**Ranks and Subdegrees of the Action of the Product of Three Alternating Groups on the Cartesian Product of Three Sets of Ordered Tuples**

**ABSTRACT**

In this paper, the ranks and subdegrees of the action of the product of three alternating groups, acting on the Cartesian product of three sets of ordered –tuples, , are determined. Using combinatorial formula and mathematical induction, the rank of acting on is and the subdegrees of on are: and

**Keywords:** Group action, Cartesian product, Rank and Subdegrees.

1. **PRELIMINARIES**
   1. **Introduction**

The following notations have been used as described:

- The stabilizer of point in

- Orbit of containing

- Number of elements in set

- An ordered

- The set of ordered -elements of

- Number of elements in set

- Cartesian product of three alternating groups

- Cartesian product of three sets of ordered tuples.

Given three sets and such that , and . Then, the sets of ordered from these sets are: , and .

The Cartesian product of sets of ordered , and is defined as the set of all such that , and . The ordered sets are generated using the Groups, Aligorithms and Programming software (GAP).

Higman (1964) introduced the rank of groups on finite permutation groups with a rank of . In 1970, Higman proved that the action of symmetric group on -element subsets from the set is of rank and the subdegrees are: and .

Nyaga *et al.,* (2011) proved that acts transitively on and the subdegrees are; .

Nyaga (2018) proved that the action of direct product of is transitive on the Cartesian product of sets. The rank and subdegrees associated with this action for is ; and respectively.

Mutua *et al.,* (2018) showed that the action of direct product of on to be both transitive and imprimitive for all . The associated rank for this action is when , but is for all. The subdegrees are: .

## **1.1 Definitions and Theorems**

**Definition 1.1.1. Group action (Njagi*,* 2016)**

Given a group and a non-empty set , the action of to the left of matches a unique element if such that for all and

1. *,* given that is the identity in .

When acts from the right side of , its action can similarly be denoted as such.

**Definition 1.1.2. Transitive group (Kinyanjui, et al., 2013)**

A group is termed to act transitively on a set provided for all that is, the action gives only a one orbit.

**Definition 1.1.3. Stabilizer of an Element (Rose, 1978)**

Let and a group act on . The stabilizer of in is given by

**Definition 1.1.4. Fixed point (Njagi, 2016)**

Given a non-empty set and group acting on with . The set of elements fixed by is referred to as fixed point set of given by *.*

**Theorem 1.1.5. Orbit – Stabilizer Theorem (Rose, 1978)**

Given acts on a set ,

**Definition 1.1.6. Direct product action (Cameron *et al.,* 2008)**

Given and as permutation groups. The direct product acts on the separate union by the law and on Cartesian product by the law .

**Theorem 1.1.7. (Maraka et al,, 2024)**

The action of the product of finite alternating groups, acts transitively on the Cartesian product of finite sets of ordered –tuples, if and only if .

**Definition 1.2.8. Orbit (Njagi, 2016)**

For the action of a group on partitions into separate equivalence classes known as orbits. Hence, .

**Definition 1.2.9. Rank and Subdegrees (Nyaga *et al.,* 2011)**

Given the action of on a set is transitive and . The orbits of on are referred to as the suborbits. The rank of on is the number of those suborbits and their sizes are called subdegrees of on .

**Theorem 1.1.10 (Armstrong, 2013)**

The -orbit containing is given by and the stabilizer of is given by .

1. **MAIN RESULTS**

**Theorem 2.1 (Rank)**

If , then the rank of on is

**Proof:**

Let act on . Let , and .

Let .

Then, has orbits with exactly or no ordered tuples elements from . Now, there is only one way of selecting an element with exactly ordered tuples from , that is, Also, there are only three possible ways of choosing an element with exactly ordered tuples from that is, . Thus, there are orbits of containing exactly ordered tuples elements from There is only three possible ways as well of choosing an element with exactly one ordered tuple elements from , that is, . There are orbits with exactly one ordered tuple elements from .

Finally, there is only one way of choosing an element with no ordered tuple from . That is, there are orbit with no ordered tuple from . The rank of on is:

.

