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Polynomial Solutions of Heun's Differential Equations in Nonstandard Analysis

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Authors contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In a paper of the present author, solutions of inhomogeneous and homogeneous Heun's differential equations, are obtained with the aid of nonstandard analysis. By using the solutions of homogeneous Heun's differential equations given there, polynomial solutions of homogeneous Heun's differential equations are derived in a form different from those presented in the past .

Keywords: Heun's differential equation; nonstandard analysis; complementary solution; polynimial solution

2020 Mathematics Subject Classification: 34E18; 26A33; 34M25; 34A08

1 Introduction

In a series of papers, Morita and Sato [1, 2] and Morita [3, 4, 5, 6] studied the problem of obtaining solutions of inhomogeneous and homogeneous differential equations by using the Green's function and nonstandard analysis.

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In the preceding paper [6], solutions of inhomogeneous and homogeneous Heun's differential equation are given.

In [7] and [8], solutions and also polynomial solutions of homogeneous Heun's differential equation are presented.

In the present paper, in Section 3 and 3.1, polynomial solutions of homogeneous Heun's differential equation are obtained by using the solutions of homogeneous Heun's differential equation given in [6].

We note that the polynomial solutions presented in this paper are not those given in [7] and [8].

We give here some notations to be used in the following sections. $\mathbb{Z}$ is the set of all integers, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$ and $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$.

We use $(z)_k$ for $z \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $k \in \mathbb{Z}_{>-1}$, which denote $(z)_k = \prod_{l=0}^{k-1}(z+l) = \frac{\Gamma(z+k)}{\Gamma(z)}$ for $k \in \mathbb{Z}_{>0}$ and $(z)_0 = 1$ for $k = 0$.

We use the step function $H(t)$ for $t \in \mathbb{R}$, which is equal to 1 if $t > 0$, and to 0 if $t \le 0$, and h_k, which denotes $h_k = 1$ if $k \in \mathbb{Z}_{>-1}$, and $h_k = 0$ if $k \in \mathbb{Z}_{<0}$.

2 Heun's Differential Equation

Before writing homogeneous Heun's differential equation, we present a related differential equation given by

$$\begin{aligned} p({}_RD_t, t)u(t) := \{&(t-t_3)(t-t_1)(t-t_2)\frac{d^2}{dt^2} \\ &+[\gamma_3(t-t_1)(t-t_2) + \gamma_1(t-t_2)(t-t_3) + \gamma_2(t-t_3)(t-t_1)]\frac{d}{dt} \\ &+(\alpha_1\beta_1 t - q_2)\}u(t) = 0, \end{aligned} \tag{1}$$

where t_1, t_2, t_3, γ_1, γ_2, γ_3, α_1, β_1 and q_2 are constants.

In the preceding paper [6], we used $\alpha_1\beta_1 q_1$ in place of q_2 in this equation, following [7]. Here we use q_2, following [8].

We express Equation (1) as follows:

$$\begin{aligned} p({}_RD_t, t)u(t) = [&(A_0 + A_1 t + A_2 t^2 + A_3 t^3)\frac{d^2}{dt^2} + (B_0 + B_1 t + B_2 t^2)\frac{d}{dt} \\ &+(C_0 + C_1 t)]u(t) = 0, \end{aligned} \tag{2}$$

where

$$\begin{aligned} &A_0 = -t_3 t_1 t_2, \quad A_1 = t_3 t_1 + t_1 t_2 + t_2 t_3, \quad A_2 = -t_3 - t_1 - t_2, \quad A_3 = 1, \\ &B_0 = \gamma_3 t_1 t_2 + \gamma_1 t_2 t_3 + \gamma_2 t_3 t_1, \quad B_1 = -\gamma_3(t_1 + t_2) - \gamma_1(t_2 + t_3) - \gamma_2(t_3 + t_1), \\ &B_2 = \gamma_3 + \gamma_1 + \gamma_2, \quad C_0 = -q_2, \quad C_1 = \alpha_1\beta_1. \end{aligned} \tag{3}$$

Homogeneous Heun's equation is a special one of Equation (1), in which $t_1 = 1$, $t_3 = 0$ and $\gamma_2 = \alpha_1 + \beta_1 + 1 - \gamma_1 - \gamma_3$.

