Original Research Article

Polynomial Solutions of Heun's Differential Equations in Nonstandard Analysis

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Abstract

In a paper of the present author, solutions of inhomogeneous and homogeneous Heun's differential equations, are obtained with the aid of nonstandard analysis. By using the solutions of homogeneous Heun's differential equations given there, polynomial solutions of homogeneous Heun's differential equations are derived.

1 Introduction

In a series of papers, Morita and Sato [1, 2] and Morita [3, 4, 5, 6] studied the problem of obtaining solutions of inhomogeneous and homogeneous differential equations by using the Green's function and nonstandard analysis.

In the preceding paper [6], solutions of inhomogeneous and homogeneous Heun's differential equation are given.

In the present paper, it is shown how polynomial solutions of homogeneous Heun's differential equation are obtained from the solutions of homogeneous Heun's differential equation given in [6].

We give here some notations to be used in the following sections. $\mathbb{Z}$ is the set of all integers, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$ and $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$.

We use $(z)_k$ for $z \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $k \in \mathbb{Z}_{>-1}$, which denote $(z)_k = \prod_{l=0}^{k-1}(z+l) = \frac{\Gamma(z+k)}{\Gamma(z)}$ for $k \in \mathbb{Z}_{>0}$ and $(z)_0 = 1$ for $k = 0$.

We use the step function $H(t)$ for $t \in \mathbb{R}$, which is equal to 1 if $t > 0$, and to 0 if $t \leq 0$, and h_k, which denotes $h_k = 1$ if $k \in \mathbb{Z}_{>-1}$, and $h_k = 0$ if $k \in \mathbb{Z}_{<0}$.

2   Heun's Differential Equation

Before writing homogeneous Heun's differential equation, we present a related differential equation given by

$$p(_RD_t,t)u(t) := \{(t-t_3)(t-t_1)(t-t_2)\frac{d^2}{dt^2}$$
$$+[\gamma_3(t-t_1)(t-t_2) + \gamma_1(t-t_2)(t-t_3) + \gamma_2(t-t_3)(t-t_1)]\frac{d}{dt}$$
$$+(\alpha_1\beta_1 t - q_2)\}u(t) = 0, \tag{1}$$

where t_1, t_2, t_3, γ_1, γ_2, γ_3, α_1, β_1 and q_2 are constants.

We express this equation as follows:

$$p(_RD_t,t)u(t) = [(A_0 + A_1t + A_2t^2 + A_3t^3)\frac{d^2}{dt^2} + (B_0 + B_1t + B_2t^2)\frac{d}{dt}$$
$$+(C_0 + C_1t)]u(t) = 0, \tag{2}$$

where

$$A_0 = -t_3t_1t_2, \quad A_1 = t_3t_1 + t_1t_2 + t_2t_3, \quad A_2 = -t_3 - t_1 - t_2, \quad A_3 = 1,$$
$$B_0 = \gamma_3t_1t_2 + \gamma_1t_2t_3 + \gamma_2t_3t_1, \quad B_1 = -\gamma_3(t_1+t_2) - \gamma_1(t_2+t_3) - \gamma_2(t_3+t_1),$$
$$B_2 = \gamma_3 + \gamma_1 + \gamma_2, \quad C_0 = -q_2, \quad C_1 = \alpha_1\beta_1. \tag{3}$$

Homogeneous Heun's equation is a special one of Equation (1), in which $t_1 = 1$, $t_3 = 0$ and $\gamma_2 = \alpha_1 + \beta_1 + 1 - \gamma_1 - \gamma_3$.

As a consequence, hompgeneous Heun's equation is expressed by the equation:

$$p_{He}(t,_RD_t)u(t) := [(A_1t + A_2t^2 + A_3t^3)\frac{d^2}{dt^2} + (B_0 + B_1t + B_2t^2)\frac{d}{dt} + (C_0 + C_1t)]u(t)$$
$$= \{[t_2t - (1+t_2)t^2 + t^3]\frac{d^2}{dt^2} + [\gamma_3t_2 + B_1t + (\alpha_1 + \beta_1 + 1)t^2]\frac{d}{dt}$$
$$-q_2 + \alpha_1\beta_1 t\}u(t) = 0, \tag{4}$$

where

$$B_1 = -[\gamma_3(1+t_2) + \gamma_1t_2 + \gamma_2] = -(\gamma_3t_2 + \gamma_1t_2 - \gamma_1 + \alpha_1 + \beta_1 + 1). \tag{5}$$

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Remark 2.1. In the preceding paper [6], we used $\alpha_1 \beta_1 q_1$ in place of q_2 in Equations (1), (3) and (4), following [7]. Here we use q_2, following [8].

