Numerical resolution of the hepatitis C model using the SOME Blaise ABBO numerical method

Original Research Article

Abstract

Aims/ objectives: We have described a model of hepatitis C (HCV). It is a system of nonlinear fractional differential equations. We studied convergence and then used the SOME Blaise ABBO (SBA) method to successfully apply to this system.

Keywords: fractional equation system, SBA method, EDO.

2020 Mathematics Subject Classification: 44Axx; 26A33; 34A08

1 Introduction

Hepatitis C is a viral disease transmitted mainly by blood. HCV is a public health problem because chronic viral hepatitis C affects 71 million people worldwide, most of whom contracted the disease through blood. It is responsible for excess mortality, mainly due to cirrhosis, followed by hepatocarcinoma.

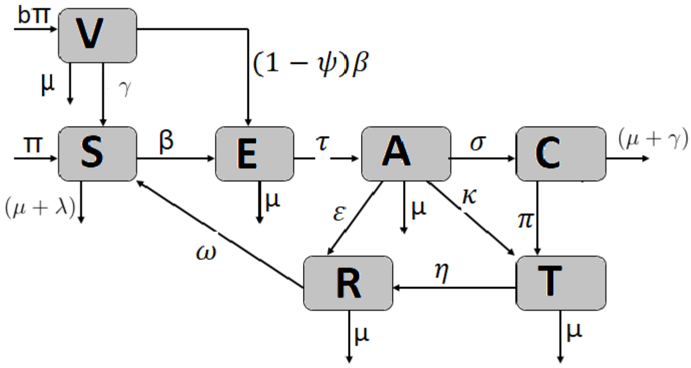
The main objective of this paper is to determine a model of hepatitis C (HCV) in the form of a system of nonlinear fractional differential equations. We will also use the SOME Blaise ABBO (SBA) method, to solve this model, as it is a numerical method that bypasses the computation of Adomian polynomials.

2 Description of the proposed hepatitis C model

To realize the HCV model, we will combine the work of our predecessors to develop a mathematical model to assess the effect of vaccination and antiviral treatment on the spread of HCV. This model is denoted VSEACTR (vaccinated, susceptible, exposed, acute, chronic, treated and resistant).

To build the model, we will consider a population of size N subdivided into seven compartments, that is seven sub-populations:

V	*vaccinated individuals*
S	*susceptible individuals*
E	*exposed individuals*
A	*infected individuals acute*
C	*chronically infected individuals*
T	*treated individuals*
R	*resistant individuals*



This defines the parameters of the proposed model:

Parameters	Parameter name
β_1	rate of transmission among the acutely infected
β_2	transmission rate of chronically infected
β_3	treaty transmission rate
γ	vaccine failure rate
τ	rate of progession to acute stage from exposure
σ	rate of transition from acute to chronic stage
λ	without vertical transmission rate
π	treatment rates for chronically infected patients
κ	treatment rate for acutely infected patients
η	cure rate for treated individuals
ω	immunity loss rate
ψ	vaccine efficacy
ε	natural recovery rate for acute state

The mathematical formulation of this diagram is represented by the system of non-linear differential *equations defined on* $[0;T]$ *by:*

$$\left\{\begin{array}{lll}\dfrac{d^{\alpha}V(t)}{dt^{\alpha}} & = & b\Pi-(\mu+\gamma)\,V(t)-(1-\psi)\left[\beta_1E(t)+\beta_2A(t)+\beta_3C(t)\right]V(t)\\ \dfrac{d^{\alpha}S(t)}{dt^{\alpha}} & = & \Pi+\gamma V(t)-(\mu+\lambda)S(t)-\left[\beta_1E(t)+\beta_2A(t)+\beta_3C(t)\right]S(t)+\omega R(t)\\ \dfrac{d^{\alpha}E(t)}{dt^{\alpha}} & = & \left[\beta_1E(t)+\beta_2A(t)+\beta_3C(t)\right]S(t)+(1-\psi)\left[\beta_1E(t)+\beta_2A(t)+\beta_3C(t)\right]V(t)\\ & & -(\tau+\mu)\,E(t)\\ \dfrac{d^{\alpha}A(t)}{dt^{\alpha}} & = & \tau E(t)-(\varepsilon+\mu+\kappa+\sigma)\,A(t)\\ \dfrac{d^{\alpha}C(t)}{dt^{\alpha}} & = & \sigma A(t)-(\pi+\mu+\gamma)\,C(t)\\ \dfrac{d^{\alpha}T(t)}{dt^{\alpha}} & = & \pi C(t)+\kappa A(t)-(\eta+\mu)\,T(t)\\ \dfrac{d^{\alpha}R(t)}{dt^{\alpha}} & = & \eta T(t)+\varepsilon A(t)-(\omega+\mu)\,R(t)\end{array}\right. \tag{2.1}$$

with $\beta=(\beta_1E(t)+\beta_2A(t)+\beta_3C(t))$

3 Convergence and uniqueness

Consider the general form of the following fractional ordinary differential equation system:

$$(S):\left\{\begin{array}{l}\dfrac{d^{\alpha}f_i(t)}{dt^{\alpha}}=R(f_i(t))+N(f_i(t))\\ \\ f_i(0)=p_i\end{array}\right.,\ \ \forall\, i=1,...,n \tag{3.1}$$

with $0<\alpha\leq 1$
Let's put $L_tf_i(t)=\dfrac{d^{\alpha}f_i(t)}{dt^{\alpha}}$
we have:

$$L_tf_i(t)=R(f_i(t))+N(f_i(t)),\ \ \forall\, i=1,...,n \tag{3.2}$$

Let's apply $L_t^{-1}(.)=I_0^{\alpha}(.)$ *the fractional integral to (**??**), we have:*

$$f_i(t)=p_i+I_0^{\alpha}(R(f_i(t)))+I_0^{\alpha}(N(f_i(t))),\ \ \forall\, i=1,...,n \tag{3.3}$$

*Applying the method of successive approximations to (**??**), we have:*

$$f_i^k(t)=p_i+I_0^{\alpha}(R(f_i^k(t)))+I_0^{\alpha}(N(f_i^{k-1}(t))),\ \ \forall\, i=1,...,n;\ k\geq 1 \tag{3.4}$$

*From (**??**), we obtain the following SBA algorithm:*

$$(SBA):\left\{\begin{array}{l}f_{i,0}^k(t)=p_i+I_0^{\alpha}(N(f_i^{k-1}(t))),\ \ \forall\, i=1,...,n;\ k\geq 1\\ \\ f_{i,n+1}^k(t)=I_0^{\alpha}(R(f_{i,n}^k(t))),\ \ \forall\, i=1,...,n;\ n\geq 0\end{array}\right. \tag{3.5}$$

Theorem
Suppose that $\forall k\geq 1$, $N(f_i^{k-1}(t))=0$, $\left|\dfrac{M_iT^{\alpha}}{\Gamma(\alpha+1)}\right|<1$, $p_i\in C(\mathbb{R}^n)$,

$f_i(t) \in C(\Omega)$, the p_i and f_i are respectively bounded by m_i and M_i such that $\exists m_i = sup|p_i|$ and $\exists M_i = \underset{t\in\Omega}{sup}|f_i(t)| > 0$ or $\Omega = \mathbb{R}^n \times [0;T]; \forall i = 1,...,n.$
then the SBA algorithm is convergent and problem (S) has a unique solution.