As a consequence, homogeneous Heun's equation is expressed by the equation:

$$\begin{aligned} p_{He}(t, {}_RD_t)u(t) &:= [(A_1 t + A_2 t^2 + A_3 t^3)\frac{d^2}{dt^2} + (B_0 + B_1 t + B_2 t^2)\frac{d}{dt} + (C_0 + C_1 t)]u(t) \\ &= \{[t_2 t - (1+t_2)t^2 + t^3]\frac{d^2}{dt^2} + [\gamma_3 t_2 + B_1 t + (\alpha_1 + \beta_1 + 1)t^2]\frac{d}{dt} \\ &\quad -q_2 + \alpha_1\beta_1 t\}u(t) = 0, \end{aligned} \tag{4}$$

where

$$B_1 = -[\gamma_3(1+t_2) + \gamma_1 t_2 + \gamma_2] = -(\gamma_3 t_2 + \gamma_1 t_2 - \gamma_1 + \alpha_1 + \beta_1 + 1). \tag{5}$$

Comparing Equation (4) with Equation (1.1.1) in [8], variable t and constants t_2, γ_1, γ_2, γ_3, α_1, β_1 and q_2 appear in the present paper, in place of z and a, δ, ϵ, γ, α, β and q in [8]. In [7], $\alpha\beta h$ appears in place of q in [8].

3 Complementary Solutions

We recall here complementary solutions of Equation (4), by using the solutions given in Section 4 of [6], and then show that special ones of them are polynomial solutions of Equation (4).

In Section 4 of [6], if $\gamma_3 \notin \mathbb{Z}_{<1}$, a complementary solution $u(t)$ of Equation (4) is expressed in three ways, as follows:

$$u(t) = p_0 \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{k!} t^k H(t) = p_0 \sum_{k=0}^{\infty} P_k \frac{1}{(\gamma_3)_k k!} (\frac{t}{t_2})^k H(t) = p_0 \sum_{k=0}^{\infty} a_k t^k H(t), \tag{6}$$

where p_0 is any number, and $\tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$ satisfy $\tilde{p}_0 = 1$ and

$$\tilde{p}_k = \frac{1}{t_2(k-1+\gamma_3)}[\tilde{p}_{k-1}Q_k(0) - h_{k-2}\tilde{p}_{k-2}R_k(0)], \quad k \in \mathbb{Z}_{>0}, \tag{7}$$

where $Q_k(0)$ and $R_k(0)$ are given by

$$\begin{aligned} Q_k(0) &:= Q_k(0,0) = [(1+t_2)(k-2) - B_1](k-1) + q_2 \\ &= [(1+t_2)(k-2+\gamma_3) - \gamma_3 - \gamma_1 + \gamma_1 t_2 + \alpha_1 + \beta_1 + 1](k-1) + q_2, \quad k \in \mathbb{Z}_{>0}, \end{aligned} \tag{8}$$

$$\begin{aligned} R_k(0) &:= R_k(0,0) = [[(k-3) + \alpha_1 + \beta_1 + 1](k-2) + \alpha_1\beta_1](k-1) \\ &= (k-2+\alpha_1)(k-2+\beta_1)(k-1), \quad k \in \mathbb{Z}_{>1}. \end{aligned} \tag{9}$$

Equation (6) shows that P_k are related with $\tilde{p}_k$ by $P_k = t_2^k(\gamma_3)_k\tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$. By using these relations in Equation (7), we see that $P_0 = 1$ and

$$P_k = P_{k-1}Q_k(0) - t_2(k-2+\gamma_3)h_{k-2}P_{k-2}R_k(0), \quad k \in \mathbb{Z}_{>0}. \tag{10}$$

Equation (6) shows that a_k are related with $\tilde{p}_k$ by $a_k = \tilde{p}_k\frac{1}{k!}$ for $k \in \mathbb{Z}_{>-1}$. By using these, we confirm that a_k satisfy $a_0 = 1$ and

$$\begin{aligned} a_k &= \frac{1}{k!}\tilde{p}_k = \frac{1}{k!} \cdot \frac{1}{t_2(k-1+\gamma_3)}[(k-1)! \cdot a_{k-1}Q_k(0) - h_{k-2}(k-2)! \cdot a_{k-2}R_k(0)]. \\ &= \frac{1}{(k-1+\gamma_3)kt_2}[a_{k-1}Q_k(0) - \frac{1}{k-1}h_{k-2}a_{k-2}R_k(0)]. \end{aligned} \tag{11}$$

