# Gaussian Numbers with Generalized Pandita Numbers Components

**Abstract:** In this study, we introduce and investigate a new class of numerical sequences in the complex domain—Gaussian generalized Pandita numbers—which extend the classical theory of linear recurrence relations. In particular, we focus on two distinct cases: the Gaussian Pandita numbers and the Gaussian Pandita-Lucas numbers. For these sequences, we derive and present a comprehensive set of mathematical results, including recurrence relations, closed-form expressions via Binet-type formulas, ordinary and exponential generating functions. In addition, we establish various algebraic identities, provide matrix representations, and prove generalized forms of Simpson's formula. Summation identities are also developed to further explore the structural and analytical properties of these numbers. The findings contribute to the broader theory of Gaussian integer sequences and open new directions for applications in discrete mathematics and computational number theory.

Keywords: Pandita numbers, Pandita-Lucas numbers,Gaussian Pandita numbers, Gaussian Pandita-Lucas numbers, Binet's formulas, generating functions, exponential generating functions.

## 1. Introduction

In this section, we give some preliminary result on Pandita numbers.

The generalized Pandita sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations as

$$W_n = 2W_{n-1} - W_{n-2} + W_{n-3} - W_{n-4}. \tag{1.1}$$

with the initial values $W_0, W_1, W_2, W_3$ are not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + 2W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}.$$

for $n=1,2,3,....$ Therefore, recurrence (1.1) holds for all integer $n$. Soykan has conducted a study on this particular sequence, for more details, see [47]

Characteristic equation of $\{W_n\}$ is

$$x^4-2x^3+x^2-x+1=(x^3-x^2-1)(x-1)=0.$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{1}{3}+\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3},\\
\beta &= \frac{1}{3}+\omega\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\omega^2\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3},\\
\gamma &= \frac{1}{3}+\omega^2\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\omega\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3},\\
\delta &= 1.
\end{aligned}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= 2,\\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= 1,\\
\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= 1,\\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Note also that

$$\begin{aligned}
\alpha+\beta+\gamma &= 1,\\
\alpha\beta+\alpha\gamma+\beta\gamma &= 0,\\
\alpha\beta\gamma &= 1.
\end{aligned}$$

For $n=1,2,3....$ Hence, recurrence (1.1) is true for all integer $n$.

For the fourth-order recurrence relations has been studied by many authors, for more detail see [54, 55, 49, 50, 53, 52, 59, 48, 47, 56].

We now present Binet's formula for the generalized Pandita numbers.

THEOREM 1.1. *[47]Binet formula of generalized Pandita numbers can be presented as follows:*

$$\begin{aligned} W_n &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)\alpha^n}{3\alpha-2} \\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)\beta^n}{3\beta-2} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)\gamma^n}{3\gamma-2} \\ &-W_3+W_2+W_0. \end{aligned}$$

Now we define two special cases of the sequence $\{W_n\}$ as follows: The Pandita sequence $\{P_n\}_{n\geq 0}$ and the Pandita-Lucas sequence $\{S_n\}_{n\geq 0}$ are respectively defined by the fourth-order recurrence relations as:

$$P_n = 2P_{n-1}-P_{n-2}+P_{n-3}-P_{n-4}, \qquad P_0=0, P_1=1, P_2=2, P_3=3, \tag{1.2}$$

$$S_n = 2S_{n-1}-S_{n-2}+S_{n-3}-S_{n-4}, \qquad S_0=4, S_1=2, S_2=2, S_3=5. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$, $\{S_n\}_{n\geq 0}$, can be extended to negative subscripts by defining,

$$\begin{aligned} P_{-n} &= P_{-(n-1)}-P_{-(n-2)}+2P_{-(n-3)}-P_{-(n-4)}, \\ S_{-n} &= S_{-(n-1)}-S_{-(n-2)}+2S_{-(n-3)}-S_{-(n-4)}. \end{aligned}$$

for $n=1,2,3,...$ respectively. As a result, recurrences (1.2)-(1.3) hold for all integer $n$. Binet's formulas of $P_n$ and $S_n$ are given as follows.

COROLLARY 1.2. *For all integers $n$, Binet's formula of Pandita and Pandita-Lucas numbers are*

$$P_n = \frac{\alpha^{n+3}}{3\alpha-2}+\frac{\beta^{n+3}}{3\beta-2}+\frac{\gamma^{n+3}}{3\gamma-2}-1,$$

and

$$S_n = \alpha^n+\beta^n+\gamma^n+1.$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $W_n$.

LEMMA 1.3. *Suppose that $f_{W_n}(z)=\sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Pandita sequence $\{W_n\}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-2W_0)x+(W_2-2W_1+W_0)x^2+(W_3-2W_2+W_1-W_0)x^3}{1-2x+x^2-x^3+x^4}.$$

Proof. Take $r=2, s=-1, t=1, u=-1$ in Lemma 47. □

Next, we give some information about Gaussian sequences from literature.

We provide some Gaussian numbers that satisfy second-order and third-order recurrence relations.

- Horadam [20] introduced Gaussian Fibonacci numbers and defined by

$$GF_n = F_n + iF_{n-1}$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact, he defined these numbers as $GF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.).

- Pethe and Horadam [31] introduced Gaussian generalized Fibonacci numbers by

$$GF_n = F_n + iF_{n-1}$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Halıcı and Öz [19] studied Gaussian Pell and Pell Lucas numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1} \\ GQ_n &= Q_n + iQ_{n-1} \end{aligned}$$

where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

- Aşcı and Gürel [1] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

$$\begin{aligned} GJ_n &= J_n + iJ_{n-1} \\ Gj_n &= j_n + ij_{n-1} \end{aligned}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

- Taşçı [64] introduced and studied Gaussian Mersenne numbers defined by

$$GM_n = M_n + iM_{n-1}$$

where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

- Taşçı [66] introduced and studied Gaussian balancing and Gaussian Lucas Balancing numbers given by, respectively,

$$\begin{aligned} GB_n &= B_n + iB_{n-1} \\ GC_n &= C_n + iC_{n-1} \end{aligned}$$

where $B_n = 6B_{n-1} - BJ_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6Cj_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

- Ertaş and Yılmaz [17] studied Gaussian Oresme numbers and defined them as

$$GS_n = S_n + iS_{n-1}$$

where oresme numbers are given by $S_n = S_{n-1} - \frac{1}{4}S_{n-2}$, $S_0 = 0$, $S_1 = \frac{1}{2}$.

Now, we present some Gaussian numbers with third order recurrence relations.

- Soykan at al [57] presented Gaussian generalized Tribonacci numbers given by

$$GW_n = W_n + iW_{n-1}$$

where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition $W_0$, $W_1$, $W_2$.

- Taşcı [65] studied Gaussian Padovan and Gaussian Pell- Padovan numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1} \\ GR_n &= R_n + iR_{n-1} \end{aligned}$$

where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

- Cerda-Morales [7] defined Gaussian third-order Jacobsthal numbers as

$$GJ_n = J_n + iJ_{n-1}$$

where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_2 = 1$.