Proof: *we have the following SBA algorithm:*

$$\begin{cases} f_{i,0}^k(t) = p_i + I_0^\alpha(N(f_i^{k-1}(t))), \ \ \forall\, i = 1,...,n;\ k \geq 1 \\ f_{i,n+1}^k(t) = I_0^\alpha(R(f_{i,n}^k(t))), \ \ \forall\, i = 1,...,n;\ n > 0 \end{cases} \tag{3.6}$$

or even

$$\begin{cases} f_{i,0}^k(t) = p_i, \ \ \forall\, i = 1,...,n;\ k \geq 1 \\ f_{i,n+1}^k(t) = I_0^\alpha(R(f_{i,n}^k(t))), \ \ \forall\, i = 1,...,n;\ n > 0 \end{cases} \tag{3.7}$$

$$\begin{cases} \left|f_{i,0}^k(t)\right| &=& |p_i| \leq m_i;\ i = 1,...,n;\ k \geq 1 \\ \left|f_{i,1}^k(t)\right| &=& \left|I_0^\alpha(R(f_{i,0}^k(t)))\right| \leq \dfrac{M_iT^\alpha}{\Gamma(\alpha+1)};\ i = 1,...,n;\ k \geq 1 \\ \left|f_{i,2}^k(t)\right| &=& \left|I_0^\alpha(R(f_{i,1}^k(t)))\right| \leq \left(\dfrac{M_iT^\alpha}{\Gamma(\alpha+1)}\right)^2;\ i = 1,...,n;\ k \geq 1 \\ \left|f_{i,3}^k(t)\right| &=& \left|I_0^\alpha(R(f_{i,2}^k(t)))\right| \leq \left(\dfrac{M_iT^\alpha}{\Gamma(\alpha+1)}\right)^3;\ i = 1,...,n;\ k \geq 1 \\ \vdots &=& \vdots \\ \left|f_{i,n}^k(t)\right| &=& \left|I_0^\alpha(R(f_{i,n-1}^k(t)))\right| \leq \left(\dfrac{M_iT^\alpha}{\Gamma(\alpha+1)}\right)^n;\ i = 1,...,n;\ k \geq 1;\ \ n > 0 \end{cases} \tag{3.8}$$

*Summing member by member (**??**), we obtain:*

$$\sum_{n=0}^{+\infty} |f_{i,n}^k(t)| = m_i + \frac{M_iT^\alpha}{\Gamma(\alpha+1) - M_iT^\alpha};\ i = 1,...,n;\ k \geq 1;\ \ n > 0$$

from $\sum_{n=0}^{+\infty} |f_{i,n}^k(t)|$ is absolutely convergent by series $\sum_{n=0}^{+\infty} f_{i,n}^k(t)$ is simply convergent.

Uniqueness of solution
*Let be $f_{i,n}^k(t)$, $g_{i,n}^k(t)$ two solutions of (**??**) with*
$f_{i,n}^k(t) \neq g_{i,n}^k(t)$ and for f and g we have the following algorithms:

$$\begin{cases} f_{i,0}^k(t) = p_i, \ \ \forall\, i = 1,...,n;\ k \geq 1 \\ f_{i,n+1}^k(t) = I_0^\alpha(R(f_{i,n}^k(t))), \ \ \forall\, i = 1,...,n;\ n > 0 \end{cases} \tag{3.9}$$

and

$$\begin{cases} g_{i,0}^k(t) = p_i, \ \ \forall\, i = 1,...,n;\ k \geq 1 \\ g_{i,n+1}^k(t) = I_0^\alpha(R(g_{i,n}^k(t))), \ \ \forall\, i = 1,...,n;\ n > 0 \end{cases} \tag{3.10}$$

*Differentiating between (**??**) and (**??**) yields:*

$$\begin{cases} f_{i,0}^k(t) - g_{i,0}^k(t) &=& p_i - p_i = 0 \Rightarrow f_{i,0}^k(t) = g_{i,0}^k(t) \\ f_{i,1}^k(t) - g_{i,1}^k(t) &=& I_0^\alpha(R(f_{i,0}^k(t))) - I_0^\alpha(R(g_{i,0}^k(t))) = 0 \Rightarrow f_{i,1}^k(t) = g_{i,1}^k(t) \\ f_{i,2}^k(t) - g_{i,2}^k(t) &=& I_0^\alpha(R(f_{i,1}^k(t))) - I_0^\alpha(R(g_{i,1}^k(t))) = 0 \Rightarrow f_{i,2}^k(t) = g_{i,2}^k(t) \\ f_{i,3}^k(t) - g_{i,3}^k(t) &=& I_0^\alpha(R(f_{i,2}^k(t))) - I_0^\alpha(R(g_{i,2}^k(t))) = 0 \Rightarrow f_{i,3}^k(t) = g_{i,3}^k(t) \\ \vdots &=& \vdots \\ f_{i,n}^k(t) - g_{i,n}^k(t) &=& I_0^\alpha(R(f_{i,n-1}^k(t))) - I_0^\alpha(R(g_{i,n-1}^k(t))) = 0 \Rightarrow f_{i,n}^k(t) = g_{i,n}^k(t) \end{cases}$$

therefore $\forall n \geq 0$ we have $f_{i,n}^{k}(t) - g_{i,n}^{k}(t) = 0 \Rightarrow f_{i,n}^{k}(t) = g_{i,n}^{k}(t)$; or according to the hypothesis $f_{i,n}^{k}(t) \neq g_{i,n}^{k}(t)$; which is contradictory, so the system solution is unique.

4 Application of the SBA method to the resolution of the model of hepatitis C

Numerical resolution of the nonlinear fractional model of hepatitis C using the SBA method. Let's start by reviewing the system:

$$
\left\{
\begin{array}{rcl}
\frac{d^{\alpha}V(t)}{dt^{\alpha}} & = & b\Pi - (\mu+\gamma)\,V(t) - (1-\psi)\left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] E(t) \\
\frac{d^{\alpha}S(t)}{dt^{\alpha}} & = & \Pi + \gamma V(t) - (\mu+\lambda) S(t) - \left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] S(t) + \omega R(t) \\
\frac{d^{\alpha}E(t)}{dt^{\alpha}} & = & \left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] S(t) + (1-\psi)\left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] V(t) \\
 & & -\,(\tau+\mu)\,E(t) \\
\frac{d^{\alpha}A(t)}{dt^{\alpha}} & = & \tau E(t) - (\varepsilon + \mu + \kappa + \sigma)\,A(t) \\
\frac{d^{\alpha}C(t)}{dt^{\alpha}} & = & \sigma A(t) - (\pi + \mu + \gamma)\,C(t) \\
\frac{d^{\alpha}T(t)}{dt^{\alpha}} & = & \pi C(t) + \kappa A(t) - (\eta + \mu)\,T(t) \\
\frac{d^{\alpha}R(t)}{dt^{\alpha}} & = & \eta T(t) + \varepsilon A(t) - (\omega + \mu)\,R(t) \\
V(0) & = & V_0,\ \ S(0) = S_0,\ \ E(0) = E_0,\ \ A(0) = A_0 \\
C(0) & = & C_0,\ \ T(0) = T_0,\ \ R(0) = R_0
\end{array}
\right. \tag{4.1}
$$

Assuming there are no newly infected individuals, we will assume that $\Pi = 0$.

