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Associative Algebras Satisfying Quadratic Equations

Original Research Article

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Abstract

This work classifies associative algebras over a field K that are generated by a finite set G and satisfy a polynomial identity of the form $X^2 = aX + b$, where a and b are elements of K and X varies either over all elements of the algebra or over all elements of the multiplicative semigroup S generated by G. The results obtained were validated computationally using the GAP system.

Keywords: Associative algebra, Polynomial identity, Nilpotency index, Nagata-Higman theorem

2020 Mathematics Subject Classification: 16R10; 16R40; 16R50; 16W50

1 Introduction

Associative algebras play a crucial role in algebraic structures and have numerous applications in various fields of mathematics and physics. Beyond their fundamental importance in algebraic structures, the classification of associative algebras that satisfy polynomial identities has applications in various other areas, as we can see in [1, 2, 3]. For example, such algebraic structures can contribute to coding theory, offering new ways to construct error-correcting codes with the aim of improving coding efficiency. In Quantum Mechanics, in Physics, these algebras can contribute to the understanding of symmetries and operator algebras to provide new algebraic approaches for physical systems. Furthermore, the classification and computational validation of algebras using the GAP system directly contribute to computational algebra, providing a basis for the development of algorithms and tools for symbolic computation.

When a field F has characteristic 0 and A is an F-algebra, we define A as a nil-algebra if there exists a positive integer n such that $a^n = 0$ for every $a \in A$. In this case, n is called the nil-index of A. An algebra A is said to be nilpotent of index m if any product of m elements of A is zero.

In [4], Nagata proved that if A is a nil-algebra with nil-index n, then A is nilpotent of index $N(n)$ for some $N(n) \in \mathbb{N}$. Higman [5] provided a refined result, showing that $n^2 \leq N(n) \leq 2^n - 1$. Razmyslov [6] improved Higman's upper bound to $N(n) \geq n^2$, and Kuzmin [7] enhanced the lower bound to $N(n) \geq \frac{n(n+1)}{2}$. Woo Lee [8] demonstrated that a nil-algebra A with nil-index 3 is nilpotent of index 6. Other recent works on algebras satisfying polynomial identities can be found in [9, 10, 11, 12].

This paper focuses on classifying and characterizing associative algebras generated by a set $G = \{x_1, x_2, \cdots, x_m\}$ over a field K that satisfy a quadratic polynomial identity of the form $X^2 = aX + b$, where $a, b \in K$ and X can vary over the elements of A or over a multiplicative semigroup S generated by G.

2 The case $X^2 = k$

Lemma 2.1. *Let S be the set of words in $\{x_1, x_2, \cdots, x_m\}$ and A an Associative Algebra over a field K containing S satisfying $X^2 = k$, $\forall X \in S$, for some $k \in K$. Then $k \in \{0, 1\}$.*

Proof. Since $k = (x^2)^2 = k^2$, then $k \in \{0, 1\}$. □

2.1 The case $X^2 = 0$

Theorem 2.2. *Let $A = \langle x_1, x_2, x_3, \ldots, x_m \rangle$ over a field of characteristic $p \neq 2$, satisfying $X^2 = 0$, $\forall X \in A$, then A is nilpotent of index at most 3 and the dimension of A is at most $\dfrac{m(m+1)}{2}$.*

Proof. Let $x, y, z \in A$, then

$$(x+y)^2 = x^2 + xy + yx + y^2 = 0 \Rightarrow xy = -yx \tag{2.1}$$

and

$$(x+yz)^2 = x^2 + xyz + yzx + (yz)^2 = xyz + yzx = 0 \Rightarrow xyz = -yzx \tag{2.2}$$

Now observe that

$$xyz \overset{(2.1)}{=} -(yxz) \overset{(2.1)}{=} -(y(-zx)) = yzx \tag{2.3}$$

By (2.3) and (2.2), we obtain that

$$2xyz = 0. \tag{2.4}$$

As A is an Algebra of characteristic $p \neq 2$ and $2xyz = 0$, it follows that $xyz = 0$. Therefore A is nilpotent of index at most 3.