- Yılmaz and Soykan [73] presented Gaussian Guglielmo and Guglielmo-Lucas numbers by written respectively,

$$\begin{aligned} GT_n &= T_n + iT_{n-1} \\ GH_n &= H_n + iH_{n-1} \end{aligned}$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$, and $H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, H_0 = 3, H_1 = 3, H_2 = 3$.

- Dikmen [13] presented Gaussian Leonardo and Leonardo-Lucas numbers by written respectively,

$$\begin{aligned} Gl_n &= l_n + il_{n-1} \\ GH_n &= H_n + iH_{n-1} \end{aligned}$$

where $l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3$, and $H_n = 2H_{n-1} - H_{n-3}, H_0 = 3, H_1 = 2, H_2 = 4$.

- Ayrılma and Soykan [2] presented Gaussian Edouard and Edouard-Lucas numbers by written respectively,

$$\begin{aligned} GE_n &= E_n + iE_{n-1} \\ GK_n &= K_n + iK_{n-1} \end{aligned}$$

where $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, E_0 = 0, E_1 = 1, E_2 = 7$, and $K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, K_0 = 3, K_1 = 7, K_2 = 35$.

- Soykan at al [58] presented Gaussian Bigollo and Bigollo-Lucas numbers by written respectively,

$$
\begin{aligned}
GB_n &= B_n + iB_{n-1} \\
GC_n &= C_n + iC_{n-1}
\end{aligned}
$$

where $B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, B_0 = 0, B_1 = 1, B_2 = 4$, and $C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, C_0 = 3, C_1 = 4, C_2 = 6$.

- Eren and Soykan [16] describe Gaussian Woodall and Woodall-Lucas numbers by written respectively,

$$
\begin{aligned}
GR_n &= R_n + iR_{n-1} \\
GC_n &= C_n + iC_{n-1}
\end{aligned}
$$

where $R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}$, $R_0 = -1, R_1 = 1, R_2 = 7,$, and $C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}$, $C_0 = 1, C_1 = 3, C_2 = 9$.

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence $W_n$.

Lemma 1.4. *Suppose that $f_{GW_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Pandita sequence $\{W_n\}$.*

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$
\begin{aligned}
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+1)W_1 - W_0)}{3\alpha - 2} e^{\alpha x} \\
&+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+1)W_1 - W_0)}{3\beta - 2} e^{\beta x} \\
&+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+1)W_1 - W_0)}{3\gamma - 2} e^{\gamma x} \\
&+ (-W_3 + W_2 + W_0)e^x.
\end{aligned}
$$

Proof: Using the Binet's formula of generating Pandita numbers we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \Big( \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+1)W_1 - W_0)\alpha^n}{3\alpha - 2} \\
&+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+1)W_1 - W_0)\beta^n}{3\beta - 2} \\
&+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+1)W_1 - W_0)\gamma^n}{3\gamma - 2} - W_3 + W_2 + W_0 \Big) \frac{x^n}{n!}
\end{aligned}
$$

$$\begin{aligned} &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{3\alpha-2}\sum_{n=0}^{\infty}\alpha^n\frac{x^n}{n!} \\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{3\beta-2}\sum_{n=0}^{\infty}\beta^n\frac{x^n}{n!} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{3\gamma-2}\sum_{n=0}^{\infty}\gamma^n\frac{x^n}{n!} \\ &+(-W_3+W_2+W_0)\sum_{n=0}^{\infty}\frac{x^n}{n!} \end{aligned}$$

$$\begin{aligned} &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{3\alpha-2}e^{\alpha x} \\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{3\beta-2}e^{\beta x} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{3\gamma-2}e^{\gamma x}+(-W_3+W_2+W_0)e^x. \ \square \end{aligned}$$

The previous Lemma 1.4 gives the following results as particular examples.

COROLLARY 1.5. *Exponential generating function of Pandita and Pandita-Lucas numbers*

**a):** $\sum\limits_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{\alpha^{n+3}}{3\alpha-2}+\frac{\beta^{n+3}}{3\beta-2}+\frac{\gamma^{n+3}}{3\gamma-2}-1)\frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}}{3\alpha-2}+\frac{\beta^3 e^{\beta x}}{3\beta-2}+\frac{\gamma^3 e^{\gamma x}}{3\gamma-2}-e^x.$

**b):** $\sum\limits_{n=0}^{\infty} S_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n+\beta^n+\gamma^n+1)\frac{x^n}{n!} = e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^x.$

## 2. Gaussian Generalized Pandita Numbers

This section introduces the Gaussian generalized Pandita numbers and explores key properties, including Binet's formula and their generating function.

Gaussian generalized Pandita numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2, GW_3)\}_{n\geq 0}$ are defined by

$$GW_n = 2GW_{n-1} - GW_{n-2} + GW_{n-3} - GW_{n-4}. \tag{2.1}$$

with the initial conditions

$$\begin{aligned} GW_0 &= W_0 + i(W_0 - W_1 + 2W_2 - W_3), \\ GW_1 &= W_1 + iW_0, \\ GW_2 &= W_2 + iW_1, \\ GW_3 &= W_3 + iW_2. \end{aligned}$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = GW_{-(n-1)} - GW_{-(n-2)} + 2GW_{-(n-3)} - GW_{-(n-4)}. \tag{2.2}$$

for $n = 1, 2, 3, ....$ Thus, recurrence (2.1) hold for all integer $n$. Note that for all integers $n$, we get

$$GW_n = W_n + iW_{n-1}, \tag{2.3}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}. \tag{2.4}$$

The first few generalized Gaussian Pandita numbers with positive subscript and negative subscript are presented in the following table.

Table 1. The first few generalized Gaussian Pandita numbers with positive subscript

| $n$ | $GW_n$ |
|---|---|
| 0 | $W_0 + i(W_0 - W_1 + 2W_2 - W_3)$ |
| 1 | $W_1 + iW_0$ |
| 2 | $W_2 + iW_1$ |
| 3 | $W_3 + iW_2$ |
| 4 | $W_1 - W_0 - W_2 + 2W_3 + iW_3$ |
| 5 | $W_1 - 2W_0 - W_2 + 3W_3 + i(W_1 - W_0 - W_2 + 2W_3)$ |

and with a negative subscript shown in Table 2

| $n$ | $GW_{-n}$ |
|---|---|
| 0 | $W_0 + i(W_0 - W_1 + 2W_2 - W_3)$ |
| 1 | $W_0 - W_1 + 2W_2 - W_3 + i(W_1 + W_2 - W_3)$ |
| 2 | $W_1 + W_2 - W_3 + i(W_0 + W_1 - W_2)$ |
| 3 | $W_0 + W_1 - W_2 + i(2W_0 - 2W_1 + 2W_2 - W_3)$ |
| 4 | $2W_0 - 2W_1 + 2W_2 - W_3 + i(3W_2 - 2W_3)$ |
| 5 | $3W_2 - 2W_3 + i(3W_1 - 2W_2)$ |

We can define two special cases of $GW_n$ : $GW_n(0, 1, 2+i, 3+2i) = GP_n$ is the sequence of Gaussian Pandita numbers , $GW_n(4+i, 2+4i, 2+2i, 5+2i) = GS_n$ is the sequence of Gaussian Pandita-Lucas numbers.

