An Extension to Implicit Second Derivative Method with Variable Step number for Direct Solution of Second Order Ordinary Differential Equations

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ABSTRACT

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| **This study developed a high – order numerical method, specifically, an implicit second derivative multistep method for direct solution of Initial Value Problems (IVPs) of second order Ordinary Differential Equations (ODEs).**  **Development of variants of the second derivative multistep method required order of derivative (*l*) to be made constant at 2 while the step – number (*k*) was varied from 1 – 3. Taylors series was adopted as the basis function and the derived methods were verified for accuracy, consistency, stability and convergence. Computational efficiency of the derived methods was verified using standard benchmark IVPs of second order ODEs.**  **Verification of the basic properties showed that the derived methods are accurate, zero - stable, consistent and convergent. Error analysis between numerical and exact solutions revealed the level of accuracy of the derived methods, which showed that three – step second derivative (*k=3, l=2*) method had better accuracy than one - step second derivative method (*k=1, l=2*) and two – step second derivative method (*k=2, l=2*) while the two – step second derivative method (*k=2, l=2*) had better accuracy than one - step second derivative method (*k=1, l=2*) and also competes favorably with three – step second derivative method (*k=3, l=2*). The results showed that accuracy of the methods increased as the step - number (*k*) increased, hence three – step second derivative (*k=3, l=2*) method was recommended for direct solution of second order IVPs** |

*Keywords:* *Implicit, Derivative, Step – number, Order, Stability, Direct Solution*

1. INTRODUCTION

**Higher order differential equations frequently arise in engineering systems including mechanical vibrations, electrical circuits and fluid dynamics, they also occur in many other fields of study. The need to develop numerical methods for direct solution of higher order differential equation against the conventional method of reduction to a system of first order differential equations cannot be overemphasized. Different numerical methods have been developed by several researchers which can be efficiently implemented in engineering, science and other fields, for example, [1] developed a class of Implicit Multiderivative Linear Multistep Method (IMLMM) for numerical solution of non – stiff and stiff ODEs. A continuous implicit linear multistep method was developed by [2] for solving IVPs of first order ODEs. [3] developed a numerical method for solving first order ODEs using collocation approach while [4] developed a diagonally implicit block method with two off – step points for solving stiff IVP. [5] developed an order eight one – step implicit block algorithm for solving both linear and non – linear problems, to mention a few.**

**The conventional approach of reducing higher order ODE to a system of first order ODE before using the appropriate numerical method to solve is tedious and inefficient hence the need for numerical methods capable of direct solution of such problems. Over the years, researchers have been investigating on the best approach to give a direct solution to higher order differential equations. [6] developed a block multiderivative method for solving first and second order ODEs using interpolation and collocation techniques. [7] developed a block method for solving higher order ODEs using power series on an implicit one – step approach, the research utilized collocation and interpolation. [8] developed high order diagonally implicit Runge – Kutta – Nystrom (RKN) method for solving second order IVPs. [9] also developed an implicit three – step method with multiderivative capabilities to handle general second order IVPs. [10] developed a method using 4th order Runge Kutta method capable of solving second order IVPs of ODEs. [11] used Adomian Decomposition Method (ADM) for solving higher order ODE, to mention a few.**

**In this study, a second derivative method with variable step – number capable of solving second order ODEs directly was developed. Variants of the method were used to solve some sampled IVPs of second order ODEs. A comparative study of the results was conducted and optimal accuracy was determined by comparing the numerical solution with exact solution. However, further research will be carried out to compare the method having optimal accuracy with some existing methods of the same order of accuracy.**

2. methodology

Derivation of the variant methods and verification of the basic properties are discussed.

