**Comparison of two-parameter burr type (X) distributions and Gamma-Weibull distributions: Its Application of Heights of Students of Akwa Ibom State University, Nigeria**

**ABSTRACT**

This paper introduces a non-nested test statistic for comparing the gamma-Weibull and the two-parameter burr type (x) distributions using the likelihood ratio test statistic and the non-nested model test statistic. The test statistic obtained is applied to the heights of 617 students collected from the Medical Centre of Akwa Ibom State University. The parameters estimate of the two-parameter Burr Type (X) and gamma-Weibull distributions were obtained using the maximum likelihood method. Some exploratory analyses were carried out using the density plots of the gamma-Weibull and the two-parameter Burr type (x) distributions. The result was compared to the critical values obtained at various levels of significance. It was observed that the four-parameter gamma-Weibull distribution is not equivalent to the two-parameter burr type (x) for heights of students at level of significance. R codes are provided for implementation. The four-parameter Gamma-Weibull distribution is better for the heights of 617 students of Akwa Ibom State University than the two-parameter Burr Type (X) distribution.

**Keywords: Gamma-Weibull Distribution, Two-Parameter Burr Type (X) Distribution, Non-nested Model Test Statistic, Likelihood Ratio Test Statistic, Maximum Likelihood Estimation**

**1.0 INTRODUCTION**

Several extended Burr-type X distributions have been formed in the past decade. These distributions are widely used in modeling lifetime data as their hazard functions can fit various shapes, such as bathtub, decreasing, and increasing. The Burr-type X distribution, a well-known model in survival analysis, is one of twelve new distributions introduced by Burr in 1942 using a differential equation approach (Oh et al., 2024). According to Surles and Padgett, its initial form is a one-parameter distribution and has been extended to a two-parameter distribution by adding a scale parameter, known as two-parameter Burr-type X (BX) (Surles & Padgett, 2001). Eugene, et al (2002), introduced a generalized class of distributions based on the logit of the beta random variable with cumulative distribution function(cdf) and probability density function(pdf). Nadarajah and Kotz (2005) defined and studied the three-parameter beta-exponential distribution, obtained the mean, variance, skewness, kurtosis, mean deviation about the mean, mean deviation about the median, moment generating function, characteristic function, cumulant generating function, Renyi and Shannon entropies and other properties. As noted by (Nadarajah and Kotz,2005), the beta-exponential distribution is tractable and can be used as a model for failure time data.

Famoye et al (2005) introduced and studied the beta-Weibull distribution and used the likelihood ratio test statistic in comparing the Weibull to that of the beta-Weibull distribution using the likelihood ratio test statistic. The likelihood ratio test statistic for the Weibull and beta-Weibull distributions, as noted by Famoye et al (2005) is

1

The Weibull distribution is a special case of the beta-Weibull distribution when . The and are the likelihood functions of the Weibull and beta-Weibull distribution over the parameter space . . The two-parameter Burr Type (X) [Johnson et al. (1994, page 54)] is a special case of the beta-Weibull distribution when and . The pdf of the two-parameter Burr Type (X) is

2

Akinsete et al. (2008) introduced a four-parameter beta-Pareto distribution (BPD) and showed that the Pareto distribution is a special case of equation (BPD) by setting and presented many other special cases, obtained the hazard function, the mean deviation, the Renyi and Shannon entropies, the moments of the BPD and suggested the maximum likelihood method for estimating of the BPD parameters. Akinsete et al (2008) fitted the BPD to the data sets from two rivers, namely; exceedances of flood peaks, discussed in (Choulakian and Stephens, 2001) and the flood data illustrated by (Mudholkar and Huston,1996). The data was fitted to four distributions; the Pareto distribution, the three-parameter Weibull, the generalized Pareto distribution and the beta-Pareto distribution. It was observed that the BPD provided the best fit to the flood data, followed successively by the GPD and the three-Paremeter Weibull distribution. The BPD is unimodal.