So Gaussian Pandita numbers are defined by

$$GP_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}. \tag{2.5}$$

with the initial conditions

$$GP_0 = 0, GP_1 = 1, GP_2 = 2+i, GP_3 = 3+2i.$$

Gaussian Pandita-Lucas numbers are defined by

$$GS_n = 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}. \tag{2.6}$$

with the initial conditions

$$GS_0 = 4+i, GS_1 = 2+4i, GS_2 = 2+2i, GS_3 = 5+2i.$$

That for all integer we have

$$GP_n = P_n + iP_{n-1},$$

$$GS_n = S_n + iS_{n-1}.$$

The initial values of the Gaussian Pandita and Gaussian Pandita–Lucas numbers, for both positive and negative subscripts, are presented in Table 3.

Table 3. Gaussian Pandita numbers, Gaussian Pandita-Lucas numbers, with positive and negative subscripts, special cases of generalized Pandita numbers.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $GP_n$ | $0$ | $1$ | $2+i$ | $3+2i$ | $5+3i$ | $8+5i$ | $12+8i$ |
| $GP_{-n}$ | $0$ | $0$ | $-i$ | $-1-i$ | $-1$ | $-i$ | $-1-2i$ |
| $GS_n$ | $4+i$ | $2+4i$ | $2+2i$ | $5+2i$ | $6+5i$ | $7+6i$ | $11+7i$ |
| $GS_{-n}$ | $4+i$ | $1-i$ | $-1+4i$ | $4+3i$ | $3-4i$ | $-4+2i$ | $2+8i$ |

Next, we present the Binet's formula for the Gaussian generalized Pandita numbers.

THEOREM 2.1. *The Binet's formula for the Gaussian generalized Pandita numbers is*

$$\begin{aligned} GW_n = {} & \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)\alpha^n}{3\alpha-2} \\ & +\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)\beta^n}{3\beta-2} \\ & +\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)\gamma^n}{3\gamma-2}-W_3+W_2+W_0 \\ & +i(\frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)\alpha^{n-1}}{3\alpha-2} \\ & +\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)\beta^{n-1}}{3\beta-2} \\ & +\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)\gamma^{n-1}}{3\gamma-2}-W_3+W_2+W_0). \end{aligned}$$

Proof. The proof follows from (1.1) and (2.3). □

The following results are immediate consequences of the preceding Theorem.

COROLLARY 2.2. *For all integers $n$, we have following identities:*

**(a):** $GP_n=\frac{\alpha^{n+3}}{3\alpha-2}+\frac{\beta^{n+3}}{3\beta-2}+\frac{\gamma^{n+3}}{3\gamma-2}-1+i(\frac{\alpha^{n+2}}{3\alpha-2}+\frac{\beta^{n+2}}{3\beta-2}+\frac{\gamma^{n+2}}{3\gamma-2}-1).$

**(b):** $GS_n=\alpha^n+\beta^n+\gamma^n+1+i(\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}+1).$

The next Theorem presents the generating function of Gaussian generalized Pandita numbers.

THEOREM 2.3. *Let $f_{GW_n}(x)=\sum\limits_{n=0}^{\infty}GW_nx^n$ donate the generating function of Gaussian generalized Pandita numbers is given as follows:*

$f_{GW_n}(z)=\sum\limits_{n=0}^{\infty}GW_nx^n=\frac{1}{1-2x+x^2-x^3+x^4}GW_0+(GW_1-2GW_0)x+(GW_2-2GW_1+GW_0)x^2+(GW_3-2GW_2+GW_1-GW_0)x^3.$

Proof. Using the definition of Gaussian Pandita numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain $(1-2x+x^2-x^3+x^4)f_{GW_n}(x)$

$$\begin{aligned}
& (1-2x+x^2-x^3+x^4)f_{GW_n}(x) \\
= & \sum_{n=0}^{\infty} GW_n x^n - 2x\sum_{n=0}^{\infty} GW_n x^n + x^2\sum_{n=0}^{\infty} GW_n x^n - x^3\sum_{n=0}^{\infty} GW_n x^n + x^4\sum_{n=0}^{\infty} GW_n x^n, \\
= & \sum_{n=0}^{\infty} GW_n x^n - 2\sum_{n=0}^{\infty} GW_n x^{n+1} + \sum_{n=0}^{\infty} GW_n x^{n+2} - \sum_{n=0}^{\infty} GW_n x^{n+3} + \sum_{n=0}^{\infty} GW_n x^{n+4}, \\
= & \sum_{n=0}^{\infty} GW_n x^n - 2\sum_{n=1}^{\infty} GW_{(n-1)} x^n + \sum_{n=2}^{\infty} GW_{(n-2)} x^n - \sum_{n=3}^{\infty} GW_{(n-3)} x^n + \sum_{n=4}^{\infty} GW_{(n-4)} x^n, \\
= & (GW_0 + GW_1 x + GW_2 x^2 + GW_3 x^3) - 2(GW_0 x + GW_1 x^2 + GW_2 x^3) + (GW_0 x^2 + GW_1 x^3) - GW_0 x^3 \\
& + \sum_{n=4}^{\infty} (GW_n - 2GW_{n-1} - GW_{n-2} - GW_{n-3} + GW_{n-4}) x^n, \\
= & GW_0 + (GW_1 - 2GW_0)x + (GW_2 - 2GW_1 + GW_0)x^2 + (GW_3 - 2GW_2 + GW_1 - GW_0)x^3.
\end{aligned}$$

And rearranging above equation, we get (2.3). □

The following results are immediate consequences of the preceding Theorem.

Corollary 2.4. *For all integers $n$, we have following identities:*

**(a):** $f_{GP_n}(z) = \sum\limits_{n=0}^{\infty} GP_n x^n = \dfrac{ix^2+x}{x^4-x^3+x^2-2x+1}$.

**(b):** $f_{GS_n}(z) = \sum\limits_{n=0}^{\infty} GS_n x^n = -\dfrac{(1-i)\,x^3-(2-5i)\,x^2+(6-2i)\,x-4-i}{x^4-x^3+x^2-2x+1}$.

Theorem (2.3) gives the following results as special cases,

$(1-2x+x^2-x^3+x^4)f_{GP_n}(x) = GP_0 + (GP_1 - 2GP_0)x + (GP_2 - 2GP_1 + GP_0)x^2 + (GP_3 - 2GP_2 + GP_1 - GP_0)x^3 = ix^2 + x$,

$(1-2x+x^2-x^3+x^4)f_{GS_n}(x) = GS_0 + (GS_1 - 2GS_0)x + (GS_2 - 2GS_1 + GS_0)x^2 + (GS_3 - 2GS_2 + GS_1 - GS_0)x^3 = -(1-i)\,x^3 + (2-5i)\,x^2 - (6-2i)\,x + 4 + i$

Next, we give the exponential Gaussian generating function of $\sum\limits_{n=0}^{\infty} GW_n x^n$ of the sequence $GW_n$.