2.1 Derivation of the Variant Methods

**The general implicit second derivative multistep method is of the form;**

**’  (2.1.1)**

**with local truncation error:**

**(2.1.2)**

**The development of variant methods adopted Taylor Series expansion of the variables**

**given as;**

**(2.1.3)**

**Accuracy of order P was imposed on the local truncation error and the resulting equations were solved for parameters and to generate the required variant methods. The order of accuracy of the methods were determined by substituting the results of parameters and into the original equations.**

**2.1.1 Derivation of one - step second derivative method**

**As derived in [1]**

**Setting *k=1, l=2* in (2.1.1) gives**

**(2.1.4)**

**with local truncation error**

**(2.1.5)**

**adopting Taylor’s series expansion of , and , imposing accuracy of order 4 on and solving gives the one – step, second derivative method:**

**(2.1.6)**

**which resolves into**

**(2.1.7)**

**2.1.2 Derivation of two - step second derivative method**

**As derived in [1]**

**Setting *k=2, l=2* in (2.1.1) gives**

**(2.1.8)**

**with local truncation error**

**(2.1.9)**

**adopting Taylor’s series expansion of , , , , and , imposing accuracy of order 7 on and solving gives two – step, second derivative method**

**(2.1.10)**

**which resolves into**

**(2.1.11)**

**2.1.3 Derivation of three - step second derivative method**

**Setting *k=3, l=2* in (2.1.1) gives**

**(2.1.12)**

**with local truncation error**

**(2.1.13)**

**adopting Taylor’s series expansion of , ,, , and and solving gives:**

**(2.1.14)**

**where**

** -**

**(2.1.15)**

**imposing accuracy of order 10 on to have**

**C0 = C1 = C2 = C3 = - - - = C10= 0.**

**solving gives;**

**= -1, =, = , β10 = , β11 , β12 = , β13 = , β20 = , β21 = , β22 = and β23 =**

**putting these values into equation (2.1.12) gives three – step, second derivative method of the form:**

**(2.1.16)**

**which resolves into**

**(2.1.17)**

**2.2 Verification of Basic Properties of the methods**

**According to [12]), a good numerical method for solution of ODEs is required to be accurate, consistent, stable and convergent.**

**One – step second derivative method (*k=1, l=2*) and two - step second derivative method (*k=2, l=2*) were investigated for these basic properties in [1] and [13], the results showed the methods are accurate, consistent, zero - stable and convergent hence recommended for solving IVPs of ODEs.**

**Investigating the basic properties for three - step second derivative method (*k=3, l-=2*)**

**2.2.1 Accuracy**

**According to [14], a numerical method is accurate if and only if order P of the local truncation error of the method is greater than or equal to 1, that is, ()**

**= -1, =, = , β10 = , β11 , β12 = , β13 = , β20 = , β21 = , β22 = and β23 =**

**Substituting these values into equation (2.1.15) gives**

**C0 = C1 = C2 = C3 = - - - = C10 = 0.**

**Since order of accuracy P 1, the method is accurate.**

**2.2.2 Consistency**

**According to [12], a linear Multistep Method is consistent if the parameters and satisfy the following conditions:**

1. **Order**
2. **=**

**For the three – step second derivative method**

**Order 1.**

**= 0() +1() +2 () +3 () and**  **=**

**= 0 + - 2() + 3(1) = - - +**

**= =**

**Since the three conditions are satisfied, the method is consistent.**

**2.2.3 Zero-stability.**

**According to [15], a linear multistep method is said to be zero – stable if the root of its first characteristic polynomial has modulus less than or equal to 1.**

**For the three-step second derivative method;**

**whose first characteristic polynomial is**

**and its roots are [roots are within a unit circle]**

**Hence, the method is zero - stable.**

**2.2.4 Convergence.**

**According to [16], a necessary and sufficient condition for a linear multistep method to be convergent is that it must be consistent and zero - stable**

**The method is consistent and zero-stable, therefore it is convergent.**

**Verification of the basic properties showed that all the derived methods are accurate, consistent, zero – stable and convergent.**

**3. RESULTS AND DISCUSSION**

To determine the level of accuracy, efficiency and effectiveness of the derived methods, they were translated into computer algorithms which were used to solve some sampled second order IVPs of ODEs. The results are presented in tables 1 – 8

Test problems:

Problem 1

Exact Solution:

Problem 2

Exact Solution:

Problem 3

Exact Solution:

Table 1: Table of result for Problem 1 using one - step second derivative method

|  |  |  |  |
| --- | --- | --- | --- |
| x | Exact solution | Numerical solution (k=1,l=2) | Error |
| 0.01 | 2.02989950080200000 | 2.0241690962099124 | 1.5730405e-2 |
| 0.02 | 2.05959601330000000 | 2.0533385974988554 | 4.6257416e-2 |
| 0.03 | 2.08908656859995454 | 2.0833385974988554 | 7.5747972e-2 |
| 0.04 | 2.11836821693618597 | 2.1176786272496516 | 1.06689590e-1 |
| 0.05 | 2.14743803092449483 | 2.1474380309244948 | 1.36588876e-1 |
| 0.06 | 2.17629310284922760 | 2.1758491554291694 | 1.66273077e-1 |
| 0.07 | 2.20493054800694353 | 2.2030200256233074 | 1.95739310e-1 |
| 0.08 | 2.23334750189505883 | 2.2134191237691043 | 2.24984711e-1 |
| 0.09 | 2.26154112364473246 | 2.2173627914914115 | 2.54006437e-1 |
| 0.10 | 2.28950859311221131 | 2.2175346868835076 | 2.82801669e-1 |

Table 2: Table of result for Problem 1 using two - step second derivative method

|  |  |  |  |
| --- | --- | --- | --- |
| X | Exact solution | Numerical solution (k=2,l=2) | Error |
| 0.02 | 2.059403980079791159077366 | 2.059596013413154794924862 | 1.92033333363635847496e-4 |
| 0.04 | 2.117631682543199279943770 | 2.118368215881858358986782 | 7.36533338659079043012e-4 |
| 0.06 | 2.174706399247587207272367 | 2.176293099308742128692596 | 1.58670006115421420229e-3 |
| 0.08 | 2.230650960840640873512662 | 2.233347494512756831876043 | 2.69653367211595836338e-3 |
| 0.10 | 2.285487745892932785081218 | 2.289508580496535989111567 | 4.02083460360320403035e-3 |
| 0.12 | 2.339238689849648635424120 | 2.344753893574490584398248 | 5.51520372484194897413e-3 |
| 0.14 | 2.391925293805057854953988 | 2.39906133635983788951861 | 7.13604255292593397873e-3 |
| 0.16 | 2.443568633101238003075694 | 2.452409186593991788016295 | 8.84055349075378494060e-3 |
| 0.18 | 2.494189365768493406735113 | 2.504776105853715364210840 | 1.05867400852219574757e-2 |
| 0.20 | 2.543807740768840325353087 | 2.556141148067666908626631 | 1.23334072988265832735e-2 |

Table 3 Table of result for Problem 1 using three - step second derivative method

|  |  |  |  |
| --- | --- | --- | --- |
| X | Exact solution | Numerical solution (k=3,l=2) | Error |
| 0.03 | 2.089086568104681737852859 | 2.989086568105462014849896 | 7.80276997037790900e-13 |
| 0.06 | 2.176293099306178514748371 | 2.176293099308742128692597 | 2.56361394422537253e-12 |
| 0.09 | 2.261411136125666902127752 | 2.261411136180216552594407 | 5.36495923816317049e-12 |
| 0.12 | 2.344753893564612309785866 | 2.344753893574490584398248 | 9.87827461238264053e-12 |
| 0.15 | 2.425856553277375321123404 | 2.425856553292882379631383 | 1.55069168397340419e-11 |
| 0.18 | 2.504776105831454149027613 | 2.504776105837153642108403 | 2.22612151832272046e-11 |
| 0.21 | 2.581441528955698277155790 | 2.581441528986595201658414 | 3.08969245026248898e-11 |
| 0.24 | 2.655783829447225751487598 | 2.655783828985462974943727 | 4.07403997859669539e-11 |
| 0.27 | 2.727736102746477518229316 | 2.727736102798274412229325 | 5.17968940000094225e-11 |
| 0.30 | 2.797233598170353290450113 | 2.797233598235307640600583 | 6.48774714504694933e-11 |