The equation system becomes:

$$\left\{\begin{array}{rcl}
\dfrac{d^\alpha V(t)}{dt^\alpha} & = & -(\mu+\gamma)\,V(t) - (1-\psi)\left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] E(t) \\
\dfrac{d^\alpha S(t)}{dt^\alpha} & = & \gamma V(t) - 7\mu S(t) - \left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] S(t) + \omega R(t) \\
\dfrac{d^\alpha E(t)}{dt^\alpha} & = & \left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] S(t) + (1-\psi)\left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] V(t) \\
 & & -(\tau+\mu)\,E(t) \\
\dfrac{d^\alpha A(t)}{dt^\alpha} & = & \tau E(t) - (\varepsilon+\mu+\kappa+\sigma)\,A(t) \\
\dfrac{d^\alpha C(t)}{dt^\alpha} & = & \sigma A(t) - (\pi+\mu+\gamma)\,C(t) \\
\dfrac{d^\alpha T(t)}{dt^\alpha} & = & \pi C(t) + \kappa A(t) - (\eta+\mu)\,T(t) \\
\dfrac{d^\alpha R(t)}{dt^\alpha} & = & \eta T(t) + \varepsilon A(t) - (\omega+\mu)\,R(t) \\
V(0) & = & V_0,\;\; S(0) = S_0,\;\; E(0) = E_0,\;\; A(0) = A_0 \\
C(0) & = & C_0,\;\; T(0) = T_0,\;\; R(0) = R_0
\end{array}\right. \tag{4.2}$$

let's talk:

$$\left\{\begin{array}{rcl}
N_1(E,A,C) & = & -(1-\psi)\left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] E(t) \\
N_2(E,A,C) & = & -\left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] S(t) \\
N_3(E,A,C) & = & \left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] S(t) + (1-\psi)\left[\beta_1 E(t) + \beta_2 A(t) + \beta_3 C(t)\right] V(t)
\end{array}\right. ,$$

Assuming the initial conditions are the same and the other parameters are expressed as a function of μ, then we have:

$\mu = \gamma = \varepsilon$; $\kappa = \sigma = \dfrac{1}{2}\mu$; $\eta = 2\mu$; $\omega = 3\mu$;

$V_0 = S_0 = R_0$; $E_0 = A_0 = C_0 = T_0 = R_0$; $\lambda = 6\mu$ *et*

$L(V(t)) = \dfrac{d^\alpha V(t)}{dt^\alpha}$, $L(S(t)) = \dfrac{d^\alpha S(t)}{dt^\alpha}$, $L(E(t)) = \dfrac{d^\alpha E(t)}{dt^\alpha}$, $L(A(t)) = \dfrac{d^\alpha A(t)}{dt^\alpha}$

$L(C(t)) = \dfrac{d^\alpha C(t)}{dt^\alpha}$, $L(T(t)) = \dfrac{d^\alpha T(t)}{dt^\alpha}$, $L(R(t)) = \dfrac{d^\alpha R(t)}{dt^\alpha}$

*the system (**??**) then becomes:*

$$\begin{cases} L(V(t)) &=& -2\mu V(t) + N_1(E,A,C) \\ L(S(t)) &=& \mu V(t) - 7\mu S(t) + 3\mu R(t) + N_2(E,A,C) \\ L(E(t)) &=& N_3(E,A,C) - 2\mu E(t) \\ L(A(t)) &=& \mu E(t) - 3\mu A(t) \\ L(C(t)) &=& \frac{1}{2}\mu A(t) - \frac{5}{2}\mu C(t) \\ L(T(t)) &=& \frac{1}{2}\mu C(t) + \frac{1}{2}\mu A(t) - 3\mu T(t) \\ L(R(t)) &=& 2\mu T(t) + \mu A(t) - 5\mu R(t) \end{cases} \tag{4.3}$$

*Let's apply $L^{-1}(.) = I_0^\alpha(.)$, the fractional integral to (**??**), we obtain:*

$$\begin{cases} V(t) &=& V_0 - 2\mu I_0^\alpha(V(t)) + I_0^\alpha(N_1(E,A,C)) \\ S(t) &=& S_0 + \mu I_0^\alpha(V(t)) - 7\mu I_0^\alpha(S(t)) + 4\mu I_0^\alpha(R(t)) + I_0^\alpha(N_2(E,A,C)) \\ E(t) &=& E_0 + I_0^\alpha(N_3(E,A,C)) - 2\mu I_0^\alpha(E(t)) \\ A(t) &=& A_0 + \mu I_0^\alpha(E(t)) - 3\mu I_0^\alpha(A(t)) \\ C(t) &=& C_0 + \frac{1}{2}\mu I_0^\alpha(A(t)) - \frac{5}{2}\mu I_0^\alpha(C(t)) \\ T(t) &=& T_0 + \frac{1}{2}\mu I_0^\alpha(C(t)) + \frac{1}{2}\mu I_0^\alpha(A(t)) - 3\mu I_0^\alpha(T(t)) \\ R(t) &=& R_0 + 2\mu I_0^\alpha(T(t)) + \mu I_0^\alpha(A(t)) - 5\mu I_0^\alpha(R(t)) \end{cases} \tag{4.4}$$

*Applying the method of successive approximations to (**??**), we obtain:*

$$\begin{cases} V^k(t) &=& V_0 - 2\mu I_0^\alpha(V^k(t)) + I_0^\alpha(N_1(E^{k-1},A^{k-1},C^{k-1})), \;\; k \geq 1 \\ S^k(t) &=& S_0 + \mu I_0^\alpha(V^k(t)) - 7\mu I_0^\alpha(S^k(t)) + 4\mu I_0^\alpha(R^k(t)) \\ && +I_0^\alpha(N_2(E^{k-1},A^{k-1},C^{k-1})), \;\; k \geq 1 \\ E^k(t) &=& E_0 + I_0^\alpha(N_3(E^{k-1},A^{k-1},C^{k-1})) - 2\mu I_0^\alpha(E^k(t)), \;\; k \geq 1 \\ A^k(t) &=& A_0 + \mu I_0^\alpha(E^k(t)) - 3\mu I_0^\alpha(A^k(t)), \;\; k \geq 1 \\ C^k(t) &=& C_0 + \frac{1}{2}\mu I_0^\alpha(A^k(t)) - \frac{5}{2}\mu I_0^\alpha(C^k(t)), \;\; k \geq 1 \\ T^k(t) &=& T_0 + \frac{1}{2}\mu I_0^\alpha(C^k(t)) + \frac{1}{2}\mu I_0^\alpha(A^k(t)) - 3\mu I_0^\alpha(T^k(t)), \;\; k \geq 1 \\ R^k(t) &=& R_0 + 2\mu I_0^\alpha(T^k(t)) + \mu I_0^\alpha(A^k(t)) - 5\mu I_0^\alpha(R^k(t)), \;\; k \geq 1 \end{cases} \tag{4.5}$$