Lemma 2.5. *Suppose that $f_{GW_n}(x) = \sum\limits_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ is the exponential Gaussian generating function of the generalized Pandita sequence $\{GW_n\}$.*

Then $\sum\limits_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!} &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{3\alpha-2}e^{\alpha x} \\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{3\beta-2}e^{\beta x} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{3\gamma-2}e^{\gamma x}+(-W_3+W_2+W_0)e^{x} \\ &+i(\frac{1}{\alpha}\frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{3\alpha-2}e^{\alpha x} \\ &+\frac{1}{\beta}\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{3\beta-2}e^{\beta x} \\ &+\frac{1}{\gamma}\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{3\gamma-2}e^{\gamma x}+(-W_3+W_2+W_0)e^{x}).\end{aligned}$$

Proof. The proof follows from the Binet's formula of $GW_n$ and $GW_n = GW_n + iGW_{n-1}$ (Lemma1.4). □

The previous Lemma 2.5 gives the following results as particular examples.

Corollary 2.6. *Exponential Gaussian generating function of Pandita and Pandita-Lucas numbers*

**a):** $\sum\limits_{n=0}^{\infty} GP_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty}(\frac{\alpha^{n+3}}{3\alpha-2}+\frac{\beta^{n+3}}{3\beta-2}+\frac{\gamma^{n+3}}{3\gamma-2}-1+i(\frac{\alpha^{n+2}}{3\alpha-2}+\frac{\beta^{n+2}}{3\beta-2}+\frac{\gamma^{n+2}}{3\gamma-2}-1))\frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}}{3\alpha-2}+\frac{\beta^3 e^{\beta x}}{3\beta-2}+\frac{\gamma^3 e^{\gamma x}}{3\gamma-2}-e^x+i(\frac{\alpha^2 e^{\alpha x}}{3\alpha-2}+\frac{\beta^2 e^{\beta x}}{3\beta-2}+\frac{\gamma^2 e^{\gamma x}}{3\gamma-2}-e^x)).$

**b):** $\sum\limits_{n=0}^{\infty} GS_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty}(\alpha^n+\beta^n+\gamma^n+1+i(\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}+1))\frac{x^n}{n!} = e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^x+i(\frac{1}{\alpha}e^{\alpha x}+\frac{1}{\beta}e^{\beta x}+\frac{1}{\gamma}e^{\gamma x}+e^x).$

## 3. Obtaining Binet Formula From Generating Function

We next find Binet's formula generalized Gaussian Pandita number $\{GW_n\}$ by the use of generating function for $GW_n$.

Theorem 3.1. *Binet's formula of generalized Gaussian Pandita numbers)*

$$GW_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \tag{3.1}$$

where

$$\begin{aligned} q_1 &= W_0\alpha^3+(W_1-2W_0)\,\alpha^2+(W_0-2W_1+W_2)\,\alpha-W_0+W_1-2W_2+W_3, \\ q_2 &= W_0\beta^3+(W_1-2W_0)\,\beta^2+(W_0-2W_1+W_2)\,\beta-W_0+W_1-2W_2+W_3, \\ q_3 &= W_0\gamma^3+(W_1-2W_0)\,\gamma^2+(W_0-2W_1+W_2)\,\gamma-W_0+W_1-2W_2+W_3, \\ q_4 &= W_0\delta^3+(W_1-2W_0)\,\delta^2+(W_0-2W_1+W_2)\,\delta-W_0+W_1-2W_2+W_3. \end{aligned}$$

*Proof.* Let

$$h(x)=x^{4}-x^{3}+x^{2}-2x+1.$$

Then for some $\alpha,\beta,\gamma$ and $\delta$ we write

$$h(x)=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).$$

i.e.,

$$x^{4}-x^{3}+x^{2}-2x+1=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \tag{3.2}$$

Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives $\alpha,\beta,\gamma$ and $\delta$ as the roots of

$$h(\frac{1}{x})=\frac{1}{x^{2}}-\frac{2}{x}-\frac{1}{x^{3}}+\frac{1}{x^{4}}+1=0.$$

This implies $x^{4}-x^{3}+x^{2}-2x+1=0$. Now, by it follows that

$$\sum_{n=0}^{\infty}GW_{n}x^{n}=\frac{(GW_{1}-GW_{0}-2GW_{2}+GW_{3})\,x^{3}+(GW_{0}-2GW_{1}+GW_{2})\,x^{2}+(GW_{1}-2GW_{0})\,x+GW_{0}}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\begin{aligned}\frac{(W_{1}-W_{0}-2W_{2}+W_{3})\,x^{3}+(W_{0}-2W_{1}+W_{2})\,x^{2}+(W_{1}-2W_{0})\,x+W_{0}}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} &= \frac{B_{1}}{(1-\alpha x)}+\frac{B_{2}}{(1-\beta x)} \\ &\quad +\frac{B_{3}}{(1-\gamma x)}+\frac{B_{4}}{(1-\delta x)}.\end{aligned} \tag{3.3}$$

So

$$\begin{aligned}&(W_{1}-W_{0}-2W_{2}+W_{3})\,x^{3}+(W_{0}-2W_{1}+W_{2})\,x^{2}+(W_{1}-2W_{0})\,x+W_{0}\\ =\;&B_{1}(1-\beta x)(1-\gamma x)(1-\delta x)+B_{2}(1-\alpha x)(1-\gamma x)(1-\delta x)\\ &+B_{3}(1-\alpha x)(1-\beta x)(1-\delta x)+B_{3}(1-\alpha x)(1-\beta x)(1-\gamma x).\end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get$W_{0}+\frac{1}{\alpha^{2}}\left(W_{0}-2W_{1}+W_{2}\right)-\frac{1}{\alpha^{3}}\left(W_{0}-W_{1}+2W_{2}-W_{3}\right)+\frac{1}{\alpha}\left(W_{1}-2W_{0}\right)=-B_{1}\left(\frac{1}{\alpha}\beta-1\right)\left(\frac{1}{\alpha}\gamma-1\right)\left(\frac{1}{\alpha}\delta-1\right).$

This gives

$$\begin{aligned}B_{1} &= \alpha^{3}(GW_{0}+\frac{1}{\alpha^{2}}\left(GW_{0}-2GW_{1}+GW_{2}\right)+\frac{1}{\alpha^{3}}\left(GW_{1}-5GW_{0}-4GW_{2}+GW_{3}\right)+\frac{1}{\alpha}\left(GW_{1}-2GW_{0}\right))\\ &= \frac{GW_{0}\alpha^{3}+\left(GW_{1}-2GW_{0}\right)\alpha^{2}+\left(GW_{0}-2GW_{1}+GW_{2}\right)\alpha-GW_{0}+GW_{1}-2GW_{2}+GW_{3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}B_{2} &= \frac{GW_{0}\beta^{3}+\left(GW_{1}-2GW_{0}\right)\beta^{2}+\left(GW_{0}-2GW_{1}+GW_{2}\right)\beta-GW_{0}+GW_{1}-2GW_{2}+GW_{3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},\\ B_{3} &= \frac{GW_{0}\gamma^{3}+\left(GW_{1}-2GW_{0}\right)\gamma^{2}+\left(GW_{0}-2GW_{1}+GW_{2}\right)\gamma-GW_{0}+GW_{1}-2GW_{2}+GW_{3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},\\ B_{4} &= \frac{GW_{0}\delta^{3}+\left(GW_{1}-2GW_{0}\right)\delta^{2}+\left(GW_{0}-2GW_{1}+GW_{2}\right)\delta-GW_{0}+GW_{1}-2GW_{2}+GW_{3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.\end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} GW_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} GW_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n) x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$GW_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

and then we get (3.1). □

## 4. Some Identities About Recurrence Relations of Gaussian Generalized Pandita Numbers

In this section, we present some identities on Gaussian Pandita, Gaussian Pandita-Lucas,