By (2.1), (2.2) and (2.3), it follows that the basis of A is a subset of

$$\left(\bigcup_{k=1}^{m} \{x_k\}\right) \bigcup \left(\bigcup_{1 \leq i < j \leq m} \{x_i x_j\}\right)$$

which has $\frac{m(m+1)}{2}$ elements. Hence, the dimension of A is at most $\frac{m(m+1)}{2}$.

□

Example 2.3. *Consider $A = \langle x, y \rangle$, where every element of the Associative Algebra satisfies the condition $X^2 = 0$.*

Thus,

$$(x+y)^2 = x^2 + y^2 + xy + yx = 0 \Rightarrow xy = -yx$$

$$(x+xy)^2 = x^2 + (xy)^2 + x^2y + xyx = 0 \Rightarrow xyx = 0$$

Indeed, we obtain that words of length greater than 2 in the semigroup S generated by x and y are equal to zero.

In this case, any element $w \in A$ can be written as

$$w = ax + by + cxy,$$

where $a, b, c \in K$.

In this case, we can take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

as we can verify by running the code in GAP found in Appendix A.

Theorem 2.4. *Let $A = \langle x_1, x_2, x_3, \ldots, x_m \rangle$ over a field of characteristic 2, satisfying $X^2 = 0, \forall X \in A$, then A is nilpotent of index at most $m + 1$ and the dimension of this algebra is less than or equal to $2^m - 1$.*

Proof. By (2.1), we obtain that a basis of A over a field F of characteristic 2, satisfying $X^2 = 0$, $\forall X \in A$, is formed by elements of the form $x_1^{\xi_1} x_2^{\xi_2} \ldots x_m^{\xi_m}$, where $\xi_1, \ldots, \xi_m \in \{0, 1\}$ and not all ξ_i are zero. Therefore, $\dim(A) \leq 2^m - 1$. □

2.2 The case $X^2 = 1$

Theorem 2.5. *Let S be the set of words in $\{x_1, x_2, \ldots, x_m\}$ and A an Associative Algebra over a field K generated by S and satisfying $X^2 = 1, \forall X \in S$, then A is an Abelian algebra and $\dim(A) = 2^m$.*

Proof. First, observe that $yx = x^2 yxy^2 = x(xy)^2 y = xy, \forall x, y \in \{x_1, \cdots, x_m\}$.

Therefore, the generators of A are of the form $x_1^{\xi_1} x_2^{\xi_2} \cdots x_m^{\xi_m}$, where $\xi_1, \cdots, \xi_m \in \{0, 1\}$.

Thus, $dim(A) = 2^m$. □

Example 2.6. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K generated by S satisfying $X^2 = 1, \forall X \in S$, then there exists a basis $\{1, x, y, xy\}$ where any element $w \in A$ can be written as*

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy,$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K$.

If $w_1 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$ and $w_2 = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$, then

$$\begin{aligned} w_1 w_2 &= (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4) \\ &+ (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_4 + \alpha_4\beta_3)x \\ &+ (\alpha_1\beta_3 + \alpha_2\beta_4 + \alpha_3\beta_1 + \alpha_4\beta_2)y \\ &+ (\alpha_1\beta_4 + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_4\beta_1)xy \end{aligned}$$

In this case, we can take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

as we can verify by running the GAP code from Appendix B.

3 The case $X^2 = kX$

Lemma 3.1. *Let S be the set of words in $\{x_1, x_2, \ldots, x_m\}$ and A an Associative Algebra over a field K containing S satisfying $X^2 = kX$, $\forall X \in S$, for some $k \in K$. Then $k \in \{0, 1\}$.*

Proof. $k^2x = k(kx) = kx^2 = (x^2)^2 = (kx)^2 = k^2(x)^2 = k^3x \Rightarrow k^2(k-1)x = 0 \Rightarrow k \in \{0, 1\}$. □

3.1 The case $X^2 = X$

Theorem 3.2. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K generated by S satisfying $X^2 = X$, $\forall X \in A$, then*

- *A is Abelian;*
- *A has characteristic 2;*
- *$dim(A) \leq 3$.*

Proof. Let $w \in A$. Thus, $(2w)^2 = 4w$ and $(2w)^2 = 2w$, which implies $2w = 0$. Therefore, A has characteristic 2.

