Fixed point results in generalized fuzzy metric space using compatible maps of type (K)

ABSTRACT

In this manuscript, we established some common fixed-point (FP) theorems in generalized-fuzzy metric spaces (M-FMS) by considering compatible self-maps of type (K). FP theory is widely extended and know-legible concept for research in various metric spaces and generalized fuzzy metric spaces in the similar sense, these results improve some existing theorems of literature. Some related examples are also proved.

Keywords: Common fixed point; Fuzzy metric space; Compatible maps of type (K); M-FMS.
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1. INTRODUCTION

Fixed point theory (FPT) is one of the most expanding fields in pure and applied mathematics. Many new nonlinear problems have been encountered in various branches of mathematics and sciences domain. FPT for solving various kind of problems in sense of uniqueness and existence of solution is very wide and interesting field. The theory of fuzzy set was initially introduced by Zadeh [16] (1965). Many authors, extend fuzzy set-in different sense like fuzzy differential operator, fuzzy integral norm and fuzzy metric space (FMS). FMS was initially defined by Kramosil and Michalek [6] (1975) using t-conorm, further by George and Veeramani [1] (1994), the modified form of the FMS was given.
Jungck [4] (1986), introduced compatible maps and proved some results in the context of metric space (MS) and in FMS given by Mishra *et al.* [8] (1994). Sedghi and Shobe [13] (2006), introduced a new space as M-FMS (Generalized FMS) and prove some FP results. Pant [9] (1994), established CPT for map which are non-commutative. Compatible maps of type (A) was firstly given by Jungck *et al.* [5] (1993). Pathak *et al.* [10] (1996), established common FP (CFP) results for compatible maps of type (P). Many mathematicians gave FP theorems in FMS in different topological properties (ref: [2], [11], [14]). Manandhar *et al.* [7] (2014), in FMS gave some FP results compatible maps of type (E).
Jha *et al.* [3] (2014), prove CFP theorems for compatible maps of type (K) in MS, further Rao and Reddy [11] (2016), extend the work in FMS for compatible maps of type (K).

In this paper, we extend FP results of Swati *et al.* [15] (2016), in generalized FMS for compatible of type (K) and prove FPT for self-map in M-FMS with some examples.

2. Preliminaries

Definition 2.1: [12] A continuous t-norm (t-conorm) is a binary operation $\hat{\mathfrak{S}}: [0,1]^2 \to [0,1]$ which satisfies the following conditions for all $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4 \in [0,1]$:
(T^1) $\hat{\mathfrak{S}}$ is continuous, commutative and associative,
(T^2) $\hat{\mathfrak{S}}(\mathfrak{d}_1, 1) = \mathfrak{d}_1$,
(T^3) $\hat{\mathfrak{S}}(\mathfrak{d}_1, \mathfrak{d}_2) \leq \hat{\mathfrak{S}}(\mathfrak{d}_3, \mathfrak{d}_4)$ whenever $\mathfrak{d}_1 \leq \mathfrak{d}_2$ and $\mathfrak{d}_3 \leq \mathfrak{d}_4$.

Definition 2.2: [1] The 3-tuple $(\breve{\mathfrak{X}}, \acute{\mathbb{M}}, \hat{\mathfrak{S}})$ is known as FM space if $\breve{\mathfrak{X}}$ is an arbitrary set, $\hat{\mathfrak{S}}$ is a t-conorm, $\acute{\mathbb{M}}$ is a fuzzy set in $\breve{\mathfrak{X}} \times \breve{\mathfrak{X}} \times [0,\infty)$ satisfies the following axioms for every $\varpi, w, \xi \in \breve{\mathfrak{X}}$ and $s, t > 0$:
(FM_1) $\acute{\mathbb{M}}(\varpi, w, t) > 0$,
(FM_2) $\acute{\mathbb{M}}(\varpi, w, t) = 1$ if and only if $\varpi = w$,
(FM_3) $\acute{\mathbb{M}}(\varpi, w, t) = \acute{\mathbb{M}}(w, \varpi, t)$,
(FM_4) $\hat{\mathfrak{S}}\left(\acute{\mathbb{M}}(\varpi, w, t), \acute{\mathbb{M}}(w, \xi, s)\right) \leq \acute{\mathbb{M}}(\varpi, \xi, t + s)$,
(FM_5) $\acute{\mathbb{M}}(\varpi, w, \cdot) : [0,\infty) \to [0,1]$ is continuous.

Definition 2.3: [8] A pair of self-maps $(\tilde{\wp}, \acute{T})$ of a FMS $(\breve{\mathfrak{X}}, \acute{\mathbb{M}}, \hat{\mathfrak{S}})$ is said to be compatible if $\lim_{m\to\infty} \acute{\mathbb{M}}(\tilde{\wp}\acute{T}\mathfrak{p}_m, \acute{T}\tilde{\wp}\mathfrak{p}_m, t) = 1$ for $t > 0$, whenever sequence $\{\mathfrak{p}_m\}$ from $\breve{\mathfrak{X}}$ s.t. $\lim_{m\to\infty} \acute{T}\mathfrak{p}_m = \lim_{m\to\infty} \tilde{\wp}\mathfrak{p}_m = \varpi$, for some $\varpi \in \breve{\mathfrak{X}}$.