*The solution of (**??**) is sought in the form:*

$$\begin{cases} V^k(t) = \sum_{n=0}^{\infty} V_n^k(t); \quad k = 1;2;3;... \\ S^k(t) = \sum_{n=0}^{\infty} S_n^k(t); \quad k = 1;2;3;... \\ E^k(t) = \sum_{n=0}^{\infty} E_n^k(t); \quad k = 1;2;3;... \\ A^k(t) = \sum_{n=0}^{\infty} A_n^k(t); \quad k = 1;2;3;... \\ C^k(t) = \sum_{n=0}^{\infty} C_n^k(t); \quad k = 1;2;3;... \\ T^k(t) = \sum_{n=0}^{\infty} T_n^k(t); \quad k = 1;2;3;... \\ R^k(t) = \sum_{n=0}^{\infty} R_n^k(t); \quad k = 1;2;3;... \end{cases} \tag{4.6}$$

*by introducing (**??**) into (**??**), we obtain the following SBA algorithm:*

$$(SBA):\begin{cases}\begin{cases} V_0^k(t)=V_0+I_0^\alpha(N_1(E^{k-1},A^{k-1},C^{k-1})), & k\geq 1\\ V_{n+1}^k(t)=-2\mu I_0^\alpha(V_n^k(t)); & n\geq 0\end{cases}\\ \begin{cases} S_0^k(t)=S_0+I_0^\alpha(N_2(E^{k-1},A^{k-1},C^{k-1})), & k\geq 1\\ S_{n+1}^k(t)=\mu I_0^\alpha(V_n^k(t))-7\mu I_0^\alpha(S_n^k(t))+4\mu I_0^\alpha(R_n^k(t)), & n\geq 0\end{cases}\\ \begin{cases} E_0^k(t)=E_0+I_0^\alpha(N_3(E^{k-1},A^{k-1},C^{k-1})), & k\geq 1\\ E_{n+1}^k(t)=-2\mu I_0^\alpha(E_n^k(t)), & n\geq 0\end{cases}\\ \begin{cases} A_0^k(t)=A_0, & k\geq 1\\ A_{n+1}^k(t)=\mu I_0^\alpha(E_n^k(t))-3\mu I_0^\alpha(A_n^k(t)), & n\geq 0\end{cases}\\ \begin{cases} C_0^k(t)=C_0, & k\geq 1\\ C_{n+1}^k(t)=\frac{1}{2}\mu I_0^\alpha(A_n^k(t))-\frac{5}{2}\mu I_0^\alpha(C_n^k(t)), & n\geq 0\end{cases}\\ \begin{cases} T_0^k(t)=T_0, & k\geq 1\\ T_{n+1}^k(t)=\frac{1}{2}\mu I_0^\alpha(C_n^k(t))+\frac{1}{2}\mu I_0^\alpha(A_n^k(t))-3\mu I_0^\alpha(T_n^k(t)), & n\geq 0\end{cases}\\ \begin{cases} R_0^k(t)=R_0, & k\geq 1\\ R_{n+1}^k(t)=2\mu I_0^\alpha(T_n^k(t))+\mu I_0^\alpha(A_n^k(t))-5\mu I_0^\alpha(R_n^k(t)), & n\geq 0\end{cases}\end{cases} \tag{4.7}$$

At step $k=1$, we obtain:

$$\begin{cases}\begin{cases} V_0^1(t)=V_0+I_0^\alpha(N_1(E^0,A^0,C^0))\\ V_{n+1}^1(t)=-2\mu I_0^\alpha(V_n^1(t)); & n\geq 0\end{cases}\\ \begin{cases} S_0^1(t)=S_0+I_0^\alpha(N_2(E^0,A^0,C^0))\\ S_{n+1}^1(t)=\mu I_0^\alpha(V_n^1(t))-7\mu I_0^\alpha(S_n^1(t))+4\mu I_0^\alpha(R_n^1(t)), & n\geq 0\end{cases}\\ \begin{cases} E_0^1(t)=E_0+I_0^\alpha(N_3(E^0,A^0,C^0))\\ E_{n+1}^1(t)=-2\mu I_0^\alpha(E_n^1(t)), & n\geq 0\end{cases}\\ \begin{cases} A_0^1(t)=A_0\\ A_{n+1}^1(t)=\mu I_0^\alpha(E_n^1(t))-3\mu I_0^\alpha(A_n^1(t)), & n\geq 0\end{cases}\\ \begin{cases} C_0^1(t)=C_0\\ C_{n+1}^1(t)=\frac{1}{2}\mu I_0^\alpha(A_n^1(t))-\frac{5}{2}\mu I_0^\alpha(C_n^1(t)), & n\geq 0\end{cases}\\ \begin{cases} T_0^1(t)=T_0\\ T_{n+1}^1(t)=\frac{1}{2}\mu I_0^\alpha(C_n^1(t))+\frac{1}{2}\mu I_0^\alpha(A_n^1(t))-3\mu I_0^\alpha(T_n^1(t)), & n\geq 0\end{cases}\\ \begin{cases} R_0^1(t)=R_0\\ R_{n+1}^1(t)=2\mu I_0^\alpha(T_n^1(t))+\mu I_0^\alpha(A_n^1(t))-5\mu I_0^\alpha(R_n^1(t)), & n\geq 0\end{cases}\end{cases} \tag{4.8}$$

*Applying Picard's principle to (**??**), we find E^0; S^0; V^0; A^0; C^0 and T^0 such that $N_1(E^0,A^0,C^0)=N_2(E^0,A^0,C^0)=N_3(E^0,A^0,C^0)=0$, we choose $E^0=S^0=V^0=A^0=C^0=T^0=0$.*

