} = -783 + 2349i.$$

6. Sum Formulas

In this section, we identify some sum formulas of generalized Gaussian Adrien numbers.

THEOREM 6.1. *For all integers $n \geq 0$, we get sum formulas below:*

(a): $\sum_{k=0}^{n} GW_k = \frac{1}{3}(-(n+3)GW_{n+3} + (2n+7)GW_{n+2} + (n+2)GW_{n+1} + (n+4)GW_n$
$+3GW_3 - 7GW_2 - 2GW_1 - GW_0)$.

(b): $\sum_{k=0}^{n} GW_{2k} = \frac{1}{3}(-(n+2)GW_{2n+2} + (2n+5)GW_{2n+1} + (n+3)GW_{2n} + (n+2)GW_{2n-1}$
$+2GW_3 - 4GW_2 - 3GW_1)$.

(c): $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{3}(-(n+1)GW_{2n+2} + (2n+5)GW_{2n+1} + (n+2)GW_{2n} + (n+2)GW_{2n-1}$
$+2GW_3 - 5GW_2 - 2GW_0)$.

Proof. It is given in Soykan [11 Theorem 3.14]. □

As a special case of the Theorem 6.1, we present following Corollary.

Corollary 6.2. *For all integers $n \geq 0$, we get sum formulas below:*

(a): $\sum_{k=0}^{n} GA_k = \frac{1}{3}(-(n+3)GA_{n+3} + (2n+7)GA_{n+2} + (n+2)GA_{n+1} + (n+4)GA_n + 1 + 2i)$.

(b): $\sum_{k=0}^{n} GA_{2k} = \frac{1}{3}(-(n+2)GA_{2n+2} + (2n+5)GA_{2n+1} + (n+3)GA_{2n} + (n+2)GA_{2n-1} + 1 + 2i)$.

(c): $\sum_{k=0}^{n} GA_{2k+1} = \frac{1}{3}(-(n+1)GA_{2n+2} + (2n+5)GA_{2n+1} + (n+2)GA_{2n} + (n+2)GA_{2n-1} + 1 + i)$.

As a special case of the Theorem 6.1, we present following Corollary.

Corollary 6.3. *For all integers $n \geq 0$, we get sum formulas below:*

(a): $\sum_{k=0}^{n} GB_k = \frac{1}{3}(-(n+3)GB_{n+3} + (2n+7)GB_{n+2} + (n+2)GB_{n+1} + (n+4)GB_n - 5 - 8i)$.

(b): $\sum_{k=0}^{n} GB_{2k} = \frac{1}{3}(-(n+2)GB_{2n+2} + (2n+5)GB_{2n+1} + (n+3)GB_{2n} + (n+2)GB_{2n-1} - 1 - 10i)$.

(c): $\sum_{k=0}^{n} GB_{2k+1} = \frac{1}{3}(-(n+1)GB_{2n+2} + (2n+5)GB_{2n+1} + (n+2)GB_{2n} + (n+2)GB_{2n-1} - 7 - i)$.

Next, we give the ordinary generating functions of some special cases of Gaussian generalized Adrien numbers.

Theorem 6.4. *The ordinary generating functions of the sequences W_{2n}, W_{2n+1} are given as follows:*

(a): $\sum_{n=0}^{\infty} GW_{2n}x^n = \dfrac{(3x^2)GW_3 + (x^3 - 8x^2 + x)GW_2 + (-3x^3)GW_1 + (x^3 + 2x^2 - 7x + 1)GW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}$.

(b): $\sum_{n=0}^{\infty} GW_{2n+1}x^n = \dfrac{(x^3 + x^2 + x)GW_3 - (3x^3 + 3x^2)GW_2 + (x^3 + 2x^2 - 7x + 1)GW_1 + (-3x^2)GW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}$.

From the last Theorem, we have the following Corollary which gives sum formula of Gaussian Adrien numbers

(Take $W_n = GA_n$ whit $GA_0 = 0, GA_1 = 1, GA_2 = 3 + i, GA_3 = 8 + 3i$.)