So, there are orbits.

**Theorem 2.2 (Subdegrees)**

The subdegrees of on are; and

**Proof:**

Let on . Then, , and .

Let so , so and so .

Let . By Theorem1.1.10 we have; .

Let; , and .

So,

The orbits of are:

1. **The suborbit with exactly ordered tuple elements from .**

Therefore,

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Therefore, .

Therefore, .

Therefore, .

1. **Suborbits with exactly ordered tuple elements from .**

Therefore, .

Therefore, .

Therefore, .

1. **Suborbits with exactly no ordered tuple element from .**

Therefore, .

**Table 1: Subdegrees of Acting on**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Length of suborbit** |  |  |  |  |
| **Number of suborbits** |  |  |  |  |

**Theorem 2.3**

The sum of the number of elements in all the orbits equals the cardinality of , that is: .

**Proof:**

Let . Therefore, we have;

But, .

So, .

**Example 2.4**

The rank of acting on is and the subdegrees are; and .

**Proof:**

From Theorem 1.1, , that, and the rank, , is given by;

.

Let act on .

Then,

and

Let , and . So, .

and

By Theorem 1.1.10, we have; .

Therefore, the orbits of are:

1. **The suborbit with exactly ordered quadruples of elements from .**

Thus,

1. **Suborbits with exactly ordered quadruples of elements from .**

Thus, .

Thus,

Thus,

1. **Suborbits with exactly ordered quadruple of elements from .**

Thus, .

Thus,

Thus,

1. **Suborbit with exactly no ordered quadruple of elements from .**

Thus, .

**Table 2: Subdegrees of Acting on**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Length of suborbit** |  |  |  |  |
| **Number of suborbits** |  |  |  |  |

From Theorem 2.3, we have:

The sum of the elements in all the orbits is:

1. **CONCLUSION**

From this research, it can be concluded that using combinatorial formula and mathematical induction, the rank of acting on , is and the subdegrees of acting on are; and

**Disclaimer (Artificial intelligence)**

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

**REFERENCES**

1. Armstrong (2013). Groups and symmetry. Springer-Verlag, New York Inc.
2. Cameron, P. J., Gewurz, D. A., and Merola, F. (2008). Product action. Discrete Math, pages 386–394.
3. Higman, D. G. (1964). Finite permutation groups of rank 3. Math Zeitschriff, 86:145– 156.
4. Higman, D.G. (1970). Characterization of families of rank 3 permutation groups by sub degrees. I. Arch.Math21: 151-156.
5. Kinyanjui, J.N., Musundi, S.W., Rimberia, J., Sitati, N.I. and Makila, P. (2013). Transitivity Action of (n=5, 6, 7, 8) on ordered and unordered pairs, IJMA-4(9), page 77-88.
6. Maraka K. Maraka, Matuya W. John, Njuguna M. Edward, Nyaga N. Lewis, "Transitivity of the Product Action of Finite Alternating Groups on Cartesian Product of Finite Ordered Sets of -tuples" Iconic Research And Engineering Journals Volume 8 Issue 1 2024 Page 612-619.
7. Mutua, A.K., Nyaga, L.N. and Gachimu, R.K. (2018). Combinatorial properties, invariants and structures of the action of on . International Journal of Sciences: Basic and Applied Research (IJSBAR),Volume 40, No. 2, pp 109-115.
8. Njagi, L.K. (2016). Ranks, subdegrees and suborbital graphs of symmetric group acting on ordered pairs. *Journal of advance research in applied science (issn: 2208- 2352)*, *3*(2), 51-70.
9. Nyaga, L. N. (2018). Transitivity of the direct product of the Alternating group acting on the cartesian product of three sets. *International journal of mathematics and its applications* (issn:2347-1557), *rn*, *55*, 7.
10. Nyaga, L., Kamuti, I. N., Mwathi, C., & Akanga, J. (2011). Ranks and subdegrees of the symmetric group acting on unordered r-element subsets. *International Journal of Pure and Applied Mathematics*, *3*(2), 147-163.
11. Rose, J. S. (1978). A course in group theory. Cambridge University Press, Cambridge.