*The algorithm (**??**) becomes:*

$$\left\{\begin{array}{l}
\left\{\begin{array}{c} V_0^1(t)=V_0 \\ V_{n+1}^1(t)=-2\mu I_0^\alpha(V_n^1(t));\;\; n\geq 0 \end{array}\right. \\
\left\{\begin{array}{c} S_0^1(t)=S_0 \\ S_{n+1}^1(t)=\mu I_0^\alpha(V_n^1(t))-7\mu I_0^\alpha(S_n^1(t))+4\mu I_0^\alpha(R_n^1(t)),\;\; n\geq 0 \end{array}\right. \\
\left\{\begin{array}{c} E_0^1(t)=E_0 \\ E_{n+1}^1(t)=-2\mu I_0^\alpha(E_n^1(t)),\;\; n\geq 0 \end{array}\right. \\
\left\{\begin{array}{c} A_0^1(t)=A_0 \\ A_{n+1}^1(t)=\mu I_0^\alpha(E_n^1(t))-3\mu I_0^\alpha(A_n^1(t)),\;\; n\geq 0 \end{array}\right. \\
\left\{\begin{array}{c} C_0^1(t)=C_0 \\ C_{n+1}^1(t)=\frac{1}{2}\mu I_0^\alpha(A_n^1(t))-\frac{5}{2}\mu I_0^\alpha(C_n^1(t)),\;\; n\geq 0 \end{array}\right. \\
\left\{\begin{array}{c} T_0^1(t)=T_0 \\ T_{n+1}^1(t)=\frac{1}{2}\mu I_0^\alpha(C_n^1(t))+\frac{1}{2}\mu I_0^\alpha(A_n^1(t))-3\mu I_0^\alpha(T_n^1(t)),\;\; n\geq 0 \end{array}\right. \\
\left\{\begin{array}{c} R_0^1(t)=R_0 \\ R_{n+1}^1(t)=2\mu I_0^\alpha(T_n^1(t))+\mu I_0^\alpha(A_n^1(t))-5\mu I_0^\alpha(R_n^1(t)),\;\; n\geq 0 \end{array}\right.
\end{array}\right. \tag{4.9}$$

Let's calculate $V_1^1(t)$*;* $S_1^1(t)$*;* $E_1^1(t)$*;* $A_1^1(t)$*;* $C_1^1(t)$*;* $T_1^1(t)$ *and* $R_1^1(t)$
We have:

$$V_1^1(t)=-2\mu I_0^\alpha(V_0^1(t))=-2\mu I_0^\alpha(V_0)=V_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}$$

$$\begin{aligned} S_1^1(t) &= \mu I_0^\alpha(V_0^1(t))-7\mu I_0^\alpha(S_0^1(t))+4\mu I_0^\alpha(R_0^1(t)) \\ &= \mu S_0\frac{t^\alpha}{\Gamma(\alpha+1)}-7\mu S_0\frac{t^\alpha}{\Gamma(\alpha+1)}+4\mu S_0\frac{t^\alpha}{\Gamma(\alpha+1)}=S_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \end{aligned}$$

$$E_1^1(t)=-2\mu I_0^\alpha(E_0^1(t))=-2\mu I_0^\alpha(E_0)=E_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}$$

$$\begin{aligned} A_1^1(t) &= \mu I_0^\alpha(E_0^1(t))-3\mu I_0^\alpha(A_0^1(t))=\mu I_0^\alpha(E_0)-3\mu I_0^\alpha(A_0) \\ &= \mu A_0\frac{t^\alpha}{\Gamma(\alpha+1)}-3\mu A_0\frac{t^\alpha}{\Gamma(\alpha+1)}=A_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \end{aligned}$$

$$\begin{aligned} C_1^1(t) &= \frac{1}{2}\mu I_0^\alpha(A_0^1(t))-\frac{5}{2}\mu I_0^\alpha(C_0^1(t))=\frac{1}{2}\mu I_0^\alpha(A_0)-\frac{5}{2}\mu I_0^\alpha(C_0) \\ &= \frac{1}{2}\mu C_0\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{5}{2}\mu C_0\frac{t^\alpha}{\Gamma(\alpha+1)}=C_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \end{aligned}$$

$$\begin{aligned} T_1^1(t) &= \frac{1}{2}\mu I_0^\alpha(C_0^1(t))+\frac{1}{2}\mu I_0^\alpha(A_0^1(t))-3\mu I_0^\alpha(T_0^1(t))=\frac{1}{2}\mu I_0^\alpha(C_0)+\frac{1}{2}\mu I_0^\alpha(A_0)-3\mu I_0^\alpha(T_0) \\ &= \frac{1}{2}\mu T_0\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{1}{2}\mu T_0\frac{t^\alpha}{\Gamma(\alpha+1)}-3\mu T_0\frac{t^\alpha}{\Gamma(\alpha+1)}=T_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \end{aligned}$$

$$\begin{aligned} R_1^1(t) &= 2\mu I_0^\alpha(T_0^1(t)) + \mu I_0^\alpha(A_0^1(t)) - 5\mu I_0^\alpha(R_0^1(t)) = 2\mu I_0^\alpha(T_0) + \mu I_0^\alpha(A_0) - 5\mu I_0^\alpha(R_0) \\ &= 2\mu R_0 \frac{t^\alpha}{\Gamma(\alpha+1)} + \mu R_0 \frac{t^\alpha}{\Gamma(\alpha+1)} - 5\mu R_0 \frac{t^\alpha}{\Gamma(\alpha+1)} = R_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \end{aligned}$$

In summary, we have:

$$\begin{cases} V_1^1(t) &= V_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\ S_1^1(t) &= S_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\ E_1^1(t) &= E_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\ A_1^1(t) &= A_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\ C_1^1(t) &= C_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\ T_1^1(t) &= T_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\ R_1^1(t) &= R_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \end{cases}$$

Let's calculate $V_2^1(t)$*;* $S_2^1(t)$*;* $E_2^1(t)$*;* $A_2^1(t)$*;* $C_2^1(t)$*;* $T_2^1(t)$ *and* $R_2^1(t)$
We have:

$$V_2^1(t) = -2\mu I_0^\alpha(V_1^1(t)) = -2\mu I_0^\alpha\left(V_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) = V_0 \frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}$$

$$\begin{aligned} S_2^1(t) &= \mu I_0^\alpha(V_1^1(t)) - 7\mu I_0^\alpha(S_1^1(t)) + 4\mu I_0^\alpha(R_1^1(t)) \\ &= -2\mu^2 S_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 14\mu^2 S_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 8\mu^2 S_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = S_0 \frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \end{aligned}$$

$$E_2^1(t) = -2\mu I_0^\alpha(E_1^1(t)) = -2\mu I_0^\alpha\left(E_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) = E_0 \frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}$$

$$\begin{aligned} A_2^1(t) &= \mu I_0^\alpha(E_1^1(t)) - 3\mu I_0^\alpha(A_1^1(t)) = \mu I_0^\alpha\left(E_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) - 3\mu I_0^\alpha\left(E_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) \\ &= -2\mu^2 A_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 6\mu^2 A_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = A_0 \frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \end{aligned}$$

$$\begin{aligned} C_2^1(t) &= \frac{1}{2}\mu I_0^\alpha(A_1^1(t)) - \frac{5}{2}\mu I_0^\alpha(C_1^1(t)) = \frac{1}{2}\mu I_0^\alpha\left(A_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) - \frac{5}{2}\mu I_0^\alpha\left(C_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) \\ &= -\mu^2 C_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 5\mu^2 C_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = C_0 \frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \end{aligned}$$

$$\begin{aligned} T_2^1(t) &= \frac{1}{2}\mu I_0^\alpha(C_1^1(t)) + \frac{1}{2}\mu I_0^\alpha(A_1^1(t)) - 3\mu I_0^\alpha(T_1^1(t)) \\ &= \frac{1}{2}\mu I_0^\alpha\left(C_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) + \frac{1}{2}\mu I_0^\alpha\left(A_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) - 3\mu I_0^\alpha\left(T_0 \frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) \\ &= -\mu^2 T_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \mu^2 T_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 6\mu^2 T_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = T_0 \frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \end{aligned}$$

$$\begin{aligned}
R_2^1(t) &= 2\mu I_0^\alpha(T_1^1(t)) + \mu I_0^\alpha(A_1^1(t)) - 5\mu I_0^\alpha(R_1^1(t))\\
&= 2\mu I_0^\alpha\left(T_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) + \mu I_0^\alpha\left(A_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right) - 5\mu I_0^\alpha\left(R_0\frac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)}\right)\\
&= -4\mu^2 R_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2\mu^2 R_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 10\mu^2 R_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} = R_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}
\end{aligned}$$