Corollary 6.5. *For $n \geq 0$ Gaussian Adrien numbers have the following properties:*

(a): $\sum_{n=0}^{\infty} GA_{2n}x^n = \dfrac{ix^3 + ix^2 + (3+i)\,x}{x^4 + 2x^3 + 3x^2 - 7x + 1}$.

(b): $\sum_{n=0}^{\infty} GA_{2n+1}x^n = \dfrac{x^2 + (1+3i)\,x + 1}{x^4 + 2x^3 + 3x^2 - 7x + 1}$.

From the last Theorem, we have the following Corollary which gives sum formula of Gaussian Adrien-Lucas numbers

(Take $W_n = Gb_n$ whit $GB_0 = 4, GB_1 = 3 + 4i, GB_2 = 7 + 3i, GB_3 = 18 + 7i$.)

Corollary 6.6. *For $n \geq 0$ Gaussian Adrien-Lucas numbers have the following properties:*

(a): $\sum_{n=0}^{\infty} GB_{2n}x^n = \dfrac{(2-9i)\,x^3 + (6-3i)\,x^2 - (21-3i)\,x + 4}{x^4 + 2x^3 + 3x^2 - 7x + 1}$.

(b): $\sum_{n=0}^{\infty} GB_{2n+1}x^n == \dfrac{2ix^3 - (9-6i)\,x^2 - (3+21i)\,x + 3 + 4i}{x^4 + 2x^3 + 3x^2 - 7x + 1}$.

7. Matrix Formulation of GW_n

In this section, we review the matrix representation of generalized Gaussian Adrien numbers
We define the square matrix M of order 4 as

$$M=\begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detM = 1$. Note that

$$M^n=\begin{pmatrix} A_{n+1} & -A_n-A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1}-A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2}-A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3}-A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}.$$

for the proof see[17].

Then we give the following lemma.

Lemma 7.1. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} GW_{n+3} \\ GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix}=\begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

Proof. The identitiy(7.1) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}=\begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0\begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix},$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identitiy is true

$$\begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix}=\begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k\begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n=k+1$, we get

$$\begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix}$$
$$= \begin{pmatrix} GW_{k+4} \\ GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}.$$

Consequently, by mathematical induction on n, the proof completed. □

We define

$$N_{Gw} = \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix}, \quad (7.1)$$

$$E_{Gw} = \begin{pmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1-} & GW_{n-2} & GW_{n-3} \end{pmatrix}. \quad (7.2)$$

Now, we have the following theorem with N_{Gw} and E_{Gw}

THEOREM 7.2. *Using N_{Gw} and E_{Gw} , we get*

$$A^n N_{Gw} = E_{Gw}.$$

Proof. Note that we get

$$\begin{aligned} A^{n}N_{Gw} &= \begin{pmatrix} A_{n+1} & -A_{n}-A_{n-2} & -A_{n-1} & -A_{n} \\ A_{n} & -A_{n-1}-A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2}-A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3}-A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix} \begin{pmatrix} GW_{3} & GW_{2} & GW_{1} & GW_{0} \\ GW_{2} & GW_{1} & GW_{0} & GW_{-1} \\ GW_{1} & GW_{0} & GW_{-1} & GW_{-2} \\ GW_{0} & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}. \end{aligned}$$