Alzaatreh et al. (2013) introduced the Gamma-Weibull distribution with the generalized gamma and gamma distribution as special cases. The probability density function of the gamma-Weibull distribution is given by

3

A test statistic is a mathematical expression used by users of statistics in making decisions and drawing inferences about a population of interest. The test statistic when used with statistical table values and applied to datasets gives a useful insight into policy formulations and decision adjustments. Some of the mathematical expressions like that of the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), likelihood ratio and a couple of others, help in selecting the best model for a given phenomenon.

Michael et al. (2019), Michael et al. (2017), Michael et al. (2021), Michael and Iseh (2025), Michael et al. (2025) used the maxLik package in R introduced (Henningsen and Toomet, 2009) for maximum likelihood estimation of a model’s parameters of the Normal, Gamma, Log-normal and Logistic distributions.

This paper developed a test statistic for comparing the gamma-Weibull and the two-parameter Burr Type (X) distributions using the likelihood ratio test statistic approach and non-nested model test statistic techniques and employs the maxLik package for parameter estimation. The density plots of the functions are presented and the application made to the heights of 617 students of Akwa Ibom State University (AKSU) Main Campus, Ikot Akpaden, Nigeria.

**2 Methodology**

**2.1 Likelihood Function for Multiparameter Case.**

Hogg et al (2013) stated the likelihood function for the multiparameter case as follows:

Let , be independent and identically distributed with probability density function or probability mass function , where n is a fixed positive integer and . The likelihood function and its log are given by

4

5

is a vector of parameters and is the parameter space

**2.2 A Likelihood Ratio Test Statistic**

Wackerly et al (2008) defined the likelihood ratio test statistic as follows:

Define 6

A likelihood ratio test of versus employs as a test statistic and the rejection region is determined by and .

A value of close to zero indicates that the likelihood of the sample is much smaller under the null hypothesis than it is under the alternative hypothesis . Therefore, the data suggest favouring over . The actual value of is chosen so that alpha ( achieves the desired result.

denote the maximum (actually the supremum) of the likelihood function for all .

and can be composite because they both might contain multiple values of the parameter of interest or because other unknown parameters may be present

denote the vector of all k parameters. That is,

**2.3 A large-sample likelihood ratio test**

Let have joint likelihood function

Let denote the number of free parameters that are specified by

And let p denote the number of free parameters specified by the statement . Then, for large has approximately a distribution with df.

The theorem allows us to use the table of the to find the rejection regions with fixed when n is large. Notice that is a decreasing function of . Because the likelihood ratio test specified that we use RR: , this rejection may be rewritten as

RR: . For some large sample sizes, if we desire an -level test, theorem implies that . That is, a large sample likelihood ratio test has a rejection region given by

RR: where is based on df.

It is important to realize that large sample likelihood ratio tests are based on , where is the original likelihood function given by

7

**2.4 Test Statistic for Non-Nested Models**

Let and be any two probability distributions, with and the corresponding vectors of parameters and the subscripts and denoting the number of parameters. To compare the two probability distributions, let consider the hypothesis:

8

against

9

The likelihood-ratio statistic for testing (9), that is against is defined as

10

If and are nested, the statistic in (10) has a chi-square distribution with degree of freedom . If and are not nested, then the statistic in (10) is different from chi-square distribution.

To test the two non-nested probability distributions in null hypothesis in (8), Vuong (1986) proposed the test statistic

11

Where

12

According to Vuong (1986), for a non-nested model, is approximately standard normal distribution under in (8).

**2.5 Decision Rule**

Large-Sample Level Hypothesis Tests

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Rejection Region: 15

At significant level , is rejected in favor of if and is rejected in favor of if . If , we fail to reject .

**2.6 Data Collection**

The heights of students’ data were obtained from the Medical Centre of Akwa Ibom State University, Akwa Ibom State, Nigeria.

**3.0 Results**

**3.1 Estimation of Models Parameters**

**3.1.1 Likelihood Function of the Gamma-Weibull Distribution**

The pdf of the gamma-Weibull distribution is given by

16

The likelihood function is defined by

The log-likelihood of the gamma-Weibull distribution denoted is defined by

17

**4.5.2 Estimation of the gamma-Weibull Distribution Parameters**

The maximum likelihood estimators of the gamma-Weibull distribution parameters , and are obtained by differentiating the log-likelihood function in (17) with respect to the parameters , and and equating to zero, we have

18

19

20

21

The numerical solution of the nonlinear equations (18)-(21), is the maximum likelihood estimates of the parameters , and denoted by .