In summary, we have:

$$\begin{cases}
V_2^1(t) &= V_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\\
S_2^1(t) &= S_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\\
E_2^1(t) &= E_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\\
A_2^1(t) &= A_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\\
C_2^1(t) &= C_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\\
T_2^1(t) &= T_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\\
R_2^1(t) &= R_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}
\end{cases}$$

Let's calculate $V_3^1(t)$*;* $S_3^1(t)$*;* $E_3^1(t)$*;* $A_3^1(t)$*;* $C_3^1(t)$*;* $T_3^1(t)$ *and* $R_3^1(t)$
We have:

$$V_3^1(t) = -2\mu I_0^\alpha(V_2^1(t)) = -2\mu I_0^\alpha\left(V_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) = V_0\frac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)}$$

$$\begin{aligned}
S_3^1(t) &= \mu I_0^\alpha(V_2^1(t)) - 7\mu I_0^\alpha(S_2^1(t)) + 4\mu I_0^\alpha(R_2^1(t))\\
&= 4\mu^3 S_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 28\mu^3 S_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 16\mu^3 S_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = S_0\frac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)}
\end{aligned}$$

$$E_3^1(t) = -2\mu I_0^\alpha(E_2^1(t)) = -2\mu I_0^\alpha\left(E_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) = E_0\frac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)}$$

$$\begin{aligned}
A_3^1(t) &= \mu I_0^\alpha(E_2^1(t)) - 3\mu I_0^\alpha(A_2^1(t)) = \mu I_0^\alpha\left(E_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) - 3\mu I_0^\alpha\left(E_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right)\\
&= 4\mu^3 A_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 12\mu^3 A_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = A_0\frac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)}
\end{aligned}$$

$$\begin{aligned}
C_3^1(t) &= \frac{1}{2}\mu I_0^\alpha(A_2^1(t)) - \frac{5}{2}\mu I_0^\alpha(C_2^1(t)) = \frac{1}{2}\mu I_0^\alpha\left(A_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) - \frac{5}{2}\mu I_0^\alpha\left(C_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right)\\
&= 2\mu^3 C_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 10\mu^3 C_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = C_0\frac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)}
\end{aligned}$$

$$\begin{aligned}
T_3^1(t) &= \frac{1}{2}\mu I_0^\alpha(C_2^1(t)) + \frac{1}{2}\mu I_0^\alpha(A_2^1(t)) - 3\mu I_0^\alpha(T_2^1(t))\\
&= \frac{1}{2}\mu I_0^\alpha\left(C_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) + \frac{1}{2}\mu I_0^\alpha\left(A_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) - 3\mu I_0^\alpha\left(T_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right)\\
&= 2\mu^3 T_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 2\mu^3 T_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 12\mu^3 T_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = T_0\frac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)}
\end{aligned}$$

$$
\begin{aligned}
R_3^1(t) &= 2\mu I_0^\alpha(T_2^1(t)) + \mu I_0^\alpha(A_2^1(t)) - 5\mu I_0^\alpha(R_2^1(t)) \\
&= 2\mu I_0^\alpha\left(T_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) + \mu I_0^\alpha\left(A_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) - 5\mu I_0^\alpha\left(R_0\frac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)}\right) \\
&= 8\mu^3 R_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 4\mu^3 R_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 20\mu^3 R_0\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = R_0\frac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)}
\end{aligned}
$$

In summary, we have:

$$
\begin{cases}
V_3^1(t) &= V_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
S_3^1(t) &= S_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
E_3^1(t) &= E_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
A_3^1(t) &= A_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
C_3^1(t) &= C_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
T_3^1(t) &= T_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
R_3^1(t) &= R_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)}
\end{cases}
$$

Recursively, we have:

$$
\begin{cases}
V_0^1(t) &= V_0 \\
V_1^1(t) &= V_0\dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
V_2^1(t) &= V_0\dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
V_3^1(t) &= V_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &= \vdots \\
V_n^1(t) &= V_0\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{cases}
;
\quad
\begin{cases}
S_0^1(t) &= S_0 \\
S_1^1(t) &= S_0\dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
S_2^1(t) &= S_0\dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
S_3^1(t) &= S_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &= \vdots \\
S_n^1(t) &= S_0\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{cases}
$$

$$
\begin{cases}
E_0^1(t) &= E_0 \\
E_1^1(t) &= E_0\dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
E_2^1(t) &= E_0\dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
E_3^1(t) &= E_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &= \vdots \\
E_n^1(t) &= E_0\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{cases}
;
\quad
\begin{cases}
A_0^1(t) &= A_0 \\
A_1^1(t) &= A_0\dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
A_2^1(t) &= A_0\dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
A_3^1(t) &= A_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &= \vdots \\
A_n^1(t) &= A_0\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{cases}
$$

$$\begin{cases} C_0^1(t) &=& C_0 \\ C_1^1(t) &=& C_0\dfrac{(-2\mu t^{\alpha})}{\Gamma(\alpha+1)} \\ C_2^1(t) &=& C_0\dfrac{(-2\mu t^{\alpha})^2}{\Gamma(2\alpha+1)} \\ C_3^1(t) &=& C_0\dfrac{(-2\mu t^{\alpha})^3}{\Gamma(3\alpha+1)} \\ \vdots &=& \vdots \\ C_n^1(t) &=& C_0\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} \end{cases} ; \quad \begin{cases} T_0^1(t) &=& T_0 \\ T_1^1(t) &=& T_0\dfrac{(-2\mu t^{\alpha})}{\Gamma(\alpha+1)} \\ T_2^1(t) &=& T_0\dfrac{(-2\mu t^{\alpha})^2}{\Gamma(2\alpha+1)} \\ T_3^1(t) &=& T_0\dfrac{(-2\mu t^{\alpha})^3}{\Gamma(3\alpha+1)} \\ \vdots &=& \vdots \\ T_n^1(t) &=& T_0\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} \end{cases}$$

$$\begin{cases} R_0^1(t) &=& R_0 \\ R_1^1(t) &=& R_0\dfrac{(-2\mu t^{\alpha})}{\Gamma(\alpha+1)} \\ R_2^1(t) &=& R_0\dfrac{(-2\mu t^{\alpha})^2}{\Gamma(2\alpha+1)} \\ R_3^1(t) &=& R_0\dfrac{(-2\mu t^{\alpha})^3}{\Gamma(3\alpha+1)} \\ \vdots &=& \vdots \\ R_n^1(t) &=& R_0\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} \end{cases}$$