where

$$\begin{aligned}
a_{11} &= A_{n+1}GW_{3}+(-A_{n}-A_{n-2})GW_{2}+(-A_{n-1})GW_{1}+(-A_{n})GW_{0}, \\
a_{12} &= A_{n+1}GW_{2}+(-A_{n}-A_{n-2})GW_{1}+(-A_{n-1})GW_{0}+(-A_{n})GW_{-1}, \\
a_{13} &= A_{n+1}GW_{1}+(-A_{n}-A_{n-2})GW_{0}+(-A_{n-1})GW_{-1}+(-A_{n})GW_{-2}, \\
a_{14} &= A_{n+1}GW_{0}+(-A_{n}-A_{n-2})GW_{-1}+(-A_{n-1})GW_{-2}+(-A_{n})GW_{-3}, \\
a_{21} &= A_{n}GW_{3}+(-A_{n-1}-A_{n-3})GW_{2}+(-A_{n-2})GW_{1}+(-A_{n-1})GW_{0}, \\
a_{22} &= A_{n}GW_{2}+(-A_{n-1}-A_{n-3})GW_{1}+(-A_{n-2})GW_{0}+(-A_{n-1})GW_{-1}, \\
a_{23} &= A_{n}GW_{1}+(-A_{n-1}-A_{n-3})GW_{0}+(-A_{n-2})GW_{-1}+(-A_{n-1})GW_{-2}, \\
a_{24} &= A_{n}GW_{0}+(-A_{n-1}-A_{n-3})GW_{-1}+(-A_{n-2})GW_{-2}+(-A_{n-1})GW_{-3}, \\
a_{31} &= A_{n-1}GW_{3}+(-A_{n-2}-A_{n-4})GW_{2}+(-A_{n-3})GW_{1}+(-A_{n-2})GW_{0}, \\
a_{32} &= A_{n-1}GW_{2}+(-A_{n-2}-A_{n-4})GW_{1}+(-A_{n-3})GW_{0}+(-A_{n-2})GW_{-1}, \\
a_{33} &= A_{n-1}GW_{1}+(-A_{n-2}-A_{n-4})GW_{0}+(-A_{n-3})GW_{-1}+(-A_{n-2})GW_{-2}, \\
a_{34} &= A_{n-1}GW_{0}+(-A_{n-2}-A_{n-4})GW_{-1}+(-A_{n-3})GW_{-2}+(-A_{n-2})GW_{-3}, \\
a_{41} &= A_{n-2}GW_{3}+(-A_{n-3}-A_{n-5})GW_{2}+(-A_{n-4})GW_{1}+(-A_{n-3})GW_{0}, \\
a_{42} &= A_{n-2}GW_{2}+(-A_{n-3}-A_{n-5})GW_{1}+(-A_{n-4})GW_{0}+(-A_{n-3})GW_{-1}, \\
a_{43} &= A_{n-2}GW_{1}+(-A_{n-3}-A_{n-5})GW_{0}+(-A_{n-4})GW_{-1}+(-A_{n-3})GW_{-2}, \\
a_{44} &= A_{n-2}GW_{0}+(-A_{n-3}-A_{n-5})GW_{-1}+(-A_{n-4})GW_{-2}+(-A_{n-3})GW_{-3}.
\end{aligned}$$

Using the Theorem 4.6 the proof is done. □

By taking $GW_n = GA_n$ with GA_0, GA_1, GA_2, GA_3 in (7.1) and (7.2)

$GW_n = GB_n$ with GB_0, GB_1, GB_2, GB_3 in (7.1) and (7.2) respectively, we get:

$$N_{GA} = \begin{pmatrix} 8+3i & 3+i & 1 & 0 \\ 3+i & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; E_{GA} = \begin{pmatrix} GA_{n+3} & GA_{n+2} & GA_{n+1} & GA_0 \\ GA_{n+2} & GA_{n+1} & GA_n & GA_{n-1} \\ GA_{n+1} & GA_n & GA_{n-1} & GA_{n-2} \\ GA_n & GA_{n-1} & GA_{n-2} & GA_{n-3} \end{pmatrix},$$

$$N_{GB} = \begin{pmatrix} 18+7i & 7+3i & 3+4i & 4 \\ 7+3i & 3+4i & 4 & 4i \\ 3+4i & 4 & 4i & -2 \\ 4 & -4i & -2 & 9-2i \end{pmatrix}, \; E_{GB} = \begin{pmatrix} GB_{n+3} & GB_{n+2} & GB_{n+1} & GB_0 \\ GB_{n+2} & GB_{n+1} & GB_n & GB_{n-1} \\ GB_{n+1} & GB_n & GB_{n-1} & GB_{n-2} \\ GB_n & GB_{n-1} & GB_{n-2} & GB_{n-3} \end{pmatrix}.$$

From Theorem [7.2], we can write the following corollary.