4.9.1 Re-parameterization of the Gamma-Weibull distribution

The pdf of gamma-Weibull distribution is given by

22

Let , and be the maximum likelihood estimates (mle) for the parameters , of the gamma-Weibull distribution obtained iteratively.

The probability density function(pdf) of the gamma-Weibull distribution with the mle of denoted by , and is

23

Then, the maximum likelihood function of the gamma-Weibull distribution is defined as

The log-likelihood function of the gamma-Weibull distribution denoted by is

24

4.8.1 Likelihood Function of **the Two Parameters Burr- Type (X) Distribution**

When and , the beta-Weibull distribution reduces to the two-parameter Burr-Type(X) distribution [Johnson et al, (1994, page 54)] with density function.

25

The likelihood function of the **Two Parameters Burr- Type (X) Distribution**

Taking the natural log and denoting it by , we have the log-likelihood function

26

4.8.2 Estimation of the two parameters Burr-Type(X) Distribution Parameters

The maximum likelihood estimators of the gamma-Weibull distribution parameters , are obtained by differentiating the log-likelihood function (26) with respect to the parameters and and equating to zero, we have

27

28

The numerical solution of the nonlinear equations (27) - (28), is the maximum likelihood estimates of the parameters and denoted by .

4.9.4 Re-parameterization of the two parameters Burr-Type (X) distribution

When and , the beta-Weibull distribution reduces to the two-parameter Burr-Type(X) distribution [Johnson et al,(1994,page 54)] with density function.

29

Let be the maximum likelihood estimates (mle) for the parameters of the beta-exponential distribution obtained iteratively.

The probability density function(pdf) of the beta-exponential distribution with the mle of denoted by , is

The likelihood function of the **Two Parameters Burr- Type (X) Distribution**

Taking the natural log and denoting it by , we have the log-likelihood function

30

4.10 The likelihood Ratio Statistic

4.10.1 The likelihood Ratio Statistic for the two parameters Burr-Type (X) distribution and gamma-Weibull distribution

Given two distributions, the two parameters Burr-Type (X) distribution and the gamma-Weibull distribution and a data set, it is interesting to test which of the distributions that a given data set is from and which best fit with the hypothesis stated as follows:

31

32

The two parameters Burr-Type (X) distribution and gamma-Weibull distribution

Let be two parameters Burr-Type (X) distribution parameters and let be the set of gamma-Weibull distribution parameters. The parameter space, ,not necessary a subset of the parameter space

The likelihood ratio statistic for testing against is given by

33

Where is the likelihood function for the two parameter Bur-Type (X) distribution and is the likelihood function for the gamma-Weibull distribution. The estimators (, , are the maximum likelihood estimates (mle) of of the gamma-Weibull distribution that maximized the distribution. The estimators are the maximum likelihood estimates(mle) of of the two-parameters Burr-Type (X) distribution that maximized the distribution.

and

The numerator and denominator of (33) can be re-written as

34

By taking the log of (34),

35

By multiplying equation (35) by (-2), we have,

36

Under very general conditions, the quantity has an approximate chi-square distribution with 2 degrees of freedom and the chi-square value is compared to the Right Hand Side of (36). This enables the hypothesis in (31) to be tested at a given level of significance for nested models.

* 1. **The Variance and Test Statistics**

**4.11.1 The Variance and Test Statistics for** the two parameters Burr-Type (X) distribution and gamma-Weibull distribution

The variance statistic is given by

37

is also the variance of the +limiting normal distribution of the likelihood ratio () statistic.

For the two-parameter Burr-Type(X) distribution against the gamma-Weibull Distribution,

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Test Statistic: 41

42

The test statistic, ,tend in distribution to a normal with mean 0 and variance 1.

Decision rule:

43

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**3.7 Parameters Estimates of The Two-parameter Burr Type (X) and Gamma-Weibull Distributions.**

The maximum likelihood estimates of the two-parameter Burr Type (X) and Gamma-Weibull distributions parameters and log-likelihood.