Definition 2.4: [5] A pair of self-maps $(\tilde{\wp}, \acute{T})$ of a FMS $(\breve{\mathfrak{X}}, \acute{\mathbb{M}}, \hat{\mathfrak{S}})$ is said to be compatible of type (A) if $\lim_{m\to\infty} \acute{\mathbb{M}}(\tilde{\wp}\acute{T}\mathfrak{p}_m, \acute{T}\acute{T}\mathfrak{p}_m, t) = 1$ and $\lim_{m\to\infty} \acute{\mathbb{M}}(\acute{T}\tilde{\wp}\mathfrak{p}_m, \tilde{\wp}\tilde{\wp}\mathfrak{p}_m, t) = 1$ for $t > 0$, whenever sequence $\{\mathfrak{p}_m\}$ from $\breve{\mathfrak{X}}$ s.t. $\lim_{m\to\infty} \acute{T}\mathfrak{p}_m = \lim_{m\to\infty} \tilde{\wp}\mathfrak{p}_m = \varpi$, for some $\varpi \in \breve{\mathfrak{X}}$.

Definition 2.5: [10] A pair of self-maps $(\tilde{\wp}, \acute{T})$ of a FMS $(\breve{\mathfrak{X}}, \acute{\mathbb{M}}, \hat{\mathfrak{S}})$ is said to be compatible of type (P) if $\lim_{m\to\infty} \acute{\mathbb{M}}(\tilde{\wp}\tilde{\wp}\mathfrak{p}_m, \acute{T}\acute{T}\mathfrak{p}_m, t) = 1$ for $t > 0$, whenever sequence $\{\mathfrak{p}_m\}$ from $\breve{\mathfrak{X}}$ s.t. $\lim_{m\to\infty} \acute{T}\mathfrak{p}_m = \lim_{m\to\infty} \tilde{\wp}\mathfrak{p}_m = \varpi$, for some $\varpi \in \breve{\mathfrak{X}}$.

Definition 2.6: [7] A pair of self-maps $(\tilde{\wp}, \acute{T})$ of a FMS $(\breve{\mathfrak{X}}, \acute{\mathbb{M}}, \hat{\mathfrak{S}})$ is said to be compatible of type (E) if $\lim_{m\to\infty} \acute{\mathbb{M}}(\tilde{\wp}\tilde{\wp}\mathfrak{p}_m, \tilde{\wp}\acute{T}\mathfrak{p}_m, t) = \acute{T}\varpi$ and $\lim_{m\to\infty} \acute{\mathbb{M}}(\acute{T}\acute{T}\mathfrak{p}_m, \acute{T}\tilde{\wp}\mathfrak{p}_m, t) = \tilde{\wp}\varpi$, for all $t > 0$, whenever sequence $\{\mathfrak{p}_m\}$ from $\breve{\mathfrak{X}}$ s.t. $\lim_{m\to\infty} \acute{T}\mathfrak{p}_m = \lim_{m\to\infty} \tilde{\wp}\mathfrak{p}_m = \varpi$, for some $\varpi \in \breve{\mathfrak{X}}$.

Definition 2.7: [11] A pair of self-maps $(\tilde{\wp}, \acute{T})$ of a FMS $(\breve{\mathfrak{X}}, \acute{\mathbb{M}}, \hat{\mathfrak{S}})$ is said to be compatible of type (K) iff $\lim_{m\to\infty} \acute{\mathbb{M}}(\tilde{\wp}\tilde{\wp}\mathfrak{p}_m, \acute{T}\varpi, t) = 1$ and $\lim_{m\to\infty} \acute{\mathbb{M}}(\acute{T}\acute{T}\mathfrak{p}_m, \tilde{\wp}\varpi, t) = 1$, for any $t > 0$, whenever sequence $\{\mathfrak{p}_m\}$ from $\breve{\mathfrak{X}}$ s.t. $\lim_{m\to\infty} \acute{T}\mathfrak{p}_m = \lim_{m\to\infty} \tilde{\wp}\mathfrak{p}_m = \varpi$, for some $\varpi \in \breve{\mathfrak{X}}$.