Comparing Equation (4) with Equation (1.1.1) in [8], variable t and constants t_2, γ_1, γ_2, γ_3, α_1, β_1 and q_2 appear in the present paper, in place of z and a, δ, ϵ, γ, α, β and q in [8]. In [7], $\alpha\beta h$ appears in place of q in [8].

3   Complementary Solutions

In Section 4 of [6], if $\gamma_3 \notin \mathbb{Z}_{<1}$, a complementary solution $u(t)$ of Equation (4) is expressed in three ways, as follows:

$$u(t) = p_0 \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{k!} t^k H(t) = p_0 \sum_{k=0}^{\infty} P_k \frac{1}{(\gamma_3)_k k!} (\frac{t}{t_2})^k H(t) = p_0 \sum_{k=0}^{\infty} a_k t^k H(t), \tag{6}$$

where p_0 is any number, and $\tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$ satisfy $\tilde{p}_0 = 1$ and

$$\tilde{p}_k = \frac{1}{t_2(k-1+\gamma_3)}[\tilde{p}_{k-1}Q_k(0) - h_{k-2}\tilde{p}_{k-2}R_k(0)], \quad k \in \mathbb{Z}_{>0}, \tag{7}$$

where $Q_k(0)$ and $R_k(0)$ are given by

$$\begin{aligned} Q_k(0) &:= Q_k(0,0) = [(1+t_2)(k-2) - B_1](k-1) + q_2 \\ &= [(1+t_2)(k-2+\gamma_3) - \gamma_3 - \gamma_1 + \gamma_1 t_2 + \alpha_1 + \beta_1 + 1](k-1) + q_2, \quad k \in \mathbb{Z}_{>0}, \end{aligned} \tag{8}$$

$$\begin{aligned} R_k(0) &:= R_k(0,0) = [[(k-3) + \alpha_1 + \beta_1 + 1](k-2) + \alpha_1\beta_1](k-1) \\ &= (k-2+\alpha_1)(k-2+\beta_1)(k-1), \quad k \in \mathbb{Z}_{>1}. \end{aligned} \tag{9}$$

Equation (6) shows that P_k are related with $\tilde{p}_k$ by $P_k = t_2^k (\gamma_3)_k \tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$. By using these relations in Equation (7), we see that $P_0 = 1$ and

$$P_k = P_{k-1}Q_k(0) - t_2(k-2+\gamma_3)h_{k-2}P_{k-2}R_k(0), \quad k \in \mathbb{Z}_{>0}. \tag{10}$$

Equation (6) shows that a_k are related with $\tilde{p}_k$ by $a_k = \tilde{p}_k \frac{1}{k!}$ for $k \in \mathbb{Z}_{>-1}$. By using these, we confirm that a_k satisfy $a_0 = 1$ and

$$\begin{aligned} a_k &= \frac{1}{k!}\tilde{p}_k = \frac{1}{k!} \cdot \frac{1}{t_2(k-1+\gamma_3)}[(k-1)! \cdot a_{k-1}Q_k(0) - h_{k-2}(k-2)! \cdot a_{k-2}R_k(0)]. \\ &= \frac{1}{(k-1+\gamma_3)kt_2}[a_{k-1}Q_k(0) - \frac{1}{k-1}h_{k-2}a_{k-2}R_k(0)]. \end{aligned} \tag{11}$$

Equation (11) is given by Equations (8.3)$\sim$(8.5) in [7] and by Equations (3.3.1)$\sim$(3.3.3c) in [8].

Theorem 3.1. *We choose $n \in \mathbb{Z}_{>0}$ and put $\alpha_1 = -n+1$, $\beta_1 = -n$ and*

$$q_2 = -n[(n-1+\gamma_1+\gamma_3)t_2 - n+1 - \gamma_1]. \tag{12}$$

By using these in Equations (8) *and* (9)*, we obtain $R_{n+1}(0) = R_{n+2}(0) = 0$ and $Q_{n+1}(0) = 0$, and then by using these in Equation* (11)*, we confirm $a_{n+1} = a_{n+2} = 0$. As a consequence, we have $a_l = 0$ for $l \in \mathbb{Z}_{>n}$. When we adopt these values, the solution given by Equation* (6) *is a polynomial of degree n.*

3.1 Complementary solution, II

In Section 4.1 of [6], if $\gamma_3 \notin \mathbb{Z}_{>0}$, another complementary solution $u(t)$ of Equation (4) is expressed in three ways, as follows:

$$u(t) = p_0 \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{\Gamma(2 - \gamma_3 + k)} t^{1-\gamma_3+k} H(t) = p_0 \frac{1}{\Gamma(2 - \gamma_3)} t^{1-\gamma_3} \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{(2 - \gamma_3)_k} t^k H(t)$$

$$= p_0 \sum_{k=0}^{\infty} P_k \frac{1}{t_2^k k! \cdot \Gamma(2 - \gamma_3 + k)} t^{1-\gamma_3+k} H(t) = p_0 \frac{1}{\Gamma(2 - \gamma_3)} t^{1-\gamma_3} \sum_{k=0}^{\infty} a_k t^k H(t), \qquad (13)$$