Table 1: Maximum Likelihood Estimation of the model parameters for the Heights of Students data and the measures AIC and BIC

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| MODEL |  |  |  |  |
| Gamma-Weibull | 2.138 | 0.000001367 | 5.857 | 9.968 |
| **Two-parameter Burr Type (X)** | 6248.4871 |  | 0.5478 |  |

Figure 1: Plots of Gamma-Weibull distribution and Heights of Students



Figure 2: Plots of **Two-parameter Burr Type (X)** distribution and Heights of Students



**3.10 The Variance Statistic Value for the gamma-Weibull and the Beta-Exponential Distributions**

Table 2: Variance Statistic of the Pairwise two-parameter Burr Type (X), Gamma-Weibull and the two-parameter Burr-Type (X) distributions

|  |  |  |
| --- | --- | --- |
| Models | Gamma-Weibull distribution | **Two-parameter Burr Type (X)** |
| Gamma-Weibull distribution |  | 3.03842 |
| **Two-parameter Burr Type (X)** | 3.03842 |  |

**3.11 The Test Statistic Value for Gamma-Weibull and Two-parameter Burr Type (X)**

Table 3: The test statistics values for the Pairwise two-parameter Burr Type (X), Gamma-Weibull and **Two-parameter Burr Type (X)** distributions

|  |  |  |
| --- | --- | --- |
| Models | Gamma-Weibull distribution | **Two-parameter Burr Type (X)** |
| Gamma-Weibull distribution |  | 6.229009 |
| **Two-parameter Burr Type (X)** | -6.229009 |  |

**3.12 Significance Levels and Critical Values**

Table 4: Critical values related to significance levels

|  |  |  |
| --- | --- | --- |
| **S/N** | **Significance Level** | **Critical Value** |
| **1** | **0.0001** | 3.916221 |
| **2** | **0.0011** | 3.279115 |
| **3** | **0.0021** | 3.088821 |
| **4** | **0.0031** | 2.969387 |
| **5** | **0.0041** | 2.881164 |
| **6** | **0.0051** | 2.810734 |
| **7** | **0.0061** | 2.751871 |
| **8** | **0.0071** | 2.701156 |
| **9** | **0.0081** | 2.656508 |
| **10** | **0.0091** | 2.616559 |
| **11** | **0.0101** | 2.580362 |
| **12** | **0.0111** | 2.547234 |
| **13** | **0.0121** | 2.516665 |
| **14** | **0.0131** | 2.488264 |
| **15** | **0.0141** | 2.461724 |

**4.0 Discussion and Conclusion**

A mathematical expression called a test statistic for comparing two non-nested models namely; **Two-parameter Burr Type (X) distribution** (TPBTX) and gamma-Weibull distribution (GW) is introduced. The maximum likelihood estimation of the parameters of the gamma-Weibull distribution and that of **Two-parameter Burr Type (X)** distributions are presented but not in closed forms. The test statistic is applied to the heights of 617 students of Akwa Ibom State University (AKSU). The test statistic introduced follows a standard normal distribution which is a pivotal quantity. R codes is provided for implementation.

For the four-parameter gamma-Weibull distribution, the values of the parameters are respectively;

For the two-parameter Burr Type (X) distribution, the values of the parameters are respectively;.

The test statistic value for comparing **Two-parameter Burr Type (X)** distribution and gamma-Weibull distribution is -6.229009 which is lower than the critical values -3.916221 at level of significance. The two-parameter Burr Type (X) distribution (TPBTX) and gamma-Weibull distribution (GWD) are not equivalent in fitting the heights of students of the Akwa Ibom State University. The test statistics negative value indicates that a four-parameter Gamma-Weibull distribution is a better fit than the two-parameter -parameter Burr Type (X) distribution for the height of Akwa Ibom State University Students

The graphs of the Two-parameter Burr Type (X) and gamma-Weibull distributions and the heights of students are presented. The graphs resemble a bell-shaped curve.