Definition 2.8: [13] A 3-tuple $(\breve{\mathfrak{X}}, \acute{\mathcal{M}}, \hat{\mathfrak{S}})$ is said to be a generalised FMS ($\mathcal{M}$-FMS) if $\breve{\mathfrak{X}} \neq \{\emptyset\}$, $\hat{\mathfrak{S}}$ is a t-conorm, $\acute{\mathcal{M}}$ is a fuzzy set on $\breve{\mathfrak{X}}^3 \times (0,\infty)$ satisfies the following axioms for every $\varpi, w, \xi, u \in \breve{\mathfrak{X}}$ and $s, t > 0$:
(M_{FM1}) $\acute{\mathcal{M}}(\varpi, w, \xi, t) > 0$,
(M_{FM2}) $\acute{\mathcal{M}}(\varpi, w, \xi, t) = 1 \Leftrightarrow \varpi = w = \xi$,
(M_{FM3}) $\acute{\mathcal{M}}(\varpi, w, \xi, t) = \acute{\mathcal{M}}(p\{w, \varpi, \xi\}, t)$ where p is a permutation,
(M_{FM4}) $\hat{\mathfrak{S}}\left(\acute{\mathcal{M}}(\varpi, w, u, t), \acute{\mathcal{M}}(u, \xi, \xi, s)\right) \leq \acute{\mathcal{M}}(\varpi, w, \xi, t + s)$,
(M_{FM5}) $\acute{\mathcal{M}}(\varpi, w, \xi, \cdot) : (0,\infty) \to [0,1]$ is continuous.

Lemma 2.9: [13] If $(\breve{\mathfrak{X}}, \acute{\mathcal{M}}, \hat{\mathfrak{S}})$ be a generalized $\mathcal{M}$-FMS then $\acute{\mathcal{M}}(\varpi, w, \xi, t)$ is non-decreasing with respect to t, for all $t > 0$.

Definition 2.10: [13] Let $(\breve{\mathfrak{X}}, \acute{\mathcal{M}}, \hat{\mathfrak{S}})$ be an $\mathcal{M}$-FMS, for some $\varpi \in \breve{\mathfrak{X}}$ and $\{\mathfrak{p}_m\}$ be a sequence in $\breve{\mathfrak{X}}$. Then

(i) A sequence $\{\mathfrak{p}_m\}$ is said to converge to ϖ if for every $t > 0$,

$$\lim_{m\to\infty}\left(\frac{1}{\acute{\mathcal{M}}(\mathfrak{p}_m,\varpi,\varpi,t)}-1\right)=0 \text{ i.e., } \lim_{m\to\infty}\mathfrak{p}_m \to \varpi \text{ or } \mathfrak{p}_m \to \varpi \text{ as } m\to\infty.$$

(ii) A sequence $\{\mathfrak{p}_m\}$ is said to be a Cauchy sequence if for all $t > 0$ and $n \in \mathbb{N}$ we have

$$\lim_{m\to\infty}\left(\frac{1}{\acute{\mathcal{M}}(\mathfrak{p}_{m+n},\mathfrak{p}_m,\mathfrak{p}_m,t)}-1\right)=0.$$

(iii) $\mathcal{M}$-FMS $(\breve{\mathfrak{X}},\acute{\mathcal{M}},\breve{\mathfrak{S}})$ in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.11: [13] Let $(\breve{\mathfrak{X}},\acute{\mathcal{M}},\breve{\mathfrak{S}})$ be a generalized $\mathcal{M}$-FMS and if $\exists\, 0<k<1$ satisfying $\acute{\mathcal{M}}(\varpi,w,\xi,kt) \geq \acute{\mathcal{M}}(\varpi,w,\xi,t)$, for every $\varpi,w,\xi \in \breve{\mathfrak{X}}$ and $t\in(0,\infty)$ then $\varpi = w = \xi$.

3. Main Results:

In this section, we firstly state compatible maps of type (K) in $\mathcal{M}$-FMS $(\breve{\mathfrak{X}},\acute{\mathcal{M}},\breve{\mathfrak{S}})$ and we prove CFP results in $\mathcal{M}$-FMS $(\breve{\mathfrak{X}},\acute{\mathcal{M}},\breve{\mathfrak{S}})$ for the compatible of type (K) map.

Definition 3.1: A pair of self-maps $(\widetilde{\wp},\acute{T})$ of a $\mathcal{M}$-FMS $(\breve{\mathfrak{X}},\acute{\mathbb{M}},\breve{\mathfrak{S}})$ is said to be compatible of type (K) iff $\lim_{m\to\infty}\acute{\mathbb{M}}(\widetilde{\wp}\widetilde{\wp}\mathfrak{p}_m,\acute{T}\varpi,\acute{T}\varpi,t)=1$ and $\lim_{m\to\infty}\acute{\mathbb{M}}(\acute{T}\acute{T}\mathfrak{p}_m,\widetilde{\wp}\varpi,\widetilde{\wp}\varpi,t)=1$, for every $t>0$, whenever sequence $\{\mathfrak{p}_m\}$ from $\breve{\mathfrak{X}}$ s.t. $\lim_{m\to\infty}\acute{T}\mathfrak{p}_m=\lim_{m\to\infty}\widetilde{\wp}\mathfrak{p}_m=\varpi$, for some $\varpi\in\breve{\mathfrak{X}}$.

