**Ranks and Subdegrees of the Action of the Product of Three Alternating Groups on the Cartesian Product of Three Sets of Ordered Tuples**

**ABSTRACT**

In this paper, the ranks and subdegrees of the action of the product of three alternating groups, acting on the Cartesian product of three sets of ordered –tuples, , are determined. Using combinatorial formula and mathematical induction, the rank of acting on is and the subdegrees of on are: and

**Keywords:** Group action, Cartesian product, Rank and Subdegrees.

1. **PRELIMINARIES**
	1. **Introduction**

The following notations have been used as described:

 - The stabilizer of point in

 - Orbit of containing

 - Number of elements in set

 - An ordered

 - The set of ordered -elements of

 - Number of elements in set

- Cartesian product of three alternating groups

 - Cartesian product of three sets of ordered tuples.

Given three sets and such that , and . Then, the sets of ordered from these sets are: , and .

The Cartesian product of sets of ordered , and is defined as the set of all such that , and . The ordered sets are generated using the Groups, Aligorithms and Programming software (GAP).

Higman (1964) introduced the rank of groups on finite permutation groups with a rank of . In 1970, Higman proved that the action of symmetric group on -element subsets from the set is of rank and the subdegrees are: and .

Nyaga *et al.,* (2011) proved that acts transitively on and the subdegrees are; .

Nyaga (2018) proved that the action of direct product of is transitive on the Cartesian product of sets. The rank and subdegrees associated with this action for is ; and respectively.

Mutua *et al.,* (2018) showed that the action of direct product of on to be both transitive and imprimitive for all . The associated rank for this action is when , but is for all. The subdegrees are: .

## **1.1 Definitions and Theorems**

**Definition 1.1.1. Group action (Njagi*,* 2016)**

Given a group and a non-empty set , the action of to the left of matches a unique element if such that for all and

1. *,* given that is the identity in .

When acts from the right side of , its action can similarly be denoted as such.

**Definition 1.1.2. Transitive group (Kinyanjui, et al., 2013)**

A group is termed to act transitively on a set provided for all that is, the action gives only a one orbit.

**Definition 1.1.3. Stabilizer of an Element (Rose, 1978)**

Let and a group act on . The stabilizer of in is given by

**Definition 1.1.4. Fixed point (Njagi, 2016)**

Given a non-empty set and group acting on with . The set of elements fixed by is referred to as fixed point set of given by *.*

**Theorem 1.1.5. Orbit – Stabilizer Theorem (Rose, 1978)**

Given acts on a set ,

**Definition 1.1.6. Direct product action (Cameron *et al.,* 2008)**

Given and as permutation groups. The direct product acts on the separate union by the law and on Cartesian product by the law .

**Theorem 1.1.7. (Maraka et al,, 2024)**

The action of the product of finite alternating groups, acts transitively on the Cartesian product of finite sets of ordered –tuples, if and only if .

**Definition 1.2.8. Orbit (Njagi, 2016)**

For the action of a group on partitions into separate equivalence classes known as orbits. Hence, .

**Definition 1.2.9. Rank and Subdegrees (Nyaga *et al.,* 2011)**

Given the action of on a set is transitive and . The orbits of on are referred to as the suborbits. The rank of on is the number of those suborbits and their sizes are called subdegrees of on .

**Theorem 1.1.10 (Armstrong, 2013)**

The -orbit containing is given by and the stabilizer of is given by .

1. **MAIN RESULTS**

**Theorem 2.1 (Rank)**

If , then the rank of on is

**Proof:**

Let act on . Let , and .

Let .

Then, has orbits with exactly or no ordered tuples elements from . Now, there is only one way of selecting an element with exactly ordered tuples from , that is, Also, there are only three possible ways of choosing an element with exactly ordered tuples from that is, . Thus, there are orbits of containing exactly ordered tuples elements from There is only three possible ways as well of choosing an element with exactly one ordered tuple elements from , that is, . There are orbits with exactly one ordered tuple elements from .

Finally, there is only one way of choosing an element with no ordered tuple from . That is, there are orbit with no ordered tuple from . The rank of on is:

.

So, there are orbits.

**Theorem 2.2 (Subdegrees)**

The subdegrees of on are; and

**Proof:**

Let on . Then, , and .

Let so , so and so .

Let . By Theorem1.1.10 we have; .

Let; , and .

So,

The orbits of are:

1. **The suborbit with exactly ordered tuple elements from .**

Therefore,

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Therefore, .

Therefore, .

Therefore, .

1. **Suborbits with exactly ordered tuple elements from .**

Therefore, .

Therefore, .

Therefore, .

1. **Suborbits with exactly no ordered tuple element from .**

 Therefore, .

 **Table 1: Subdegrees of Acting on**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Length of suborbit** |  |  |  |  |
| **Number of suborbits** |  |  |  |  |

**Theorem 2.3**

The sum of the number of elements in all the orbits equals the cardinality of , that is: .

**Proof:**

Let . Therefore, we have;

But, .

 So, .

**Example 2.4**

The rank of acting on is and the subdegrees are; and .

**Proof:**

From Theorem 1.1, , that, and the rank, , is given by;

.

Let act on .

Then,

 and

Let , and . So, .

 and

By Theorem 1.1.10, we have; .

Therefore, the orbits of are:

1. **The suborbit with exactly ordered quadruples of elements from .**

Thus,

1. **Suborbits with exactly ordered quadruples of elements from .**

Thus, .

Thus,

Thus,

1. **Suborbits with exactly ordered quadruple of elements from .**

Thus, .

Thus,

Thus,

1. **Suborbit with exactly no ordered quadruple of elements from .**

 Thus, .

**Table 2: Subdegrees of Acting on**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Length of suborbit** |  |  |  |  |
| **Number of suborbits** |  |  |  |  |

From Theorem 2.3, we have:

 The sum of the elements in all the orbits is:

1. **CONCLUSION**

From this research, it can be concluded that using combinatorial formula and mathematical induction, the rank of acting on , is and the subdegrees of acting on are; and

**Disclaimer (Artificial intelligence)**

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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