Since

$$(x + y)^2 = x + y \tag{3.1}$$

and

$$(x + y)^2 = x^2 + y^2 + xy + yx = x + y + xy + yx \tag{3.2}$$

then

$$xy = yx. \tag{3.3}$$

Therefore, A is an Abelian algebra.

By (3.3) and by definition of A, it follows that a basis for A is a subset of $\{x, y, xy\}$. Thus, $\dim(A) \leq 3$.

□

Example 3.3. *The algebra over $\mathbb{Z}_2$ generated by x and y, where $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ satisfies the hypotheses of Theorem 3.2 and has dimension 3 as a vector space, as we verify by running the GAP code from Appendix C. .‘*

Theorem 3.4. *If S is the set of words in $\{x_1, x_2, \ldots, x_m\}$ and A is an Associative Algebra over a field K generated by S satisfying $X^2 = X$, $\forall X \in A$, then*

- *$\mathcal{A}$ is Abelian;*
- *A has characteristic 2;*

- $dim(A) \leq 2^m - 1$;

Proof. By Theorem 3.2, it follows that A is Abelian and A has characteristic 2. As A is Abelian and by definition of A, it follows that a basis of A is a subset of

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m} \mid \alpha_1, \dots, \alpha_m \in \{0,1\}, \alpha_1^2 + \cdots + \alpha_m^2 \neq 0\}$$

Thus, A has dimension at most $2^m - 1$. □

Theorem 3.5. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K generated by S satisfying $X^2 = X$, $\forall X \in S$, then $\dim(A) \leq 6$.*

Proof. It suffices to observe that the set $\{x, y, xy, yx, xyx, yxy\}$ generates the algebra A. Note that if $A = \langle x, y \rangle$, where

$$x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then A satisfies the conditions of the proposition, and $\beta = \{x, y, xy, yx, xyx, yxy\}$ is a basis for A with $\dim(A) = 6$, as we can verify by running the GAP code from Appendix E. □

4 The case $X^2 = aX + b$, with $a, b \in K^*$

Theorem 4.1. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K with characteristic 2 generated by S satisfying $X^2 = X + 1$, $\forall X \in S$, then A is the field $GF(4)$ with four elements $\{0, 1, x, 1 + x\}$.*

Proof. We have

$$(x^2y)^2 = x^2y + 1 = (x+1)y + 1 = xy + y + 1 \tag{4.1}$$

and

$$\begin{aligned} (x^2y)^2 &= ((x+1)y)^2 \\ &= (xy + y)^2 \\ &= (xy)^2 + y^2 + xy^2 + yxy \\ &= xy + 1 + y + 1 + x(y+1) + yxy \\ &= 2xy + 2 + y + x + yxy = y + x + yxy \end{aligned} \tag{4.2}$$

By (4.1) and (4.2), we have

$$yxy = xy + x + 1 \tag{4.3}$$

$$y^2xy = yxy + yx + y = xy + x + 1 + yx + y \tag{4.4}$$

$$y^2xy = (y+1)xy = yxy + xy = 2xy + x + 1 = x + 1 \tag{4.5}$$

By (4.4) and (4.5), we have

$$xy = yx + y \tag{4.6}$$

Thus, by symmetry,

$$yx = xy + x \tag{4.7}$$

By (4.6) and (4.7), we have that $x = y$. □

Theorem 4.2. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K with characteristic greater than 2 generated by S satisfying $X^2 = aX + b, \forall X \in S$ with $a, b \in K^*$, then*

(i) $b = -1$ *and* $a = 2$ *or* $a = -1$;

(ii) $\{1, x, y, xy\}$ *is a basis for A as a vector space over K;*

(iii) $dim(\mathcal{A}) = 4$.