The solution of the system at step $k = 1$ is:

$$\begin{cases} V^1(t) &=& V_0\displaystyle\sum_{n=0}^{\infty}\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} = V_0.E_{\alpha}(-2\mu t^{\alpha}) \\ S^1(t) &=& S_0\displaystyle\sum_{n=0}^{\infty}\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} = S_0.E_{\alpha}(-2\mu t^{\alpha}) \\ E^1(t) &=& E_0\displaystyle\sum_{n=0}^{\infty}\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} = E_0.E_{\alpha}(-2\mu t^{\alpha}) \\ A^1(t) &=& A_0\displaystyle\sum_{n=0}^{\infty}\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} = A_0.E_{\alpha}(-2\mu t^{\alpha}) \\ C^1(t) &=& C_0\displaystyle\sum_{n=0}^{\infty}\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} = C_0.E_{\alpha}(-2\mu t^{\alpha}) \\ T^1(t) &=& T_0\displaystyle\sum_{n=0}^{\infty}\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} = T_0.E_{\alpha}(-2\mu t^{\alpha}) \\ R^1(t) &=& R_0\displaystyle\sum_{n=0}^{\infty}\dfrac{(-2\mu t^{\alpha})^n}{\Gamma(n\alpha+1)} = R_0.E_{\alpha}(-2\mu t^{\alpha}) \end{cases}$$

With $E_{\alpha}(-2\mu t^{\alpha})$ the Mittag-Leffler function

At step $k=2$, we obtain:

$$\left\{\begin{array}{c}\left\{\begin{array}{c}V_0^2(t)=V_0+I_0^\alpha(N_1(E^1,A^1,C^1))\\ V_{n+1}^2(t)=-2\mu I_0^\alpha(V_n^2(t));\;\; n\geq 0\end{array}\right.\\ \left\{\begin{array}{c}S_0^2(t)=S_0+I_0^\alpha(N_2(E^1,A^1,C^1))\\ S_{n+1}^2(t)=\mu I_0^\alpha(V_n^2(t))-7\mu I_0^\alpha(S_n^2(t))+4\mu I_0^\alpha(R_n^2(t)),\;\; n\geq 0\end{array}\right.\\ \left\{\begin{array}{c}E_0^2(t)=E_0+I_0^\alpha(N_3(E^1,A^1,C^1))\\ E_{n+1}^2(t)=-2\mu I_0^\alpha(E_n^2(t)),\;\; n\geq 0\end{array}\right.\\ \left\{\begin{array}{c}A_0^2(t)=A_0\\ A_{n+1}^2(t)=\mu I_0^\alpha(E_n^2(t))-3\mu I_0^\alpha(A_n^2(t)),\;\; n\geq 0\end{array}\right.\\ \left\{\begin{array}{c}C_0^2(t)=C_0\\ C_{n+1}^2(t)=\dfrac{1}{2}\mu I_0^\alpha(A_n^2(t))-\dfrac{5}{2}\mu I_0^\alpha(C_n^2(t)),\;\; n\geq 0\end{array}\right.\\ \left\{\begin{array}{c}T_0^2(t)=T_0\\ T_{n+1}^2(t)=\dfrac{1}{2}\mu I_0^\alpha(C_n^2(t))+\dfrac{1}{2}\mu I_0^\alpha(A_n^2(t))-3\mu I_0^\alpha(T_n^2(t)),\;\; n\geq 0\end{array}\right.\\ \left\{\begin{array}{c}R_0^2(t)=R_0\\ R_{n+1}^2(t)=2\mu I_0^\alpha(T_n^2(t))+\mu I_0^\alpha(A_n^2(t))-5\mu I_0^\alpha(R_n^2(t)),\;\; n\geq 0\end{array}\right.\end{array}\right. \tag{4.10}$$

Let's calculate $N_1(E^1,A^1,C^1)$; $N_2(E^1,A^1,C^1)$ and $N_3(E^1,A^1,C^1)$

Let's ask $\beta_1=\beta_2=1-\mu$ and $\beta_3=2\mu-2$
we have:

$$\begin{aligned} N_1(E^1,A^1,C^1) &= -(1-\psi)\left[\beta_1E^1(t)+\beta_2A^1(t)+\beta_3C^1(t)\right]E^1(t)\\ &= -(1-\psi)\left[(1-\mu)E_0+(1-\mu)A_0+(2\mu-2)C_0\right]E_0.\left(E_\alpha(-2\mu t^\alpha)\right)^2\\ &= -(1-\psi)\left[(1-\mu)E_0+(1-\mu)E_0+(2\mu-2)E_0\right]E_0.\left(E_\alpha(-2\mu t^\alpha)\right)^2\\ &= -(1-\psi)\left[1-\mu+1-\mu+2\mu-2\right]\left(E_0.E_\alpha(-2\mu t^\alpha)\right)^2\\ &= -(1-\psi)\left[2-2+2\mu-2\mu\right]\left(E_0.E_\alpha(-2\mu t^\alpha)\right)^2\\ &= 0\end{aligned}$$

$$\begin{aligned} N_2(E^1,A^1,C^1) &= -\left[\beta_1E^1(t)+\beta_2A^1(t)+\beta_3C^1(t)\right]S^1(t)\\ &= -\left[(1-\mu)E_0+(1-\mu)A_0+(2\mu-2)C_0\right]S_0.\left(E_\alpha(-2\mu t^\alpha)\right)^2\\ &= -\left[(1-\mu)S_0+(1-\mu)S_0+(2\mu-2)S_0\right]S_0.\left(E_\alpha(-2\mu t^\alpha)\right)^2\\ &= -\left[1-\mu+1-\mu+2\mu-2\right]\left(S_0.E_\alpha(-2\mu t^\alpha)\right)^2\\ &= -\left[2-2+2\mu-2\mu\right]\left(S_0.E_\alpha(-2\mu t^\alpha)\right)^2\\ &= 0\end{aligned}$$

$$
\begin{aligned}
N_3(E^1, A^1, C^1) &= \left[\beta_1 E^1(t) + \beta_2 A^1(t) + \beta_3 C^1(t)\right] S^1(t) + \\
&\quad (1-\psi)\left[\beta_1 E^1(t) + \beta_2 A^1(t) + \beta_3 C^1(t)\right] V^1(t) \\
&= \left[(1-\mu)E_0 + (1-\mu)A_0 + (2\mu-2)C_0\right] S_0.\left(E_\alpha(-2\mu t^\alpha)\right)^2 + \\
&\quad (1-\psi)\left[(1-\mu)E_0 + (1-\mu)A_0 + (2\mu-2)C_0\right] V_0.\left(E_\alpha(-2\mu t^\alpha)\right)^2 \\
&= \left[(1-\mu)S_0 + (1-\mu)S_0 + (2\mu-2)S_0\right] S_0.\left(E_\alpha(-2\mu t^\alpha)\right)^2 + \\
&\quad (1-\psi)\left[(1-\mu)V_0 + (1-\mu)V_0 + (2\mu-2)V_0\right] V_0.\left(E_\alpha(-2\mu t^\alpha)\right)^2 \\
&= \left[1-\mu+1-\mu+2\mu-2\right]\left(S_0.E_\alpha(-2\mu t^\alpha)\right)^2 + \\
&\quad (1-\psi)\left[1-\mu+1-\mu+2\mu-2\right]\left(V_0.E_\alpha(-2\mu t^\alpha)\right)^2 \\
&= \left[2-2+2\mu-2\mu\right]\left(S_0.E_\alpha(-2\mu t^\alpha)\right)^2 + \\
&\quad (1-\psi)\left[2-2+2\mu-2\mu\right]\left(V_0.E_\alpha(-2\mu t^\alpha)\right)^2 \\
&= 0
\end{aligned}
$$