Table 4: Table of result for Problem 2 using one - step second derivative method

|  |  |  |  |
| --- | --- | --- | --- |
| x | Exact solution | Numerical solution (k=1,l=2) | Error |
| 0.01 | 0.904837418000000000 | 0.9058333333333334 | -9.9591530e-4 |
| 0.02 | 0.818730753500000000 | 0.8211173611111112 | -2.3866076e-3 |
| 0.03 | 0.740818221124284538 | 0.7449655457175928 | -4.1473246e-3 |
| 0.04 | 0.670320046907880164 | 0.6765806850120565 | -6.2606381e-3 |
| 0.05 | 0.606530660661683361 | 0.6152465680762472 | -8.7159074e-3 |
| 0.06 | 0.548811637460721165 | 0.5603204756417662 | -1.1508838e-2 |
| 0.07 | 0.496585305293378601 | 0.5112264466821800 | -1.4641141e-2 |
| 0.08 | 0.449328966040773290 | 0.4674492406434138 | -1.8120275e-2 |
| 0.09 | 0.406569661862600732 | 0.4285289320362432 | -2.1959270e-2 |
| 0.10 | 0.367879443735233612 | 0.3940560807019650 | -2.6176637e-2 |

Table.5: Table of result for Problem 2 using two - step second derivative method

|  |  |  |  |
| --- | --- | --- | --- |
| X | Exact solution | Numerical solution (k=2,l=2) | Error |
| 0.02 | 0.818730735409967688209584 | 0.818730753077981858669936 | 1.76680141704603518e-8 |
| 0.04 | 0.670319995495932487040748 | 0.670320046035639300744433 | 5.05397068137036853e-8 |
| 0.06 | 0.548811538787118002611547 | 0.548811636094026432628459 | 9.73069084300169117e-8 |
| 0.08 | 0.449328806421825392066938 | 0.449328964117221591430102 | 1.57695396199363164-e7 |
| 0.10 | 0367879208803887223372251 | 0.367879441171442321595524 | 2.32367555098223273e-7 |
| 0.12 | 0.301193889033924156286584 | 0.301194211912202096644978 | 3.22878277940358393e-7 |
| 0.14 | 0.246596532262108809250625 | 0.246596963941606476939861 | 4.31679497667689236e-7 |
| 0.16 | 0.201895955823444186546433 | 0.201896517994655408485179 | 5.62171211221938746e-7 |
| 0.18 | 0.165298169422445331429414 | 0.165298888221586538296805 | 7.18799141206867390e-7 |
| 0.20 | 0.135334376035214710005120 | 0.135335283236612691894000 | 9.07201397981888879e-7 |

Table 6: Table of result for Problem 2 using three - step second derivative method

|  |  |  |  |
| --- | --- | --- | --- |
| X | Exact solution | Numerical solution (k=3,l=2) | Error |
| 0.03 | 0.74081854085136512706 | 0.74081822068171786607 | 3.20169647260989e-7 |
| 0.06 | 0.54881263470501661682 | 0.54881163609402643263 | 9.98610990184187e-7 |
| 0.09 | 0.40657171961617261583 | 0.40656965974059911188 | 2.09857557350395e-12 |
| 0.12 | 0.30119783876996323155 | 0.30119421191220209664 | 3.62685776113491e-11 |
| 0.15 | 0.22313581672185919615 | 0.22313016014842982893 | 5.65657342936722e-11 |
| 0.18 | 0.16530720626322349019 | 0.16529888822158653830 | 8.31804116369519e-11 |
| 0.21 | 0.12246829169296383698 | 0.12245642852981910220 | 1.18634399819268e-10 |
| 0.24 | 0.09073449229852437764 | 0.09071795328941250338 | 1.65390091187427e-10 |
| 0.27 | 0.06722827534568262187 | 0.06720551273974976513 | 2.27626059328567e-10 |
| 0.30 | 0.04981817135313826428 | 0.04978706836786394298 | 3.11029852743213e-10 |