Proof. (i) Since $X^2 = aX + b$, then

$$\begin{aligned}(X^2)^2 &= (aX+b)^2 \\ &= a^2X^2 + 2abX + b^2 \\ &= a^2(aX+b) + 2abX + b^2 \\ &= (a^3+2ab)X + a^2b + b^2\end{aligned} \tag{4.8}$$

and

$$(X^2)^2 = aX^2 + b = a(aX+b) + b = a^2X + ab + b \tag{4.9}$$

Therefore, we obtain

$$\left\{\begin{array}{l} a^2 = a^3 + 2ab \\ ab + b = a^2b + b^2 \end{array}\right. \Rightarrow \left\{\begin{array}{l} a = a^2 + 2b \\ a + 1 = a^2 + b \end{array}\right. \Rightarrow \left\{\begin{array}{l} b = -1 \\ a^2 - a - 2 = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} b = -1 \\ a = 2 \text{ or } a = -1 \end{array}\right. \tag{4.10}$$

(ii) and (iii) For $X^2 = -X - 1$, we have

$$\begin{aligned}(x^2y)^2 &= -x^2y - 1 \\ &= -(-x-1)y - 1 \\ &= xy + y - 1\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}(x^2y)^2 &= [(-x-1)y]^2 \\ &= (xy+y)^2 \\ &= (xy)^2 + y^2 + xy^2 + yxy \\ &= -xy - 1 - y - 1 + x(-y-1) + yxy \\ &= -2xy - 2 - y - x + yxy\end{aligned} \tag{4.12}$$

By (4.11) and (4.12), we obtain

$$yxy = 3xy + 2y + x + 1 \tag{4.13}$$

Now,

$$\begin{aligned}xyxy &= 3x^2y + 2xy + x^2 + x \\ &= 3(-x-1)y + 2xy + (-x-1) + x \\ &= -3xy - 3y + 2xy - 1 = -xy - 3y - 1\end{aligned} \tag{4.14}$$

and

$$xyxy = -xy - 1 \tag{4.15}$$

By (4.14) and (4.15),

$$3y = 0 \tag{4.16}$$

By symmetry, we can prove that $3x = 0$. Therefore, K has characteristic 3.
Since K has characteristic 3, we obtain that

$$yxy = 2y + x + 1 \tag{4.17}$$

Thus,

$$y^2xy = y(2y + x + 1) = 2y^2 + yx + y = -2y - 2 + yx + y = -2 + yx + 2y \tag{4.18}$$

and

$$y^2xy = (-y - 1)xy = -yxy - xy = -(2y + x + 1) - xy = -2y - x - 1 - xy \tag{4.19}$$

By (4.18) and (4.19), we obtain that

$$yx + xy = -4y - x + 1 = 2y + 2x + 1 \tag{4.20}$$

For $X^2 = 2X - 1$.

$$\begin{aligned}(x^2y)^2 &= 2x^2y - 1\\ &= 2(2x - 1)y - 1\\ &= 4xy - 2y - 1\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}(x^2y)^2 &= [(2x - 1)y]^2\\ &= (2xy - y)^2\\ &= 4(xy)^2 + y^2 - 2xy^2 - 2yxy\\ &= 8xy - 4 + 2y - 1 - 2x(2y - 1) - 2yxy\\ &= 4xy - 5 + 2y + 2x - 2yxy\end{aligned} \tag{4.22}$$

By (4.21) and (4.22), we obtain that

$$yxy = 2y + x - 2 \tag{4.23}$$

Now,

$$\begin{aligned}y^2xy &= 2y^2 + yx - 2y\\ &= 4y - 2 + yx - 2y\\ &= 2y - 2 + yx\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}y^2xy &= (2y - 1)xy\\ &= 2yxy - xy\\ &= 2(2y + x - 2) - xy\\ &= 4y + 2x - 4 - xy\end{aligned} \tag{4.25}$$

By (4.24) and (4.25),

$$yx + xy = 2y + 2x - 2 \tag{4.26}$$

Equations (4.11) to (4.25) show that $\{1, x, y, xy\}$ is a basis of A as a vector space over K and ,therefore, $\dim(A) = 4$.