Equation (11) is given by Equations (8.3)∼(8.5) in [7] and by Equations (3.3.1)∼(3.3.3c) in [8].
Theorem 3.1. *We choose $n \in \mathbb{Z}_{>0}$ and put $\alpha_1 = -n+1$, $\beta_1 = -n$ and*

$$q_2 = -n[(n-1+\gamma_1+\gamma_3)t_2 - n + 1 - \gamma_1]. \tag{12}$$

By using these in Equations (8) *and* (9)*, we obtain $R_{n+1}(0) = R_{n+2}(0) = 0$ and $Q_{n+1}(0) = 0$, and then by using these in Equation* (11)*, we confirm $a_{n+1} = a_{n+2} = 0$. As a consequence, we have $a_l = 0$ for $l \in \mathbb{Z}_{>n}$. When we adopt these values, the solution given by Equation* (6) *is a polynomial of degree n.*

3.1 Complementary solution, II

In Section 3, we used the solutions given in Section 4 of [6]. We now use those in Section 4.1 of [6].

In Section 4.1 of [6], if $\gamma_3 \notin \mathbb{Z}_{>0}$, another complementary solution $u(t)$ of Equation (4) is expressed in three ways, as follows:

$$\begin{aligned} u(t) &= p_0 \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{\Gamma(2-\gamma_3+k)} t^{1-\gamma_3+k} H(t) = p_0 \frac{1}{\Gamma(2-\gamma_3)} t^{1-\gamma_3} \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{(2-\gamma_3)_k} t^k H(t) \\ &= p_0 \sum_{k=0}^{\infty} P_k \frac{1}{t_2^k k! \cdot \Gamma(2-\gamma_3+k)} t^{1-\gamma_3+k} H(t) = p_0 \frac{1}{\Gamma(2-\gamma_3)} t^{1-\gamma_3} \sum_{k=0}^{\infty} a_k t^k H(t), \end{aligned} \tag{13}$$

where p_0 is any number, and $\tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$ satisfy $\tilde{p}_0 = 1$ and

$$\tilde{p}_k = \frac{1}{t_2 k}[\tilde{p}_{k-1} Q_k(1-\gamma_3) - h_{k-2}\tilde{p}_{k-2} R_k(1-\gamma_3)], \quad k \in \mathbb{Z}_{>0}, \tag{14}$$

where $Q_k(1-\gamma_3)$ and $R_k(1-\gamma_3)$ are given by

$$\begin{aligned} Q_k(1-\gamma_3) &:= Q_k(1-\gamma_3, 0) = [(1+t_2)(k-1-\gamma_3) - B_1](k-\gamma_3) + q_2 \\ &= [(1+t_2)(k-1) - \gamma_3 - \gamma_1 + \gamma_1 t_2 + \alpha_1 + \beta_1 + 1](k-\gamma_3) + q_2, \quad k \in \mathbb{Z}_{>0}, \end{aligned} \tag{15}$$

$$\begin{aligned} R_k(1-\gamma_3) &:= R_k(1-\gamma_3, 0) = [[(k-2-\gamma_3) + \alpha_1 + \beta_1 + 1](k-1-\gamma_3) + \alpha_1\beta_1](k-\gamma_3) \\ &= (k-1-\gamma_3+\alpha_1)(k-1-\gamma_3+\beta_1)(k-\gamma_3), \quad k \in \mathbb{Z}_{>1}. \end{aligned} \tag{16}$$

Remark 3.1. We see that the second member of Equation (13) is obtained from the second member of Equation (6), by replacing k by $k+1-\gamma_3$, where $k! = \Gamma(k+1)$ is replaced by $\Gamma(k+1-\gamma_3)$ and $\tilde{p}_k$ given by Equation (7) with Equations (8) and (9) is replaced by $\tilde{p}_k$ given by Equation (14) with Equations (15) and (16).