Table 7: Table of result for Problem 3 using two - step second derivative method

|  |  |  |  |
| --- | --- | --- | --- |
| x | Exact solution | Numerical solution (k=2,l=2) | Error |
| 0.02 | -0.02020134002673593300207 | -0.02020134002675581016014 | 1.987715807134e-14 |
| 0.04 | -0.04081077419232778201197 | -0.04081077419238822675705 | 6.044474507237e-14 |
| 0.06 | -0.06183654654523709162665 | -0.06183654654535962222469 | 1.225305980312e-13 |
| 0.08 | -0.08328706767475156687882 | -0.08328706767495855443599 | 2.069875571687e-13 |
| 0.10 | -0.10517091807533293067345 | -0.10517091807564762481117 | 3.146941382471e-13 |
| 0.12 | -0.12749685157892911625732 | -0.12749685157937567147927 | 4.465552219439e-13 |
| 0.14 | -0.15027379885662376536312 | -0.15027379885722726812356 | 6.035027604436e-13 |
| 0.16 | -0.17351087099102373851658 | -0.17351087099181023501861 | 7.864965020302e-13 |
| 0.18 | -0.19721736312081364014308 | -0.19721736312181016487682 | 9.965247337420e-13 |
| 0.20 | -0.22140275815893522887831 | -0.22140275816016983392107 | 1.234605042761e-12 |

Table 8: Table of result for Problem 3 using three - step second derivative method

|  |  |  |  |
| --- | --- | --- | --- |
| X | Exact solution | Numerical solution (k=3,l=2) | Error |
| 0.03 | -0.03045453393910516183 | -0.0304545339535168556 | 3.936605830e-13 |
| 0.06 | -0.06183654654665662562 | -0.0618365464535962220 | 1.297003419e-12 |
| 0.09 | -0.09417428370794876244 | -0.0941742837052103579 | 2.738404537e-12 |
| 0.12 | -0.12749685158443065810 | -0.1274968515793756715 | 5.067943100e-12 |
| 0.15 | -0.16183424273629499846 | -0.1618342427282831226 | 8.011875860e-12 |
| 0.18 | -0.19721736313341482291 | -0.1972173631218101649 | 1.160465801e-11 |
| 0.21 | -0.23367805992793253318 | -0.2336780599567432511 | 1.623000222e-11 |
| 0.24 | -0.27124915032996747939 | -0.2712491503214046916 | 2.159205634e-11 |
| 0.27 | -0.30996445076097600955 | -0.3099644507332473639 | 2.772864565e-11 |
| 0.30 | -0.34985880761106530787 | -0.3498588075760031040 | 3.506220387e-11 |

**Discussion**

From tables 1 – 6, it was observed that the three – step second derivative (*k=3, l=2*) method has better accuracy than one - step second derivative method (*k=1, l=2*) and two – step second derivative method (*k=2, l=2*). Also, tables 7 and 8 showed that two – step second derivative method (*k=2, l=2*) and three – step second derivative method (*k=3, l=2*) have high accuracy and compete favorably with each other, hence recommended for direct solution of second order IVPs of ODEs.

In further studies, the two methods with better accuracy; (*k=2, l=2*) and (*k=3, l=2*) will be used to compare other existing methods of the same order of accuracy.

**4.CONCLUSION**

A second derivative method with variable step – size capable of solving second order ODEs directly was developed. Variants of the method were used to solve some sampled IVPs of second order ODEs. A comparative study of the results was conducted and optimal accuracy was determined by comparing the numerical solution with exact solution. Three – step second derivative (*k=3, l=2*) method had better accuracy than other methods considered in this study. However, further research will be carried out to compare this method (*k=3,l=2*) with some existing methods of the same order of accuracy.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, e.t.c.) and text - to – image generators have been used during the writing or editing of this manuscript.

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