Note that if A is an algebra satisfying the above conditions, then every element $w \in A$ can be written as

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy,$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K.$

Moreover, if $w_1 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$ and $w_2 = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$, then

$$\begin{aligned} w_1 w_2 &= (\alpha_1\beta_1 - \alpha_2\beta_2 - 2\alpha_3\beta_2 - \alpha_3\beta_3 - 2\alpha_3\beta_4 - 2\alpha_4\beta_2 - \alpha_4\beta_4) \\ &+ (\alpha_1\beta_2 + \alpha_2\beta_1 + 2\alpha_3\beta_2 + 2\alpha_2\beta_2 + \alpha_3\beta_4 + 2\alpha_4\beta_2 - \alpha_4\beta_3)x \\ &+ (\alpha_1\beta_3 - \alpha_2\beta_4 + 2\alpha_3\beta_2 + \alpha_3\beta_1 + 2\alpha_3\beta_3 + 2\alpha_3\beta_4 + \alpha_4\beta_2)y \\ &+ (\alpha_1\beta_4 + \alpha_2\beta_3 + 2\alpha_2\beta_4 - \alpha_3\beta_2 + \alpha_4\beta_1 + 2\alpha_4\beta_3 + 2\alpha_4\beta_4)xy \end{aligned}$$

In this case, we can take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -2 & 2 & 2 & -1 \\ -1 & 0 & 2 & 0 \\ -2 & 1 & 2 & 0 \end{pmatrix},$$

as we can verify by running the GAP code from Appendix F. □

Conclusion

In this work, we classify associative algebras that are generated by a finite set of elements and satisfy specific quadratic polynomial identities. The results presented not only generalize existing contributions to the theory of nil-algebras but also provide new insights into the structure and dimensional characteristics of associative algebras under various conditions. Computational validation was performed using the **GAP** software package, which confirmed the theoretical results.

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Appendix

A Verification Code for Example 2.3

```
# Define the finite field GF(3)
K := GF(3);

# Left regular representation of the two generators of the algebra as two 4x4 matrices over GL(3)
x := [[0,1,0,0],[0,0,0,0],[0,0,0,1],[0,0,0,0]] * One(K);
y := [[0,0,1,0],[0,0,0,-1],[0,0,0,0],[0,0,0,0]] * One(K);

# Construct the algebra A generated by x and y
A := Algebra(K, [x, y]);


# Check if A is nilpotent of index 3
v := true;
for i in A do
  for j in A do
    for k in A do
      if not IsZero(i*j*k) then
        v := false; break;
      fi;
    od;
    if v = false then break; fi;
  od;
  if v = false then break; fi;
od;

if v = false then
  Print("This algebra isn't nilpotent of index 3!");
else
  Print("This algebra is nilpotent of index 3!");
fi;

# Check if every element in A satisfies x^2 = 0
v := true;
```

```
for i in A do
  if not IsZero(i*i) then
    v := false; break;
  fi;
od;

if v = false then
  Print("\nThis algebra doesn't satisfy the equation X^2=0");
else
  Print("\nThis algebra satisfy the given condition!\n");
fi;


# Get a basis B for the algebra A
B := Basis(A);

# Compute the number of elements and dimension of A
s := Size(A);
t := Size(B);

# Print size and dimension
Print("\nSize of  A:", s, "\n");
Print("\nDimension of A as a vectorial space:", t, "\n");

#Creates the vector subspace over K generated by {x, y, x*y}
Sub := VectorSpace(K, [x, y, x*y]);

#Computes the dimension of the subspace Sub
dim := Dimension(Sub);
```