Appendix

R codes

*library(maxLik)*

*#####gamma-Weibull distribution ###X is the name of the data set*

*logLikgammaWeibull<-function(gammaWeibull){*

*alphagammaWeibull<-gammaWeibull[1]*

*betagammaWeibull<-gammaWeibull[2]*

*gammagammaWeibull<-gammaWeibull[3]*

*cgammaWeibull<-gammaWeibull[4]*

*agw<-log(cgammaWeibull)*

*bgw<-(alphagammaWeibull\*cgammaWeibull\*log(alphagammaWeibull))*

*cgw<-log(gamma(alphagammaWeibull))*

*dgw<-(alphagammaWeibull\*log(betagammaWeibull))*

*egw<-((alphagammaWeibull\*cgammaWeibull-1)\*(log(X)))*

*fgw<-((X/gammagammaWeibull)^cgammaWeibull)/(betagammaWeibull)*

*sum(agw-bgw-cgw-dgw+egw-fgw)*

*}*

*mlegammaWeibull<-maxLik(logLik=logLikgammaWeibull,*

*start=c(alphagammaWeibull=1,betagammaWeibull=0.2,gammagammaWeibull=2,*

*cgammaWeibull=1.4))*

*summary(mlegammaWeibull)*

*###Plot of gamma Weibull pdf*

*coef(mlegammaWeibull)*

*a=coef(mlegammaWeibull)[1]*

*b=coef(mlegammaWeibull)[2]*

*g=coef(mlegammaWeibull)[3]*

*c=coef(mlegammaWeibull)[4]*

*gwf<-(c\*((X/g)^((a\*c)-1))\*(exp(-(1/b)\*(X/g)^c)))/(g\*gamma(a)\*b^a)*

*plot(X,gwf, xlab="Heights of Students",ylab="gamma-Weibull Density",main="gamma-Weibull Density Plot")*

*####the two-parameters Burr-Type (X) distribution*

*logLiktwoparameterBurrtypeX<-function(twoparameterBurrtypeX){*

*alphatwoparameterBurrtypeX<-twoparameterBurrtypeX[1]*

*gammatwoparameterBurrtypeX<-twoparameterBurrtypeX[2]*

*sum(log(2\*alphatwoparameterBurrtypeX)-2\*log(gammatwoparameterBurrtypeX)+*

*log(X)-((X/gammatwoparameterBurrtypeX)^2)+*

*(alphatwoparameterBurrtypeX-1)\*log(1-exp(-1\*((X/gammatwoparameterBurrtypeX)^2))))*

*}*

*mletwoparameterBurrtypeX<-maxLik(logLik=logLiktwoparameterBurrtypeX,start=c(alphatwoparameterBurrtypeX=2,gammatwoparameterBurrtypeX=1))*

*summary(mletwoparameterBurrtypeX)*

*coef(mletwoparameterBurrtypeX)*

*#####answers*

*Log-Likelihood(mletwoparameterBurrtypeX)*

*at<-coef(mletwoparameterBurrtypeX)[1]*

*bt<-coef(mletwoparameterBurrtypeX)[2]*

*at;bt*

*####The two-parameter Density plot*

*ttpbtx<-2\*at\*(((1-exp(-(X/bt)^2))^(at-1)))\*(exp(-(X/bt)^2))/(bt^2)*

*ttpbtx*

*plot(X,ttpbtx,col="BLUE",xlab="Heights of Students",ylab="The Two-Parameter Burr Type X Density",main="The Two-Parameter Burr Type X Distribution")*

*plot(X,ttpbtx,xlab="Heights of Students",ylab="The Two-Parameter Burr Type X Density",main="The Two-Parameter Burr Type X Distribution")*

*##H t and g variance statistics for tpbtx and gamma-Weibull distribution*

*Atg=log(ttpbtx/gwf)*

*Btg=Atg^2*

*Ctg=sum(Btg)*

*Ctg*

*ntg=length(Atg)*

*Dtg=(Ctg)/ntg##first term result*

*##second term*

*Etg=Atg*

*Ftg=sum(Etg)*

*Gtg=(Ftg)/(ntg)*

*Htg=Gtg^2*

*vartg=Dtg-Htg# variance stats for ttp and gw*

*vartg*

*wtg=sqrt(vartg)##*

*nroottg=1/sqrt(ntg)*

*nroottg*

*NNRtg=Ftg\*nroottg/wtg*

*NNRtg##Non nested result*

*Critical Values*

*alpha<-seq(0.0001,0.1,0.001)*

*qt(1-alpha/2,617)*

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**References**

Akinsete, Alfred, Famoye, Felix & Lee, Carl (2008): The beta-Pareto distribution',Statistics,42:6,547 — 563