Example 3.2: Consider $\breve{\mathfrak{X}}=[-1,6]$ be a complete in $\mathcal{M}$-FMS and two self-maps $\widetilde{\wp},\acute{T}:\breve{\mathfrak{X}}\to\breve{\mathfrak{X}}$ be defined as:

$$\widetilde{\wp}(\varpi)=\begin{cases}3 & \text{if } \varpi\in[-1,3]-\left\{\frac{1}{6}\right\}\\ 6 & \text{if } \varpi=\frac{1}{6}\\ \frac{(4-\varpi)}{6} & \text{if } \varpi\in(3,6]\end{cases} \text{ and } \acute{T}(\varpi)=\begin{cases}\varpi & \text{if } \varpi\in\left[-1,\frac{1}{6}\right)\\ 3 & \text{if } \varpi=\frac{1}{6}\\ 6 & \text{if } \varpi\in\left(\frac{1}{6},2\right]\\ \frac{\varpi}{18} & \text{if } \varpi\in(2,6]\end{cases}.$$

Now, consider a sequence $\mathfrak{p}_m=3+\frac{1}{6m}$ from $\breve{\mathfrak{X}}$, for each non-negative integer m then

$$\lim_{m\to\infty}\widetilde{\wp}\mathfrak{p}_m=\lim_{m\to\infty}\widetilde{\wp}\left(3+\frac{1}{6m}\right)=\lim_{m\to\infty}\frac{1}{6}\left(1-\frac{1}{6m}\right)=\frac{1}{6} \text{ and}$$

$$\lim_{m\to\infty}\acute{T}\mathfrak{p}_m=\lim_{m\to\infty}\acute{T}\left(3+\frac{1}{6m}\right)=\lim_{m\to\infty}\frac{1}{18}\left(3+\frac{1}{6m}\right)=\frac{1}{6}.$$

Thus, both $\widetilde{\wp}\mathfrak{p}_m$ and $\acute{T}\mathfrak{p}_m$ converges to $\frac{1}{6}$ i.e., $\lim_{m\to\infty}\widetilde{\wp}\mathfrak{p}_m=\lim_{m\to\infty}\acute{T}\mathfrak{p}_m=\frac{1}{6}$. As, $\widetilde{\wp}\left(\frac{1}{6}\right)=6$ and $\acute{T}\left(\frac{1}{6}\right)=3$, therefore $\lim_{m\to\infty}\acute{T}\widetilde{\wp}\mathfrak{p}_m=\lim_{m\to\infty}\acute{T}\widetilde{\wp}\left(3+\frac{1}{6m}\right)=\lim_{m\to\infty}\acute{T}\left(\frac{1}{6}-\frac{1}{36m}\right)=\frac{1}{6}$,

$$\lim_{m\to\infty}\widetilde{\wp}\acute{T}\mathfrak{p}_m=\lim_{m\to\infty}\widetilde{\wp}\acute{T}\left(3+\frac{1}{6m}\right)=\lim_{m\to\infty}\widetilde{\wp}\left(\frac{1}{6}+\frac{1}{108m}\right)=3,$$

$$\lim_{m\to\infty}\widetilde{\wp}\widetilde{\wp}\mathfrak{p}_m=\lim_{m\to\infty}\widetilde{\wp}\widetilde{\wp}\left(3+\frac{1}{6m}\right)=\lim_{m\to\infty}\widetilde{\wp}\left(\frac{1}{6}-\frac{1}{36m}\right)=3=\acute{T}\left(\frac{1}{6}\right),$$

$$\lim_{m\to\infty}\acute{T}\acute{T}\mathfrak{p}_m=\lim_{m\to\infty}\acute{T}\acute{T}\left(3+\frac{1}{6m}\right)=\lim_{m\to\infty}\acute{T}\left(\frac{1}{6}+\frac{1}{108m}\right)=6=\widetilde{\wp}\left(\frac{1}{6}\right).$$

Hence, the maps are compatible of type (K) but not compatible, compatible of type (A), (P) and (E).

Theorem 3.3: Consider $(\breve{\mathfrak{X}},\acute{\mathcal{M}},\breve{\mathfrak{S}})$ be a complete $\mathcal{M}$-FMS (generalized-FMS) defined the $\zeta_1,\zeta_2,\zeta_3,\zeta_4,\Delta_5$ and Δ_6 be six self-maps on $\breve{\mathfrak{X}}$ s.t. they satisfies the following property:

($A^{3.3.1}$) $\zeta_1(\breve{\mathfrak{X}})\subset\Delta_5\zeta_3(\breve{\mathfrak{X}})$ and $\zeta_2(\breve{\mathfrak{X}})\subset\Delta_6\zeta_4(\breve{\mathfrak{X}})$,

($A^{3.3.2}$) $\zeta_1\zeta_4=\zeta_4\zeta_1$, $\zeta_2\zeta_3=\zeta_3\zeta_2$, $\zeta_3\Delta_6=\Delta_6\zeta_3$, and $\zeta_4\Delta_5=\Delta_5\zeta_4$,