B Verification Code for Example 2.6

```
# Define the finite field GF(5)
K:=GF(5);

# Left regular representation of the two generators of the algebra as two 4x4 matrices over GL(5)
x:=[[0,1,0,0],[1,0,0,0],[0,0,0,1],[0,0,1,0]]*One(K);
y:=[[0,0,1,0],[0,0,0,1],[1,0,0,0],[0,1,0,0]]*One(K);

#Construct the semigroup S generated by x and y
S:=Semigroup(x,y);


# Construct the algebra A generated by x and y
A := Algebra(K, [x, y]);


# Check if every element in S satisfies X^2 = 1

v:=true;
for i in S do
if not IsZero(i^2-One(A)) then v:=false; break; fi;
od;

if v=false then
Print("\nThis algebra doesn't satisfy the given condition!\n");
else
Print("\nThis algebra satisfy the given condition!\n");
```

```
fi;

# Get a basis B for the algebra A
B := Basis(A);

# Compute the number of elements and dimension of A
s := Size(A);
t := Size(B);

# Print size and dimension
Print("\nSize of  A:", s, "\n");
Print("\nDimension of A as a vectorial space:", t, "\n");

#Creates the vector subspace over K generated by {1, x, y, xy}
Sub := VectorSpace(K, [One(A), x, y, x*y]);

#Computes the dimension of the subspace Sub
dim := Dimension(Sub);
```

C Verification Code for Example 3.3

```
# Define the finite field GF(2)
K:=GF(2);

# Left regular representation of the two generators of the algebra as two 3x3 matrices over GL(2)
x:=[[1,0,0],[0,0,1],[0,0,1]]*One(K);
y:=[[0,0,1],[0,1,0],[0,0,1]]*One(K);


# Construct the algebra A generated by x and y
A := Algebra(K, [x, y]);


# Check if A is abelian
if (IsAbelian(A)=true) then Print("\nA is Abelian");
else Print("\nA is not Abelian"); fi;

# Check if every element in S satisfies X^2 = X
v:=true;
for i in A do
if not IsZero(i^2-i) then v:=false; break; fi;
od;

if v=false then
Print("\nThis algebra doesn't satisfy the given condition!\n");
else
Print("\nThis algebra satisfy the given condition!\n");
fi;

# Get a basis B for the algebra A
B := Basis(A);

# Compute the number of elements and dimension of A
s := Size(A);
t := Size(B);

# Print size and dimension
Print("\nSize of  A:", s, "\n");
```

```
Print("\nDimension of A as a vectorial space:", t, "\n");

#Creates the vector subspace over K generated by {x, y, xy}
Sub := VectorSpace(K, [x, y, x*y]);

#Computes the dimension of the subspace Sub
dim := Dimension(Sub);
```

D Verification Code for Theorem 3.4

```
# Define the finite field GF(2)
K:=GF(2);

# Left regular representation of the three generators of the algebra as two 7x7 matrices over GL(2)
x:=[[1,0,0,0,0,0,0],[0,0,0,1,0,0,0],[0,0,0,0,1,0,0],
[0,0,0,1,0,0,0],[0,0,0,0,1,0,0],[0,0,0,0,0,0,1],[0,0,0,0,0,0,1]
]*One(K);
y:=[[0,0,0,1,0,0,0],[0,1,0,0,0,0,0],[0,0,0,0,0,1,0],
[0,0,0,1,0,0,0],[0,0,0,0,0,0,1],[0,0,0,0,0,1,0],
[0,0,0,0,0,0,1]]*One(K);
z:=[[0,0,0,0,1,0,0],[0,0,0,0,0,1,0],[0,0,1,0,0,0,0],[0,0,0,0,0,0,1],
[0,0,0,0,1,0,0],[0,0,0,0,0,1,0],[0,0,0,0,0,0,1]]*One(K);


# Construct the algebra A generated by x, y and z
A := Algebra(K, [x, y, z]);


# Check if A is abelian
if (IsAbelian(A)=true) then Print("\nA is Abelian");
else Print("\nA is not Abelian"); fi;

# Check if every element in A satisfies X^2 = X
v:=true;
for i in A do
if not IsZero(i^2-i) then v:=false; break; fi;
od;

if v=false then
Print("\nThis algebra doesn't satisfy the given condition!\n");
else
Print("\nThis algebra satisfy the given condition!\n");
fi;

# Get a basis B for the algebra A
B := Basis(A);

# Compute the number of elements and dimension of A
s := Size(A);
t := Size(B);

# Print size and dimension
Print("\nSize of  A:", s, "\n");
Print("\nDimension of A as a vectorial space:", t, "\n");

#Creates the vector subspace over K generated by {x, y, z, xy, xz, yz, xyz}

Sub:=VectorSpace(K,[x,y,z,x*y,x*z,y*z,x*y*z]);
```

```
#Computes the dimension of the subspace Sub
dim := Dimension(Sub);
```