Theorem 4.1. *The following equations hold for all integer n*

$$\begin{aligned} GP_n &= \frac{54}{31}GS_{n+3} - \frac{41}{31}GS_{n+2} + \frac{6}{31}GS_{n+1} - \frac{50}{31}GS_n, \\ GS_n &= -GP_{n+3} + 3GP_{n+2} + GP_{n+1} - 4GP_n. \end{aligned} \tag{4.1}$$

Proof. To proof identity (4.1), we can write

$$GP_n = aGS_{n+3} + bGS_{n+2} + cGS_{n+1} + dGS_n.$$

Solving the system of equations

$$\begin{aligned} GP_0 &= aGS_3 + bGS_2 + cGS_1 + dGS_0, \\ GP_1 &= aGS_4 + bGS_3 + cGS_2 + dGS_1, \\ GP_2 &= aGS_5 + bGS_4 + cGS_3 + dGS_2, \\ GP_3 &= aGS_6 + bGS_5 + cGS_4 + dGS_3. \end{aligned}$$

we get $a = \frac{54}{31}, b = -\frac{41}{31}, c = \frac{6}{31}, d = -\frac{50}{31}$.

The other identities can be found similarly.

$$GS_n = aGP_{n+3} + bGP_{n+2} + cGP_{n+1} + dGP_n.$$

$$\begin{aligned} GS_0 &= aGP_3 + bGP_2 + cGP_1 + dGP_0, \\ GS_1 &= aGP_4 + bGP_3 + cGP_2 + dGP_1, \\ GS_2 &= aGP_5 + bGP_4 + cGP_3 + dGP_2, \\ GS_3 &= aGP_6 + bGP_5 + cGP_4 + dGP_3. \end{aligned}$$

we get $a=-1, b=3, c=1, d=-4$.

Lemma 4.2. *18, Let's assume that* $f(x)=\sum\limits_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x)=\sum_{n=0}^{\infty} a_{2n}x^n=\frac{f(\sqrt{x})+f(-\sqrt{x})}{2},$$

and

$$f_{a_{2n+1}}(x)=\sum_{n=0}^{\infty} a_{2n+1}x^n=\frac{f(\sqrt{x})-f(-\sqrt{x})}{2\sqrt{x}}.$$

respectively.

The generating functions of the even and odd-indexed Gaussian generalized Pandita sequences are provided by the following theorem.

Theorem 4.3. *The generating functions of the sequance* $GW_{2n}$ *and* $GW_{2n+1}$ *are provided by*

$$f_{GW_{2n}}(x)=\frac{GW_2\left(x^3+3x^2-x\right)+GW_0\left(2x^2+2x-1\right)-GW_1\left(x^2-x^3\right)-GW_3\left(x^3+2x^2\right)}{-x^4-x^3+x^2+2x-1}, \tag{4.2}$$

$$f_{GW_{2n+1}}(x)=\frac{GW_0\left(x^3+2x^2\right)-GW_3\left(x^3+x^2+x\right)-GW_1\left(x^3-2x+1\right)+GW_2\left(2x^3+x^2\right)}{-x^4-x^3+x^2+2x-1}. \tag{4.3}$$

Proof. We only proof (4.2). From Theorem (2.3) we can obtain following identities.

$f_{GW_n}(\sqrt{x})=\dfrac{1}{\sqrt{x^3}-x^2+x+2\sqrt{x}-1}((GW_1+GW_2-GW_3)\sqrt{x^3}+(2GW_0+GW_2-GW_3)\,x$
$+(-GW_2+2GW_0)\sqrt{x}-GW_0),$

$f_{GW_n}(-\sqrt{x})=-\frac{1}{-\sqrt{x^3}-x^2+x+2\sqrt{x}-1}((GW_0-GW_1+2GW_2-GW_3)\sqrt{x^3}$
$+(-GW_3-3GW_2+2GW_1+2GW_0)\,x+(2GW_1-GW_3)\sqrt{x}-GW_1).$

Thus, the result follows from Lemma (4.2). and the other identity can be derived analogously. □

From Theorem (4.3), we get the following Corollary.

Corollary 4.4.

$$\begin{aligned}
f_{GP_{2n}}(x) &= \frac{ix^3+(1+i)\,x^2+(2+i)\,x}{x^4+x^3-x^2-2x+1},\\
f_{GP_{2n+1}}(x) &= \frac{(1+i)\,x^2+(1+2i)\,x+1}{x^4+x^3-x^2-2x+1},\\
f_{GS_{2n}}(x) &= \frac{(1-4i)\,x^3-2x^2-6x+4+i}{x^4+x^3-x^2-2x+1},\\
f_{GS_{2n+1}}(x) &= \frac{(-1+i)\,x^3+(-5-2i)\,x^2+(1-6i)\,x+2+4i}{x^4+x^3-x^2-2x+1}.
\end{aligned}$$

From Corollary 4.4 we can obtain the following Corollary which presents the identities on Gaussian Pantida sequences.

Corollary 4.5. **a):** $(2+i)GS_{2n-2}+(1+i)\,GS_{2n-4}+iGS_{2n-6}=(4+i)GP_{2n}+(-6)\,GP_{2n-2}+(-2)GP_{2n-4}+(1-4i)\,GP_{2n-6}$.

**b):** $GS_{2n}+(1+2i)GS_{2n-2}+(1+i)\,GS_{2n-4}=(4+i)GP_{2n+1}+(-6)\,GP_{2n-1}+(-2)GP_{2n-3}+(1-4i)\,GP_{2n-5}$.

**c):** $(2+4i)GS_{2n}+(1-6i)\,GS_{2n-2}+(-5-2i)\,GS_{2n-4}+(-1+i)\,GS_{2n-6}=(4+i)GS_{2n+1}+(-6)\,GS_{2n-1}$

$+(-2)GS_{2n-3}+(1-4i)\,GS_{2n-5}$.

**d):** $GS_{2n+1}+(1+2i)\,GS_{2n-1}+(1+i)\,GS_{2n-3}=(2+4i)GP_{2n+1}+(1-6i)\,GP_{2n-1}+(-5-2i)\,GP_{2n-3}+(-1+i)\,GP_{2n-5}$.

**e):** $(2+i)\,GS_{2n-1}+(1+i)\,GS_{2n-3}+iGS_{2n-5}=(2+4i)GP_{2n}+(1-6i)\,GP_{2n-2}+(-5-2i)\,GP_{2n-4}+(-1+i)\,GP_{2n-6}$.