($A^{3.3.3}$) $(\zeta_1,\Delta_5\zeta_4)$, $(\zeta_2,\Delta_6\zeta_3)$ are compatible of type (K) where one of them is continuous,

($A^{3.3.4}$) for all $\varpi,w,\xi\in\breve{\mathfrak{X}}$ and $0<\lambda<2$ there exists constant $0<k<1$ s.t.:

$$\acute{\mathcal{M}}(\zeta_1\varpi,\zeta_2 w,\zeta_2 w,kt)$$

$$\geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\Delta_5\zeta_4\varpi,\zeta_1\varpi,\zeta_1\varpi,t),\acute{\mathcal{M}}(\Delta_6\zeta_3 w,\zeta_2 w,\zeta_2 w,t),\acute{\mathcal{M}}(\Delta_5\zeta_4\varpi,\Delta_6\zeta_3 w,\Delta_6\zeta_3 w,t),\\ \acute{\mathcal{M}}(\Delta_6\zeta_3 w,\zeta_1\varpi,\zeta_1\varpi,\lambda t),\acute{\mathcal{M}}(\Delta_5\zeta_4\varpi,\zeta_2 w,\zeta_2 w,(-\lambda+2)t)\end{matrix}\right\}.$$

Then, six self-maps $\zeta_1,\zeta_2,\zeta_3,\zeta_4,\Delta_5$ and Δ_6 have unique CFP in $\breve{\mathfrak{X}}$.

Proof: Suppose $\mathfrak{p}_0 \in \breve{\mathfrak{A}}$. From given hypothesis ($A^{3.3.1}$): $\zeta_1(\breve{\mathfrak{A}}) \subset \Delta_5\zeta_3(\breve{\mathfrak{A}})$, $\zeta_2(\breve{\mathfrak{A}}) \subset \Delta_6\zeta_4(\breve{\mathfrak{A}})$, then $\exists\ \mathfrak{p}_1, \mathfrak{p}_2 \in \breve{\mathfrak{A}}$ s.t. $\zeta_1(\mathfrak{p}_0) = \Delta_5\zeta_3(\mathfrak{p}_0) = \mathfrak{q}_0$ and $\zeta_2(\mathfrak{p}_1) = \Delta_6\zeta_4(\mathfrak{p}_2) = \mathfrak{q}_1$.
Now, we generate two-sequences $\{\mathfrak{p}_m\}$ and $\{\mathfrak{q}_m\}$ from $\breve{\mathfrak{A}}$ in such a way that

$$\zeta_1(\mathfrak{p}_{2m}) = \Delta_5\zeta_3(\mathfrak{p}_{2m+1}) = \mathfrak{q}_{2m} \text{ and } \zeta_2(\mathfrak{p}_{2m+1}) = \Delta_6\zeta_4(\mathfrak{p}_{2m+2}) = \mathfrak{q}_{2m+1}. \quad (3.1)$$

for each non-negative integer m and $\lambda = -\mu + 1$, where $0 < \mu < 1$.
Now, we show that $\{\mathfrak{q}_m\}$ is Cauchy in $\breve{\mathfrak{A}}$. From ($A^{3.3.4}$), we have

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) = \acute{\mathcal{M}}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, kt) = \acute{\mathcal{M}}(\zeta_1\mathfrak{p}_{2m}, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, kt),$$

Therefore, one can have

$$\acute{\mathcal{M}}(\zeta_1\mathfrak{p}_{2m}, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, kt)$$

$$\geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\Delta_5\zeta_4\mathfrak{p}_{2m}, \zeta_1\mathfrak{p}_{2m}, \zeta_1\mathfrak{p}_{2m}, t), \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m}, \zeta_2\mathfrak{p}_{2m}, t),\\ \acute{\mathcal{M}}(\Delta_5\zeta_4\mathfrak{p}_{2m}, \Delta_6\zeta_3\mathfrak{p}_{2m+1}, \Delta_6\zeta_3\mathfrak{p}_{2m+1}, t), \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{p}_{2m+1}, \zeta_1\mathfrak{p}_{2m}, \zeta_1\mathfrak{p}_{2m}, \lambda t),\\ \acute{\mathcal{M}}(\Delta_5\zeta_4\mathfrak{p}_{2m}, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, (-\lambda+2)t)\end{matrix}\right\},$$

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, t),\\ \acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, (-\mu+1)t),\\ \acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, (\mu+1)t)\end{matrix}\right\}.$$

By equation (2.1), we get

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, t),\\ \acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, (\mu+1)t)\end{matrix}\right\},$$

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, t),\\ \acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \mu t)\end{matrix}\right\}.$$

Letting as μ assumes to 1 and using $\acute{\mathcal{M}}$-FMS axioms, we obtain

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \min\{\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, t)\} \quad (3.2)$$

Replacing t with t/k in equation (3.2), we have

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t) \geq \min\left\{\acute{\mathcal{M}}\left(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \frac{t}{k}\right), \acute{\mathcal{M}}\left(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, \frac{t}{k}\right)\right\},$$