Equation (13) shows that P_k are related with $\tilde{p}_k$ by $P_k = t_2^k (\gamma_3)_k \tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$. By using these relations in Equation (14), we see that $P_0 = 1$ and

$$P_k = P_{k-1} Q_k(1-\gamma_3) - t_2(k-1)h_{k-2} P_{k-2} R_k(1-\gamma_3), \quad k \in \mathbb{Z}_{>0}. \tag{17}$$

Equation (13) shows that a_k are related with $\tilde{p}_k$ by $a_k = \frac{1}{(2-\gamma_3)_k}\tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$. Then we confirm that a_k satisfy $a_0 = 1$ and

$$\begin{aligned} a_k &= \frac{1}{(2-\gamma_3)_k} p_k = \frac{1}{(2-\gamma_3)_k t_2 k}[\tilde{p}_{k-1} Q_k(1-\gamma_3) - h_{k-2}\tilde{p}_{k-2} R_k(1-\gamma_3)] \\ &= \frac{1}{t_2 k(k+1-\gamma_3)}[a_{k-1} Q_k(1-\gamma_3) - \frac{1}{k-\gamma_3} h_{k-2} a_{k-2} R_k(1-\gamma_3)], \quad k \in \mathbb{Z}_{>0}. \end{aligned} \tag{18}$$

Theorem 3.2. *We choose $n \in \mathbb{Z}_{>0}$ and put $\alpha_1 = \gamma_3 - n$, $\beta_1 = \gamma_3 - n - 1$ and*

$$q_2 = (\gamma_3 - n - 1)[(n + \gamma_1)t_2 + \gamma_3 - n - \gamma_1]. \tag{19}$$

By using these in Equations (15), (16) *and* (18), *we obtain $R_{n+1}(1-\gamma_3) = R_{n+2}(1-\gamma_3) = 0$, $Q_{n+1}(1-\gamma_3) = 0$, and $a_{n+1} = a_{n+2} = 0$. As a consequence, we have $a_l = 0$ for $l \in \mathbb{Z}_{>n}$. When we adopt these values, the solution given by* (13) is a polynomial of degree n, multipied by $t^{1-\gamma_3}$.

Remark 3.2. When $\gamma_3 = 1$, the solution given by (13) agrees with that given by Equation (6).

4 Conclusion

In Sections 3 and 3.1, we obtain two complementary solutions of Heun's equation. They are expressed in three formats. The complementary solution in one format is in agreement with a solution presented in the past, given in [7] and [8].

We show that we can construct a series of polynomial solutions of homogeneous Heun's equation, in Section 3, and a series of solutions of homogeneous Heun's equation, each of which is a polynomial multiplied by $t^{1-\gamma_3}$, in Section 3.1. We note that each of these solutions is a special solution of (4) given in Section 3 or 3.1.

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Competing Interests

Author has declared that no competing interests exist.

References

[1] Morita T, Sato K. Solution of inhomogeneous differential equations with polynomial coefficients in terms of the Green's function. Mathematics. 2017; 5(62):1–24.
DOI: https://doi.org/10.3390/math5040062

[2] Morita T, Sato, K. Solution of inhomogeneous fractional differential equations with polynomial coefficients in terms of the Green's function, in nonstandard analysis. Mathematics. 2021; 9: 1944: 1–24.
DOI: https://doi.org/10.3390/math9161944

[3] Morita T. Solution of inhomogeneous differential equations with polynomial coefficients in terms of the Green's function, in nonstandard analysis. AppliedMath. 2022; 2: 379-392.
DOI: https://doi.org/10.3390/appliedmath2030022

[4] Morita T. Solution of inhomogeneous differential equations with polynomial coefficients in nonstandard analysis. J. Adv. Math. Comput. Sci. 2023; 38(3): 49–60.
DOI: https://doi.org/10.9734/jamcs/2023/v38i31751

[5] Morita T. Solution of Inhomogeneous differential equations with polynomial coefficients in nonstandard analysis, in terms of the Green's function. J. Adv. Math. Comput. Sci. 2024; 39 (1): 20–28.
DOI: https://doi.org/10.9734/JAMCS/2024/v39l11859

[6] Morita T. Solution of Inhomogeneous and homogeneous Heun's differential equations in nonstandard analysis. J. Adv. Math. Comput. Sci. 2025; 40 (1): 72–84.
DOI: https://doi.org/10.9734/jamcs/2025/v40l1963

[7] Kristenson G. Second order differential equations: special functions and their classification. Springer Science-Business Media, LLC2010; 2010: Section 8.2.

[8] Arscott FM. Part A: Heun's equations, in Heun's differential equations edited by A. Ronveaux. Oxford University Press, Oxford; 1995.