**f):** $GP_{2n}+(1+2i)\,GP_{2n-2}+(1+i)\,GP_{2n-4}=(2+i)\,GP_{2n-1}+(1+i)\,GP_{2n-3}+iGP_{2n-5}$.

Proof. From corollary (4.4) we obtain

$$(ix^3+(1+i)\,x^2+(2+i)\,x)f_{GS_{2n}}(x)=((1-4i)\,x^3-2x^2-6x+4+i)f_{GP_{2n}}(x).$$

LHS is equal to

$$\begin{aligned}
LHS &= (ix^3+(1+i)\,x^2+(2+i)\,x)\sum_{n=0}^{\infty}GS_{2n}x^n,\\
&= (2+i)\,x\sum_{n=0}^{\infty}GS_{2n}x^n+(1+i)\,x^2\sum_{n=0}^{\infty}GS_{2n}x^n+ix^3\sum_{n=0}^{\infty}GS_{2n}x^n\\
&= (2+i)\sum_{n=0}^{\infty}GS_{2n}x^{n+1}+(1+i)\sum_{n=0}^{\infty}GS_{2n}x^{n+2}+i\sum_{n=0}^{\infty}GS_{2n}x^{n+3}\\
&= (2+i)\sum_{n=1}^{\infty}GS_{2n-2}x^n+(1+i)\sum_{n=2}^{\infty}GS_{2n-4}x^n+i\sum_{n=3}^{\infty}GS_{2n-6}x^n,\\
&= (7+6i)\,x+(5+11i)\,x^2+\sum_{n=2}^{\infty}((2+i)GS_{2n-2}+(1+i)\,GS_{2n-4}+iGS_{2n-6})x^n.
\end{aligned}$$

Whereas the RHS is equal to

$$\begin{aligned}
RHS &= ((1-4i)\,x^3-2x^2-6x+(4+i))\sum_{n=0}^{\infty}GP_{2n}x^n,\\
&= (4+i)\sum_{n=0}^{\infty}GP_{2n}x^n-6x\sum_{n=0}^{\infty}GP_{2n}x^n-2x^2\sum_{n=0}^{\infty}GP_{2n}x^n+(1-4i)\,x^3\sum_{n=0}^{\infty}GP_{2n}x^n\\
&= (4+i)\sum_{n=0}^{\infty}GP_{2n}x^n+(-6)\sum_{n=0}^{\infty}GP_{2n}x^{n+1}+(-2)\sum_{n=0}^{\infty}GP_{2n}x^{n+2}+(1-4i)\sum_{n=0}^{\infty}GP_{2n}x^{n+3}\\
&= (4+i)\sum_{n=0}^{\infty}GP_{2n}x^n+(-6)\sum_{n=1}^{\infty}GP_{2n-2}x^n+(-2)\sum_{n=2}^{\infty}GP_{2n-4}x^n+(1-4i)\sum_{n=3}^{\infty}GP_{2n-6}x^n\\
&= (7+6i)\,x+(5+11i)\,x^2+\sum_{n=3}^{\infty}((4+i)GP_{2n}+(-6)\,GP_{2n-2}+(-2)GP_{2n-4}+(1-4i)\,GP_{2n-6})
\end{aligned}$$

By comparing the coefficients, the proof of the first identity (a) is done. We can prove other identity similarly. □

The following identity establishes a relationship between the Gaussian Pandita numbers and the Pandita–Lucas numbers.

COROLLARY 4.6. *For all integers $m, n$ the following identities holds:*

$$GW_{m+n} = P_{m-2}GW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})GW_{n+2} + (P_{m-3} - P_{m-4})GW_{n+1} - GW_nP_{m-3}.$$

Proof. First we assume that $m, n \geq 0$.The Theorem (4.6) can be proved by mathematical induction on $m$. If $m = 0$ we get

$$GW_n = P_{-2}GW_{n+3} + (P_{-4} - P_{-3} - P_{-5})GW_{n+2} + (P_{-3} - P_{-4})GW_{n+1} - GW_nP_{-3}.$$

which is true since $P_{-2} = 0, P = -1, P_{-4} = -1, P_{-5} = 0$. Assume that the equality holds for $m \leq k$. For $m = k + 1$, we get

$$\begin{aligned} GW_{k+1+n} &= 2GW_{n+k} - GW_{n+k-1} + GW_{n+k-2} - GW_{n+k-3}, \\ &\quad 2(P_{m-2}GW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})GW_{n+2} + (P_{m-3} - P_{m-4})GW_{n+1} - GW_nP_{m-3}) \\ &\quad -(P_{m-3}GW_{n+3} + (P_{m-5} - P_{m-4} - P_{m-6})GW_{n+2} + (P_{m-4} - P_{m-5})GW_{n+1} - GW_nP_{m-4}) \\ &\quad +(P_{m-4}GW_{n+3} + (P_{m-6} - P_{m-5} - P_{m-7})GW_{n+2} + (P_{m-5} - P_{m-6})GW_{n+1} - GW_nP_{m-5}) \\ &\quad -(P_{m-5}GW_{n+3} + (P_{m-7} - P_{m-6} - P_{m-8})GW_{n+2} + (P_{m-6} - P_{m-7})GW_{n+1} - GW_nP_{m-6}). \end{aligned}$$

Consequently, by mathematical induction on $m$, this proves Theorem 4.6.

The other cases of $m, n$ can be proved smilarly for all integers $m, n$. □

Taking $GW_n = GP_n$ or $GW_n = GS_n$ in above Theorem, respectively, we get:

COROLLARY 4.7.

$$\begin{aligned} GP_{m+n} &= P_{m-2}GP_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})GP_{n+2} + (P_{m-3} - P_{m-4})GP_{n+1} - GP_nP_{m-3}, \\ GS_{m+n} &= P_{m-2}GS_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})GS_{n+2} + (P_{m-3} - P_{m-4})GS_{n+1} - GS_nP_{m-3}. \end{aligned}$$

## 5. SIMSON'S FORMULA

This section is devoted to the presentation of Simson's formula associated with the generalized Gaussian Pandita numbers. This is a special case of [51, Theorem 4.1].

THEOREM 5.1. *For all integers $n$, we can write the following equality:*

$$\begin{vmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1} & GW_{n-2} & GW_{n-3} \end{vmatrix} = \begin{vmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{vmatrix}$$

$= (GW_3 - 2GW_2 + GW_0)(GW_3 - 2GW_1 + GW_0)(GW_3^2 - GW_2^2$

$+GW_1^2 - GW_0^2 - GW_2GW_3 - 2GW_1GW_3 + GW_1GW_2 + GW_0GW_3 + 2GW_0GW_2 - GW_0GW_1).$

Proof. Using Theorem 2.1 it can be proved by using induction use [51, Theorem 4.1]

From the Theorem 5.1 we get the following Corollary.

Corollary 5.2. *For all integers n, the Simson's formulas of Pandita and Pandita Lucas numbers are given as respectively.*

**a):**
$$\begin{vmatrix} GP_{n+3} & GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+2} & GP_{n+1} & GP_n & GP_{n-1} \\ GP_{n+1} & GP_n & GP_{n-1} & GP_{n-2} \\ GP_n & GP_{n-1} & GP_{n-2} & GP_{n-3} \end{vmatrix} = 1 - i.$$

**b):**
$$\begin{vmatrix} GS_{n+3} & GS_{n+2} & GS_{n+1} & GS_n \\ GS_{n+2} & GS_{n+1} & GS_n & GS_{n-1} \\ GS_{n+1} & GS_n & GS_{n-1} & GS_{n-2} \\ GS_n & GS_{n-1} & GS_{n-2} & GS_{n-3} \end{vmatrix} = 31 + 31i.$$

## 6. SUM FORMULAS

In this section, we identify some sum formulas of generalized Gaussian Pandita numbers.