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt)$$

$$\geq \min\left\{\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}\left(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \frac{t}{k}\right), \acute{\mathcal{M}}\left(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, \frac{t}{k}\right)\right\},$$

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \min\left\{\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}\left(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, \frac{t}{k}\right)\right\},$$

i.e., $\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt)$

$$\geq \min\left\{\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}\left(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, \frac{t}{k^2}\right), \acute{\mathcal{M}}\left(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, \frac{t}{k^2}\right)\right\},$$

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \min\left\{\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}\left(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, \frac{t}{k^2}\right)\right\}.$$

Similarly, one can get

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \min\left\{\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), \acute{\mathcal{M}}\left(\mathfrak{q}_{2m}, \mathfrak{q}_{2m+1}, \mathfrak{q}_{2m+1}, \frac{t}{k^m}\right)\right\}.$$

As, limit m tending to ∞, we have

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \min\{\acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t), 1\}.$$

$$\acute{\mathcal{M}}(\mathfrak{q}_{2m+1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, kt) \geq \acute{\mathcal{M}}(\mathfrak{q}_{2m-1}, \mathfrak{q}_{2m}, \mathfrak{q}_{2m}, t) \text{ for } t > 0.$$

Thus, for every m and $t > 0$, we say $\acute{\mathcal{M}}(\mathfrak{q}_{m+1}, \mathfrak{q}_m, \mathfrak{q}_m, kt) \geq \acute{\mathcal{M}}(\mathfrak{q}_m, \mathfrak{q}_{m-1}, \mathfrak{q}_{m-1}, t)$. Therefore,

$$\acute{\mathcal{M}}(\mathfrak{q}_{m+1}, \mathfrak{q}_m, \mathfrak{q}_m, t) \geq \acute{\mathcal{M}}\left(\mathfrak{q}_m, \mathfrak{q}_{m-1}, \mathfrak{q}_{m-1}, \frac{t}{k}\right)$$

$$> \acute{\mathcal{M}}\left(\mathfrak{q}_{m-1}, \mathfrak{q}_{m-2}, \mathfrak{q}_{m-2}, \frac{t}{k^2}\right) > \cdots > \acute{\mathcal{M}}\left(\mathfrak{q}_1, \mathfrak{q}_0, \mathfrak{q}_0, \frac{t}{k^m}\right).$$

$$\lim_{m\to\infty} \acute{\mathcal{M}}(\mathfrak{q}_{m+1}, \mathfrak{q}_m, \mathfrak{q}_m, t) = 1 \text{ for } t > 0.$$

For any p integer, we have

$$\acute{\mathcal{M}}(\mathfrak{q}_m, \mathfrak{q}_{m+p}, \mathfrak{q}_{m+p}, t)$$

$$\geq \widehat{\mathfrak{S}}\left(\acute{\mathcal{M}}\left(\mathfrak{q}_m, \mathfrak{q}_{m+1}, \mathfrak{q}_{m+1}, \frac{t}{k}\right), \acute{\mathcal{M}}\left(\mathfrak{q}_{m+1}, \mathfrak{q}_{m+2}, \mathfrak{q}_{m+2}, \frac{t}{k}\right), \ldots, \acute{\mathcal{M}}\left(\mathfrak{q}_{m+p-1}, \mathfrak{q}_{m+p}, \mathfrak{q}_{m+p}, \frac{t}{k}\right)\right)$$

$$\lim_{m\to\infty} \acute{\mathcal{M}}(\mathfrak{q}_{m+1}, \mathfrak{q}_m, \mathfrak{q}_m, t) \geq \widehat{\mathfrak{S}}(1,1,,1 \ldots, \ldots,1,1) = 1 \text{ for } t > 0.$$

Hence, $\{\mathfrak{q}_m\}$ is Cauchy sequence in $\breve{\mathfrak{U}}$, which is complete $\mathcal{M}$-FMS. Therefore, there exists $\xi \in \breve{\mathfrak{U}}$ and the sub-sequences $\{\zeta_1(\mathfrak{p}_{2m})\}$, $\{\Delta_5\zeta_3(\mathfrak{p}_{2m+1})\}$, $\{\zeta_2(\mathfrak{p}_{2m+1})\}$, $\{\Delta_6\zeta_4(\mathfrak{p}_{2m+2})\}$ also converges to $\xi \in \breve{\mathfrak{U}}$.

$$\lim_{m\to\infty} \zeta_1(\mathfrak{p}_{2m}) = \lim_{m\to\infty} \Delta_5\zeta_3(\mathfrak{p}_{2m+1}) = \lim_{m\to\infty} \zeta_2(\mathfrak{p}_{2m+1}) = \lim_{m\to\infty} \Delta_6\zeta_4(\mathfrak{p}_{2m+2}) = \xi. \tag{3.3}$$