E Verification Code for Theorem 3.5

```
# Define the finite field GF(5)
K:=GF(5);

# Left regular representation of the two generators of the algebra as two 7x7 matrices over GL(5)
x:=[[0,1,0,0,0,0,0],[0,1,0,0,0,0,0],[0,0,0,1,0,0,0],[0,0,0,1,0,0,0],
[0,0,0,0,0,1,0],[0,0,0,0,0,1,0],[0,0,0,1,0,0,0]]*One(K);
y:=[[0,0,1,0,0,0,0],[0,0,0,0,1,0,0],[0,0,1,0,0,0,0],
[0,0,0,0,0,0,1],[0,0,0,0,1,0,0],[0,0,0,0,1,0,0],[0,0,0,0,0,0,1]]*One(K);

#Construct the semigroup S generated by x and y
S:=Semigroup(x,y);


# Construct the algebra A generated by x and y
A := Algebra(K, [x, y]);


# Check if every element in S satisfies X^2 = X

v:=true;
for i in S do
if not IsZero(i^2-i) then v:=false; break; fi;
od;

if v=false then
Print("\nThis algebra doesn’t satisfy the given condition!\n");
else
Print("\nThis algebra satisfy the given condition!\n");
fi;

# Get a basis B for the algebra A
B := Basis(A);

# Compute the number of elements and dimension of A
s := Size(A);
t := Size(B);

# Print size and dimension
Print("\nSize of  A:", s, "\n");
Print("\nDimension of A as a vectorial space:", t, "\n");

#Creates the vector subspace over K generated by { x, y, xy, yx, xyx,  yxy}
Sub:=VectorSpace(K,[x,y,x*y,y*x,x*y*x,y*x*y]);

#Computes the dimension of the subspace Sub
dim := Dimension(Sub);
```

F Verification Code for Theorem 4.2

```
# Define the finite field GF(5)
```

```
K:=GF(5);

# Left regular representation of the two generators of the algebra as two 4x4 matrices over GL(5)
x:=[[0,1,0,0],[-1,2,0,0],[0,0,0,1],[0,0,-1,2]]*One(K);
y:=[ [0,0,1,0],[-2,2,2,-1],[-1,0,2,0],[-2,1,2,0]]*One(K);

#Construct the semigroup S generated by x and y
S:=Semigroup(x,y);


# Construct the algebra A generated by x and y
A := Algebra(K, [x, y]);


# Check if every element in S satisfies X^2 = 2X - 1

v:=true;
for i in S do
if not IsZero(i^2-2*i+One(A)) then v:=false; break; fi;
od;


if v=false then
Print("\nThis algebra doesn't satisfy the given condition!\n");
else
Print("\nThis algebra satisfy the given condition!\n");
fi;

# Get a basis B for the algebra A
B := Basis(A);

# Compute the number of elements and dimension of A
s := Size(A);
t := Size(B);

# Print size and dimension
Print("\nSize of  A:", s, "\n");
Print("\nDimension of A as a vectorial space:", t, "\n");

#Creates the vector subspace over K generated by { 1, x, y, xy }
Sub:=VectorSpace(K,[One(A),x,y,x*y]);

#Computes the dimension of the subspace Sub
dim := Dimension(Sub);
```