Theorem 6.1. *For all integers $n \geq 0$, we get sum formulas below*

a) $\sum\limits_{k=0}^{n} GW_k = -(n+3)GW_{n+3} + (n+4)GW_{n+2} + (n+4)GW_n + 3GW_3 - 4GW_2 - 3GW_0.$

b) $\sum\limits_{k=0}^{n} GW_{2k} = \frac{1}{3}(-3(n+2)GW_{2n+2} + (3n+8)GW_{2n+1} + 2GW_{2n} + (3n+7)GW_{2n-1} + 7GW_3 - 8GW_2 - GW_1 - 6GW_0).$

c) $\sum\limits_{k=0}^{n} GW_{2k+1} = \frac{1}{3}(-(3n+4)GW_{2n+2} + (3n+8)GW_{2n+1} + GW_{2n} + 3(n+2)GW_{2n-1} + 6GW_3 - 8GW_2 + GW_1 - 7GW_0).$

Proof. It is given in Soykan [53, Theorem 3.12]. □

As a special case of the theorem 6.1, we present the following Corollary.

Corollary 6.2. *For all integers $n \geq 0$, we get sum formulas below:*

a) $\sum\limits_{k=0}^{n} GP_k = -(n+3)GP_{n+3} + (n+4)GP_{n+2} + (n+4)GP_n + 1 + 2i.$

b) $\sum\limits_{k=0}^{n} GP_{2k} = \frac{1}{3}(-3(n+2)GP_{2n+2} + (3n+8)GP_{2n+1} + 2GP_{2n} + (3n+7)GP_{2n-1} + 4 + 6i).$

c) $\sum\limits_{k=0}^{n} GP_{2k+1} = \frac{1}{3}(-(3n+4)GP_{2n+2} + (3n+8)GP_{2n+1} + GP_{2n} + 3(n+2)GP_{2n-1} + 3 + 4i).$

As a special case of the theorem 6.1, we present the following Corollary.

Corollary 6.3. *For all integers $n \geq 0$, we get sum formulas below:*

a) $\sum\limits_{k=0}^{n} GS_k = -(n+3)GS_{n+3} + (n+4)GS_{n+2} + (n+4)GS_n - 5 - 5i.$

b) $\sum\limits_{k=0}^{n} GS_{2k} = \frac{1}{3}(-3(n+2)GS_{2n+2} + (3n+8)GS_{2n+1} + 2GS_{2n} + (3n+7)GS_{2n-1} - 7 - 12i).$

c) $\sum\limits_{k=0}^{n} GS_{2k+1} = \frac{1}{3}(-(3n+4)GS_{2n+2} + (3n+8)GS_{2n+1} + GS_{2n} + 3(n+2)GS_{2n-1} - 12 - 7i).$

Next, we give the ordinary generating functions of some special cases of Gaussian generalized Pandita numbers.

Theorem 6.4. *The ordinary generating functions of the sequences $W_{2n}$, $W_{2n+1}$ are given as follows:*

a)$\sum_{n=0}^{\infty} GW_{2n}x^n = \dfrac{GW_2\left(x^3+3x^2-x\right) + GW_0\left(2x^2+2x-1\right) - GW_1\left(x^2-x^3\right) - GW_3\left(x^3+2x^2\right)}{-x^4-x^3+x^2+2x-1}.$

b)$\sum_{n=0}^{\infty} GW_{2n+1}x^n = \dfrac{GW_0\left(x^3+2x^2\right) - GW_3\left(x^3+x^2+x\right) - GW_1\left(x^3-2x+1\right) + GW_2\left(2x^3+x^2\right)}{-x^4-x^3+x^2+2x-1}.$

From the last Theorem, we have the following Corollary which gives sum formula of Gaussian Pandita numbers (Take $W_n = GP_n$ whit $GP_0 = 0$,$GP_1 = 1, GP_2 = 2+i, GP_3 = 3+2i$ ).

Corollary 6.5. *For $n \geq 0$ Gaussian Pandita numbers have the following properties.*

a)$\sum_{n=0}^{\infty} GP_{2n}x^n = \dfrac{ix^3 + (1+i)x^2 + (2+i)x}{x^4+x^3-x^2-2x+1}.$

b)$\sum_{n=0}^{\infty} GP_{2n+1}x^n = \dfrac{(1+i)x^2 + (1+2i)x + 1}{x^4+x^3-x^2-2x+1}.$

## 7. Matrix Formulation of $\mathbf{GW}_n$

In this section, we present the matrix representation of generalized Gaussian Pandita numbers

We define the square matrix $A$ of order 4 as

$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detA = 1$. Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [56].

Then we present the following lemma.

For $n \geq 0$ the following identitiy is true:

$$\begin{pmatrix} GW_{n+3} \\ GW_{n+2} \\ GW_{n+1} \\ GW_{n} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix}.$$

Proof. The identitiy (7) can be proved by mathematical induction on $n$. If $n = 0$ we obtain

$$\begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{0} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$, we deduce that the following identitiy is true

$$\begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_{k} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix}.$$

For $n = k + 1$, we obtain

$$\begin{aligned} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix} &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_{3} \\ GW_{2} \\ GW_{1} \\ GW_{0} \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_{k} \end{pmatrix} \\ &= \begin{pmatrix} GW_{k+4} \\ GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by applying mathematical induction on $n$, the proof completed. $\square$

We define

$$N_{Gw} = \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix}, \tag{7.1}$$

$$E_{Gw} = \begin{pmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1-} & GW_{n-2} & GW_{n-3} \end{pmatrix}. \tag{7.2}$$