Corollary 7.3. *The following identities are hold:*

(a): $A^n N_{GA} = E_{GA}$.

(b): $A^n N_{GB} = E_{GB}$.

References

[1] Ayrılma, E.E., Soykan, Y., A Study On Gaussian Generalized Edouard Numbers, Asian Journal of Advanced Research and Reports, 19(5), 421-438,

[2] Dikmen, C.M., Properties of Gaussian Generalized Leonardo Numbers, Karaelmas Science and Engineering Journal,15(1), 134-145, 2025. DOI: 10.7212/karaelmasfen.1578154.

[3] Eren, O., Soykan, Y., Gaussian Generalized Woodall Numbers, Archives of Current Research International, 23(8), 48-68, 2023. DOI: 10.9734/ACRI/2023/v23i8611 DOI: 10.7212/karaelmasfen.1578154

[4] Frontczak, R., Convolutions for Generalized Tribonacci Numbers and Related Results, International Journal of Mathematical Analysis, Vol. 12, 2018, no. 7, 307 - 324.

[5] Halıcı S., Öz S., On Some Gaussian Pell And Pell-Lucas Numbers, Ordu Univ. J. Sci. Tech., 6(1), 8-18, 2016.

[6] Horadam, A.F., Complex Fibonacci Numbers and Fibonacci Quaternions, Amer. Math. Monthly, 70 , 289-291,1963.

[7] Pethe, S., Horadam, A.F., Gaussian Generalized Fibonacci Numbers, Bull. Austral. Math. Soc., 33 , 37-48., 1986.

[8] Soykan, Y., Generalized Adrien Numbers, Applied Mathematic and Computer Science,7(1),37-51

[9] Soykan Y., Sum Formulas For Generalized Tetranacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} x^k W_k$ and $\sum_{k=1}^{n} x^k W_{-k}$,Journal of Progressive Research in Mathematics 18(1), 24-26, 2021

[10] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56, 2019.

[11] Soykan, Y., Sums and Generating Functions of Special Cases of Generalized Tetranacci Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 12(4), 34-10, 2025.

[12] Soykan, Y., On Generalized 4-primes Numbers, Int. J. Adv. Appl. Math. and Mech. 7(4), 20-33, 2020, (ISSN: 2347-2529).

[13] Soykan, Y., Generalized Olivier Numbers, Asian Research Journal of Mathematics, 19(1), Page 1-22, 2023. DOI: 10.9734/ARJOM/2023/v19i1634

[14] Soykan, Y., A Study on Generalized Blaise Numbers, Asian Journal of Advanced Research and Reports, 17(1), 32-53, 2023. DOI: 10.9734/AJARR/2023/v17i1463

[15] Soykan, Y., Generalized Friedrich Numbers, Journal of Advances in Mathematics and Computer Science, 38(3), 12-31, 2023. DOI: 10.9734/JAMCS/2023/v38i31748

[16] Soykan, Y., Generalized Richard Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 10(3), 38-51, 2023

[17] Soykan, Y., Properties of Generalized (r,s,t,u)-Numbers, Earthline Journal of Mathematical Sciences, 5(2), 297-327, 2021. https://doi.org/10.34198/ejms.5221.297327

[18] Soykan, Y., Okumuş, İ., Bilgin, N.G., On Generalized Bigollo Numbers, Asian Research Journal of Mathematics, 19(8), 72-88, 2023. DOI: 10.9734/ARJOM/2023/v19i8690.

[19] Yılmaz, B., Soykan, Y., Gaussian Generalized Guglielmo Numbers, Asian Journal of Advanced Research and Reports, 17(12), 1-18, 2023. DOI: 10.9734/AJARR/2023/v17i12581