Alzaatreh, A, Lee, C, Famoye, F (2013): A new method for generating families of continuous distributions. *Metron*. **71**(1), 63–79

Choulakian, V & M.A. Stephens (2001). Goodness-of-fit for the generalized Pareto distribution, Technometrics 43(4), pp. 478–484

Eugene, N., Lee, C., Famoye, F.(2002): The beta-normal distribution and its applications. *Communication in Statistics-Theory and Methods* **31**(4), 497–512

Famoye, F., Lee, C. and Olumolade, O. (2005): The beta-Weibull distribution, *J Stat Theory and Applications, 4,121-136.*

Henningsen, A. & Toomet, O. (2009). MaxLik: Tools for Maximum Likelihood Estimation. R package version 0.5, Retrieved September 10, 2017 from [*http://CRAN.R-project.org.*](http://cran.r-project.org/)

Hogg, R.V., McKean, J.W. and Craig, A.T. (2013): *Introduction to Mathematical Statistics*, 7th Ed., USA: Pearson Education Inc, Boston.

Johnson,N., Kotz, S. & Balakrishnan, N. 1994. *Continuous Univariate Distributions*, 2nd Ed., Vol.1, Wiley, New York.

Michael, I. T., Isaac, A. A & Etim, A. C. (2025). Fitting a Normal Distribution to the Heights of Akwa Ibom State University Students Using Chi-Square. *Asian Journal of Probability and Statistics* 27(5): 42 – 49.

Michael, I. T., Ikpang, I. N. & Isaac, A. A (2017). Goodness of fit test: a chi-squared approach to fitting of a normal distribution to the weights of students of Akwa Ibom State University, Nigeria. *Asian Journal of Natural and Applied Sciences* 6(4) 107 – 113.

Michael, I. T., and Iseh, M. J. (2025). Utilizing the CHI-SQUARE Technique in Fitting a Logistic Distribution to the Heights of Students of Akwa Ibom State University. SCIREA Journal of Mathematics, VOLUME 10 ISSUE 3.

Michael, I. T., Iseh, M. J., Ikpang, I. N., George, E. U., and Bassey, D. I. (2021). A GOODNESS OF FIT TEST: USING A CHI-SQUARED APPROACH TO FIT THE LOGNORMAL PROBABILITY MODEL TO THE WEIGHTS OF STUDENTS OF AKWA IBOM STATE UNIVERSITY. International Journal of Applied Science and Research, VOLUME 4 ISSUE 6.

Michael, I. T., Usoro,A. E., Ikpang, I. N., George, E. U. & Bassey, D. I.(2019). Fitting a Gamma Distribution using a chi-squared approach to the Heights of students of Akwa Ibom State University, Nigeria. *International Journal of Advanced Statistics and Probability* 7(2) (2019) 42 – 46.

Mudholkar, G.S. & Huston, A.D.: The exponentiated Weibull family: Some properties and a flood data application, Commun. Statist. Theory Meth. 23(4) (1996), pp. 1149–1171

Nadarajah, S and Kotz, S (2005): The beta exponential distribution, *Reliability Eng. Syst. Safety* 91, pp. 689–697.

Vuong, Q. H. (1986). Likelihood ratio tests for model selection and non-nested hypothesis. *Econometrica 57*, pp. 307-333

Wackerly, D. D., Mendenhall, W. and Scheaffer, R. L. (2008). *Mathematical Statistics with Applications*, 7th Ed. U.S.A.: Thomson Brooks/Cole.

Oh, Y. L., Lim, F. P., Chen, C. Y., Ling, W. S., Loh, Y. F., & Yap, H. K. (2024). Exponentiated gamma Burr-type X distribution: model, theory, and applications. Journal of Inequalities and Applications, 2024(1), 142.

Surles, J. G., & Padgett, W. J. (2001). Inference for reliability and stress-strength for a scaled Burr type X distribution. Lifetime data analysis, 7, 187-200.