Now, we have the following theorem with $N_{Gw}$ and $E_{Gw}$

THEOREM 7.1. *Using $N_{Gw}$ and $E_{Gw}$ , we get*

$$A^n N_{Gw} = E_{Gw}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{Gw} &= \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$
\begin{aligned}
a_{11} &= GW_1(P_n - P_{n-1}) - GW_2(P_n - P_{n-1} + P_{n-2}) - GW_0 P_n + GW_3 P_{n+1} = GW_3, \\
a_{12} &= GW_0(P_n - P_{n-1}) - GW_1(P_n - P_{n-1} + P_{n-2}) - GP_n W_{-1} + GW_2 P_{n+1} = GW_2, \\
a_{13} &= GW_{-1}(P_n - P_{n-1}) - GW_0(P_n - P_{n-1} + P_{n-2}) - GP_n W_{-2} + GW_1 P_{n+1} = GW_1, \\
a_{14} &= GW_{-2}(P_n - P_{n-1}) - GW_{-1}(P_n - P_{n-1} + P_{n-2}) - GP_n W_{-3} + GW_0 P_{n+1} = GW_0, \\
a_{21} &= GW_3 P_n - GW_2(P_{n-1} - P_{n-2} + P_{n-3}) + GW_1(P_{n-1} - P_{n-2}) - GW_0 P_{n-1} = GW_2, \\
a_{22} &= GW_2 P_n - GW_{-1} P_{n-1} - GW_1(P_{n-1} - P_{n-2} + P_{n-3}) + GW_0(P_{n-1} - P_{n-2}) = GW_1, \\
a_{23} &= G(P_{n-1} - P_{n-2}) W_{-1} - GW_{-2} P_{n-1} + GW_1 P_n - GW_0(P_{n-1} - P_{n-2} + P_{n-3}) = GW_0, \\
a_{24} &= G(P_{n-1} - P_{n-2}) W_{-2} - GW_{-3} P_{n-1} + GW_0 P_n - GW_{-1}(P_{n-1} - P_{n-2} + P_{n-3}) = GW_{-1}, \\
a_{31} &= GW_1(P_{n-2} - P_{n-3}) - GW_2(P_{n-2} - P_{n-3} + P_{n-4}) - GW_0 P_{n-2} + GW_3 P_{n-1} = GW_1, \\
a_{32} &= GW_0(P_{n-2} - P_{n-3}) - GW_1(P_{n-2} - P_{n-3} + P_{n-4}) - GW_{-1} P_{n-2} + GW_2 P_{n-1} = GW_0, \\
a_{33} &= G(P_{n-2} - P_{n-3}) W_{-1} - GW_{-2} P_{n-2} - GW_0(P_{n-2} - P_{n-3} + P_{n-4}) + GW_1 P_{n-1} = GW_{-1}, \\
a_{34} &= G(P_{n-2} - P_{n-3}) W_{-2} - GW_{-3} P_{n-2} - GW_{-1}(P_{n-2} - P_{n-3} + P_{n-4}) + GW_0 P_{n-1} = GW_{-2}, \\
a_{41} &= GW_1(P_{n-3} - P_{n-4}) - GW_2(P_{n-3} - P_{n-4} + P_{n-5}) - GW_0 P_{n-3} + GW_3 P_{n-2} = GW_0, \\
a_{42} &= GW_0(P_{n-3} - P_{n-4}) - GW_1(P_{n-3} - P_{n-4} + P_{n-5}) - GW_{-1} P_{n-3} + GW_2 P_{n-2} = GW_{-1}, \\
a_{43} &= G(P_{n-3} - P_{n-4}) W_{-1} - GW_{-2} P_{n-3} - GW_0(P_{n-3} - P_{n-4} + P_{n-5}) + GW_1 P_{n-2} = GW_{-2}, \\
a_{44} &= G(P_{n-3} - P_{n-4}) W_{-2} - GW_{-3} P_{n-3} - GW_{-1}(P_{n-3} - P_{n-4} + P_{n-5}) + GW_0 P_{n-2=} GW_{-3}.
\end{aligned}
$$

Using the Theorem 4.6 the proof is done. □

By taking $GW_n = GP_n$ with $GP_0, GP_1, GP_2, GP_3$ in 7.1 and 7.2,

and

$GW_n = GS_n$ with $GS_0, GS_1, GS_2, GS_3$ in 7.1 and 7.2.

respectively, we get:

$$
N_{GP} = \begin{pmatrix} 3+2i & 2+i & 1 & 0 \\ 2+i & 1 & 0 & 0 \\ 1 & 0 & 0 & -i \\ 0 & 0 & -i & -1-i \end{pmatrix}, \; E_{GO} = \begin{pmatrix} GP_{n+3} & GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+2} & GP_{n+1} & GP_n & GP_{n-1} \\ GP_{n+1} & GP_n & GP_{n-1} & GP_{n-2} \\ GP_n & GP_{n-1} & GP_{n-2} & GP_{n-3} \end{pmatrix},
$$

$$
N_{GS} = \begin{pmatrix} 5+2i & 2+2i & 2+4i & 4+i \\ 2+2i & 2+4i & 4+i & 1-i \\ 2+4i & 4+i & 1-i & -1+4i \\ 4+i & 1-i & -1+4i & -4+3i \end{pmatrix}, \; E_{GS} = \begin{pmatrix} GS_{n+3} & GS_{n+2} & GS_{n+1} & GS_n \\ GS_{n+2} & GS_{n+1} & GS_n & GS_{n-1} \\ GS_{n+1} & GS_n & GS_{n-1} & GS_{n-2} \\ GS_n & GS_{n-1} & GS_{n-2} & GS_{n-3} \end{pmatrix}.
$$

From Theorem 7.1, we can write the following corollary.

Corollary 7.2. *The following identities are hold:*

**a):** $A^n N_{GP} = E_{GP}$.

**b):** $A^n N_{GS} = E_{GS}$.

## 8. Conclusions

Sequences defined by recurrence relations have long been a focal point of mathematical inquiry, due to their inherent structure and wide applicability across diverse fields such as physics, engineering, architecture, biology, and the arts. Classical examples include second-order integer sequences like the Fibonacci, Lucas, Pell, and Jacobsthal numbers. Notably, the Fibonacci sequence—introduced by Leonardo of Pisa in his 1202 treatise Liber Abaci—was initially formulated through a rabbit population problem, and has since become a cornerstone in the study of recursive patterns and mathematical identities.

In this context, we propose a novel extension of recurrence relations to fourth-order systems: the Gaussian generalized Pandita numbers, along with two distinguished subclasses. For these newly defined sequences, we derive Binet-type expressions, ordinary and exponential generating functions, and generalized Simson-type identities. Additionally, we present closed-form summation formulas, recurrence properties, algebraic identities, and matrix representations.

Given their foundational role in both theory and application, recurrence-based sequences offer powerful tools for modeling and analysis. Therefore, we first examine the practical uses of second-order sequences before situating our fourth-order generalizations within this broader mathematical landscape.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [3].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [6].
- For the application of Jacobsthal numbers to special matrices, see [69].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [25].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [70].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [71].
- For the applications of generalized Fibonacci numbers to binomial sums, see [68].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [33].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [34].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [11].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [32].

- For the application of higher order Jacobsthal numbers to quaternions, see [30].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [21].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [5].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [26].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [27].
- For the applications of $k$-Fibonacci and $k-$Lucas numbers to spinors, see [28].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [63].
- For the application of special cases of Horadam numbers to Neutrosophic analysis see [23].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [10].
- For the application of Pell numbers to Gaussian Hyperbolic numbers, see [24].

In the following, we explore several applications of third-order recurrence sequences across various mathematical and applied contexts.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [9] and [8], respectively.
- For the application of Tribonacci numbers to special matrices, see [72].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [60] and [4], respectively.
- For the application of Pell-Padovan numbers to groups, see [12].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [22].
- For the application of Gaussian Tribonacci numbers to various graphs, see [62].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [14].For the application of Narayan numbers to finite groups see [29].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [46].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [35].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [36].
- For the application of generalized Tribonacci numbers to Sedenions, see [37].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [38].
- For the application of generalized Tribonacci numbers to circulant matrix, see [39].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [67].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [15].

In the following lists, we outline several applications of fourth-order recurrence sequences across theoretical and applied domains.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [40].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [41].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [42].
- For the application of generalized Tetranacci numbers to binomial transform, see [43].

We now explore several applications of fifth-order sequences.

- For the application of Pentanacci numbers to matrices, see [61].
- For the application of generalized Pentanacci numbers to quaternions, see [44].
- For the application of generalized Pentanacci numbers to binomial transform, see [45].

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