Case (i) $(\zeta_1, \Delta_5\zeta_4)$ is compatible of type (K) and either $\Delta_5\zeta_4$ or ζ_1 is continuous. Now, we have

$$\lim_{m\to\infty} \zeta_1(\mathfrak{p}_{2m}) = \lim_{m\to\infty} \Delta_5\zeta_4(\mathfrak{p}_{2m+2}) = \xi \text{ i.e., } \lim_{m\to\infty} \zeta_1(\mathfrak{p}_{2m}) = \lim_{m\to\infty} \Delta_5\zeta_4(\mathfrak{p}_{2m}) = \xi,$$

since, $(\zeta_1, \zeta_5\zeta_4)$ is compatible of type (K), we get

$$\lim_{m\to\infty} \zeta_1\zeta_1(\mathfrak{p}_{2m}) = \Delta_5\zeta_4\xi \text{ and } \lim_{m\to\infty} \Delta_5\zeta_4\Delta_5\zeta_4(\mathfrak{p}_{2m}) = \zeta_1\xi.$$

Now, if map ζ_1 is continuous then $\lim_{m\to\infty} \zeta_1(\mathfrak{p}_{2m}) = \xi$ i.e., $\lim_{m\to\infty} \zeta_1\zeta_1(\mathfrak{p}_{2m}) = \zeta_1\xi$.

Therefore, $\zeta_1\xi = \Delta_5\zeta_4\xi$.

Similarly, if $\Delta_5\zeta_4$ is continuous, then $\lim_{m\to\infty} \Delta_5\zeta_4(\mathfrak{p}_{2m}) = \xi$ i.e., $\lim_{m\to\infty} \Delta_5\zeta_4\Delta_5\zeta_4(\mathfrak{p}_{2m}) = \Delta_5\zeta_4\xi$.

Therefore, $\zeta_1\xi = \Delta_5\zeta_4\xi$. (3.4)

Considering $\xi = \varpi$ and $w = \mathfrak{p}_{2m+1}$ in ($A^{3.3.4}$), one can have

$$\acute{\mathcal{M}}(\zeta_1\xi, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, kt)$$
$$\geq \min\left\{\begin{matrix} \acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \zeta_1\xi, \zeta_1\xi, t), \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, t), \\ \acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \Delta_6\zeta_3\mathfrak{p}_{2m+1}, \Delta_6\zeta_3\mathfrak{p}_{2m+1}, t), \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{p}_{2m+1}, \zeta_1\xi, \zeta_1\xi, \lambda t), \\ \acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, (-\lambda+2)t) \end{matrix}\right\}.$$

Since by equation (2.4), we get

$$\acute{\mathcal{M}}(\zeta_1\xi, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, kt)$$
$$\geq \min\left\{\begin{matrix} \acute{\mathcal{M}}(\zeta_1\xi, \zeta_1\xi, \zeta_1\xi, t), \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, t), \\ \acute{\mathcal{M}}(\zeta_1\xi, \Delta_6\zeta_3\mathfrak{p}_{2m+1}, \Delta_6\zeta_3\mathfrak{p}_{2m+1}, t), \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{p}_{2m+1}, \zeta_1\xi, \zeta_1\xi, \lambda t), \\ \acute{\mathcal{M}}(\zeta_1\xi, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, (-\lambda+2)t) \end{matrix}\right\}.$$

$$\acute{\mathcal{M}}(\zeta_1\xi, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, kt)$$
$$\geq \min\left\{\begin{matrix} 1, \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, t), \acute{\mathcal{M}}(\zeta_1\xi, \Delta_6\zeta_3\mathfrak{p}_{2m+1}, \Delta_6\zeta_3\mathfrak{p}_{2m+1}, t), \\ \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{p}_{2m+1}, \zeta_1\xi, \zeta_1\xi, \lambda t), \acute{\mathcal{M}}(\zeta_1\xi, \zeta_2\mathfrak{p}_{2m+1}, \zeta_2\mathfrak{p}_{2m+1}, (-\lambda+2)t) \end{matrix}\right\}.$$

by letting limit m tend to ∞, we arrive at

$$\acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, kt)$$
$$\geq \min\{1, \acute{\mathcal{M}}(\xi, \xi, \xi, t), \acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, t), \acute{\mathcal{M}}(\xi, \zeta_1\xi, \zeta_1\xi, \lambda t), \acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, (-\lambda+2)t)\}.$$

Since by from equation (2.3), when λ tend to 1, one can get

$$\acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, kt) \geq \min\{1, 1, \acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, t), \acute{\mathcal{M}}(\xi, \zeta_1\xi, \zeta_1\xi, \lambda t), \acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, t)\},$$
$$\acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, kt) \geq \min\{1, 1, \acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, t)\},$$
$$\acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, kt) \geq \acute{\mathcal{M}}(\zeta_1\xi, \xi, \xi, t).$$

From using Lemma 2.11, we say $\zeta_1\xi = \xi$.