The algorithm at step $k = 2$ is the same as the algorithm at step $k = 1$. So recursively we have:

$$
\left\{
\begin{array}{lcl}
V_0^2(t) &=& V_0 \\
V_1^2(t) &=& V_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
V_2^2(t) &=& V_0 \dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
V_3^2(t) &=& V_0 \dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &=& \vdots \\
V_n^2(t) &=& V_0 \dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{array}
\right. ;
\quad
\left\{
\begin{array}{lcl}
S_0^2(t) &=& S_0 \\
S_1^2(t) &=& S_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
S_2^2(t) &=& S_0 \dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
S_3^2(t) &=& S_0 \dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &=& \vdots \\
S_n^2(t) &=& S_0 \dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{array}
\right.
$$

$$
\left\{
\begin{array}{lcl}
E_0^2(t) &=& E_0 \\
E_1^2(t) &=& E_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
E_2^2(t) &=& E_0 \dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
E_3^2(t) &=& E_0 \dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &=& \vdots \\
E_n^2(t) &=& E_0 \dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{array}
\right. ;
\quad
\left\{
\begin{array}{lcl}
A_0^2(t) &=& A_0 \\
A_1^2(t) &=& A_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
A_2^2(t) &=& A_0 \dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
A_3^2(t) &=& A_0 \dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &=& \vdots \\
A_n^2(t) &=& A_0 \dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{array}
\right.
$$

$$
\left\{
\begin{array}{lcl}
C_0^2(t) &=& C_0 \\
C_1^2(t) &=& C_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
C_2^2(t) &=& C_0 \dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
C_3^2(t) &=& C_0 \dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &=& \vdots \\
C_n^2(t) &=& C_0 \dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{array}
\right. ;
\quad
\left\{
\begin{array}{lcl}
T_0^2(t) &=& T_0 \\
T_1^2(t) &=& T_0 \dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\
T_2^2(t) &=& T_0 \dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\
T_3^2(t) &=& T_0 \dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\
\vdots &=& \vdots \\
T_n^2(t) &=& T_0 \dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)}
\end{array}
\right.
$$

$$\begin{cases} R_0^2(t) &=& R_0 \\ R_1^2(t) &=& R_0\dfrac{(-2\mu t^\alpha)}{\Gamma(\alpha+1)} \\ R_2^2(t) &=& R_0\dfrac{(-2\mu t^\alpha)^2}{\Gamma(2\alpha+1)} \\ R_3^2(t) &=& R_0\dfrac{(-2\mu t^\alpha)^3}{\Gamma(3\alpha+1)} \\ \vdots &=& \vdots \\ R_n^2(t) &=& R_0\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} \end{cases}$$

The solution of the system at step $k = 2$ is:

$$\begin{cases} V^2(t) &=& V_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = V_0.E_\alpha(-2\mu t^\alpha) \\ S^2(t) &=& S_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = S_0.E_\alpha(-2\mu t^\alpha) \\ E^2(t) &=& E_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = E_0.E_\alpha(-2\mu t^\alpha) \\ A^2(t) &=& A_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = A_0.E_\alpha(-2\mu t^\alpha) \\ C^2(t) &=& C_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = C_0.E_\alpha(-2\mu t^\alpha) \\ T^2(t) &=& T_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = T_0.E_\alpha(-2\mu t^\alpha) \\ R^2(t) &=& R_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = R_0.E_\alpha(-2\mu t^\alpha) \end{cases}$$

With $E_\alpha(-2\mu t^\alpha)$ the Mittag-Leffler function
recursively, we have:

$$\begin{cases} V^k(t) &=& V^2(t) = ... = V_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = V_0.E_\alpha(-2\mu t^\alpha) \\ S^k(t) &=& S^2(t) = ... = S_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = S_0.E_\alpha(-2\mu t^\alpha) \\ E^k(t) &=& E^2(t) = ... = E_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = E_0.E_\alpha(-2\mu t^\alpha) \\ A^k(t) &=& A^2(t) = ... = A_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = A_0.E_\alpha(-2\mu t^\alpha) \\ C^k(t) &=& C^2(t) = ... = C_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = C_0.E_\alpha(-2\mu t^\alpha) \\ T^k(t) &=& T^2(t) = ... = T_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = T_0.E_\alpha(-2\mu t^\alpha) \\ R^k(t) &=& R^2(t) = ... = R_0\sum_{n=0}^{\infty}\dfrac{(-2\mu t^\alpha)^n}{\Gamma(n\alpha+1)} = R_0.E_\alpha(-2\mu t^\alpha) \end{cases}$$

We have:

$$\begin{cases} V(t) = \lim\limits_{k\to+\infty} V^k(t) & = & V_0.E_\alpha(-2\mu t^\alpha) \\ S(t) = \lim\limits_{k\to+\infty} S^k(t) & = & S_0.E_\alpha(-2\mu t^\alpha) \\ E(t) = \lim\limits_{k\to+\infty} E^k(t) & = & E_0.E_\alpha(-2\mu t^\alpha) \\ A(t) = \lim\limits_{k\to+\infty} A^k(t) & = & A_0.E_\alpha(-2\mu t^\alpha) \\ C(t) = \lim\limits_{k\to+\infty} C^k(t) & = & C_0.E_\alpha(-2\mu t^\alpha) \\ T(t) = \lim\limits_{k\to+\infty} T^k(t) & = & T_0.E_\alpha(-2\mu t^\alpha) \\ R(t) = \lim\limits_{k\to+\infty} R^k(t) & = & R_0.E_\alpha(-2\mu t^\alpha) \end{cases}$$

The solution of the problem for $\alpha = 1$ *is:*

$$\begin{cases} V(t) & = & V_0.e^{-2\mu t} \\ S(t) & = & S_0.e^{-2\mu t} \\ E(t) & = & E_0.e^{-2\mu t} \\ A(t) & = & A_0.e^{-2\mu t} \\ C(t) & = & C_0.e^{-2\mu t} \\ T(t) & = & T_0.e^{-2\mu t} \\ R(t) & = & R_0.e^{-2\mu t} \end{cases}$$

5 Conclusions

In this work, we gave a brief review of fractional calculations, then studied the convergence of the SBA method for a system of nonlinear fractional differential equations and gave a mathematical model of hepatitis C. Finally, we used the SBA method, which is a numerical solution method, to apply this non-linear fractional system. The analytical resolution enabled us to obtain an exact solution.

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