where p_0 is any number, and $\tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$ satisfy $\tilde{p}_0 = 1$ and

$$\tilde{p}_k = \frac{1}{t_2 k}[\tilde{p}_{k-1} Q_k(1 - \gamma_3) - h_{k-2}\tilde{p}_{k-2} R_k(1 - \gamma_3)], \quad k \in \mathbb{Z}_{>0}, \qquad (14)$$

where $Q_k(1 - \gamma_3)$ and $R_k(1 - \gamma_3)$ are given by

$$Q_k(1 - \gamma_3) := Q_k(1 - \gamma_3, 0) = [(1 + t_2)(k - 1 - \gamma_3) - B_1](k - \gamma_3) + q_2$$
$$= [(1 + t_2)(k - 1) - \gamma_3 - \gamma_1 + \gamma_1 t_2 + \alpha_1 + \beta_1 + 1](k - \gamma_3) + q_2, \quad k \in \mathbb{Z}_{>0}, \qquad (15)$$
$$R_k(1 - \gamma_3) := R_k(1 - \gamma_3, 0) = [[(k - 2 - \gamma_3) + \alpha_1 + \beta_1 + 1](k - 1 - \gamma_3) + \alpha_1 \beta_1](k - \gamma_3)$$
$$= (k - 1 - \gamma_3 + \alpha_1)(k - 1 - \gamma_3 + \beta_1)(k - \gamma_3), \quad k \in \mathbb{Z}_{>1}. \qquad (16)$$

Remark 3.1. We see that the second member of Equation (13) is obtained from the second member of Equation (6), by replacing k by $k + 1 - \gamma_3$, where $k! = \Gamma(k + 1)$ is replaced by $\Gamma(k + 1 - \gamma_3)$ and $\tilde{p}_k$ given by Equation (7) with Equations (8) and (9) is replaced by $\tilde{p}_k$ given by Equation (14) with Equations (15) and (16).

Equation (13) shows that P_k are related with $\tilde{p}_k$ by $P_k = t_2^k(\gamma_3)_k \tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$. By using these relations in Equation (14), we see that $P_0 = 1$ and

$$P_k = P_{k-1} Q_k(1 - \gamma_3) - t_2(k - 1)h_{k-2} P_{k-2} R_k(1 - \gamma_3), \quad k \in \mathbb{Z}_{>0}. \qquad (17)$$

Equation (13) shows that a_k are related with $\tilde{p}_k$ by $a_k = \frac{1}{(2-\gamma_3)_k}\tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$. Then we confirm that a_k satisfy $a_0 = 1$ and

$$a_k = \frac{1}{(2 - \gamma_3)_k} p_k = \frac{1}{(2 - \gamma_3)_k t_2 k}[\tilde{p}_{k-1} Q_k(1 - \gamma_3) - h_{k-2}\tilde{p}_{k-2} R_k(1 - \gamma_3)]$$

$$= \frac{1}{t_2 k(k + 1 - \gamma_3)}[a_{k-1} Q_k(1 - \gamma_3) - \frac{1}{k - \gamma_3} h_{k-2} a_{k-2} R_k(1 - \gamma_3)], \quad k \in \mathbb{Z}_{>0}. \qquad (18)$$

Theorem 3.2. *We choose $n \in \mathbb{Z}_{>0}$ and put $\alpha_1 = \gamma_3 - n$, $\beta_1 = \gamma_3 - n - 1$ and*

$$q_2 = (\gamma_3 - n - 1)[(n + \gamma_1)t_2 + \gamma_3 - n - \gamma_1]. \qquad (19)$$

By using these in Equations (15), (16) *and* (18)*, we obtain $R_{n+1}(1 - \gamma_3) = R_{n+2}(1 - \gamma_3) = 0$, $Q_{n+1}(1 - \gamma_3) = 0$, and $a_{n+1} = a_{n+2} = 0$. As a consequence, we have $a_l = 0$ for $l \in \mathbb{Z}_{>n}$. When we adopt these values, the solution given by* (13) *is a polynomial of degree n, multipied by $t^{1-\gamma_3}$.*
Remark 3.2. When $\gamma_3 = 1$, the solution given by (13) agrees with that given by Equation (6).

4 Conclusion

In Sections 3 and 3.1, we obtain two complementary solutions of Heun's equation. They are expressed in three formats. The complementary solution in one format is in agreement with a solution presented in the past, given in [7] and [8].

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We show that we can construct a series of polynomial solutions of homogeneous Heun's equation, in Section 3, and a series of solutions of homogeneous Heun's equation, each of which is a polynomial multiplied by $t^{1-\gamma_3}$, in Section 3.1.

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