Therefore, $\zeta_1\xi = \Delta_5\zeta_4\xi = \xi$. (3.5)

Case (ii) $(\zeta_2, \Delta_6\zeta_3)$ is compatible of type (K) and either $\Delta_6\zeta_3$ or ζ_2 is continuous. Now, we get

$$\lim_{m\to\infty} \zeta_2(\mathfrak{p}_{2m+1}) = \lim_{m\to\infty} \Delta_6\zeta_2(\mathfrak{p}_{2m+1}) = \xi,$$

since, $(\zeta_2, \Delta_6\zeta_3)$ is compatible of type (K), then we get

$$\lim_{m\to\infty} \zeta_2\zeta_2(\mathfrak{p}_{2m+1}) = \zeta_6\zeta_3\xi \text{ and } \lim_{m\to\infty} \Delta_6\zeta_3\Delta_6\zeta_3(\mathfrak{p}_{2m+1}) = \zeta_2\xi.$$

Now, if ζ_2 is continuous then $\lim_{m\to\infty} \zeta_2(\mathfrak{p}_{2m+1}) = \xi$ i.e., $\lim_{m\to\infty} \zeta_2\zeta_2(\mathfrak{p}_{2m+1}) = \zeta_2\xi$.

Also, if $\Delta_6\zeta_3$ is continuous, we obtain

$$\lim_{m\to\infty} \Delta_6\zeta_3(\mathfrak{p}_{2m+1}) = \xi \text{ i.e., } \lim_{m\to\infty} \Delta_6\zeta_3\Delta_6\zeta_3(\mathfrak{p}_{2m+1}) = \Delta_6\zeta_3\xi.$$

Therefore, $\zeta_1\xi = \Delta_5\zeta_4\xi$. (3.6)

Put $\xi = \varpi = w$ in ($A^{3.3.4}$), one can have

$$\acute{\mathcal{M}}(\zeta_1\xi, \zeta_2\xi, \zeta_2\xi, kt)$$
$$\geq \min\left\{\begin{matrix} \acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \zeta_1\xi, \zeta_1\xi, t), \acute{\mathcal{M}}(\Delta_6\zeta_3\xi, \zeta_2\xi, \zeta_2\xi, t), \acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \Delta_6\zeta_3\xi, \Delta_6\zeta_3\xi, t), \\ \acute{\mathcal{M}}(\Delta_6\zeta_3\xi, \zeta_1\xi, \zeta_1\xi, \lambda t), \acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \zeta_2\xi, \zeta_2\xi, (-\lambda+2)t) \end{matrix}\right\}.$$

Since by equation (3.5) and (3.6), we obtain

$$\acute{\mathcal{M}}(\xi,\zeta_2\xi,\zeta_2\xi,kt) \geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\xi,\zeta_1\xi,\zeta_1\xi,t),\acute{\mathcal{M}}(\zeta_2\xi,\zeta_2\xi,\zeta_2\xi,t),\acute{\mathcal{M}}(\xi,\zeta_2\xi,\zeta_2\xi,t)\\ \acute{\mathcal{M}}(\zeta_2\xi,\xi,\xi,\lambda t),\acute{\mathcal{M}}(\xi,\zeta_2\xi,\zeta_2\xi,(-\lambda+2)t)\end{matrix}\right\}.$$

as λ tend to 1, we have

$$\acute{\mathcal{M}}(\xi,\zeta_2\xi,\zeta_2\xi,kt) \geq \min\{1,1,\acute{\mathcal{M}}(\xi,\zeta_2\xi,\zeta_2\xi,t),\acute{\mathcal{M}}(\zeta_2\xi,\xi,\xi,t),\acute{\mathcal{M}}(\xi,\zeta_2\xi,\zeta_2\xi,t)\},$$

$$\acute{\mathcal{M}}(\xi,\zeta_2\xi,\zeta_2\xi,kt) \geq \acute{\mathcal{M}}(\xi,\zeta_2\xi,\zeta_2\xi,t),$$

by using Lemma 2.11, implies that $\zeta_2\xi = \xi$.

Therefore, $\zeta_1\xi = \Delta_5\zeta_4\xi = \zeta_2\xi = \Delta_6\zeta_3\xi = \xi$. (3.7)

Now, put $\xi = \varpi$ and $w = \zeta_3\xi$ in ($A^{3.3.4}$), we obtain

$$\acute{\mathcal{M}}(\zeta_1\xi,\zeta_2\zeta_3\xi,\zeta_2\zeta_3\xi,kt)$$
$$\geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\Delta_5\zeta_4\xi,\zeta_1\xi,\zeta_1\xi,t),\acute{\mathcal{M}}(\Delta_6\zeta_3\zeta_3\xi,\zeta_2\zeta_3\xi,\zeta_2\zeta_3\xi,t),\\ \acute{\mathcal{M}}(\Delta_5\zeta_4\xi,\Delta_6\zeta_3\zeta_3\xi,\Delta_6\zeta_3\zeta_3\xi,t),\\ \acute{\mathcal{M}}(\Delta_6\zeta_3\zeta_3\xi,\zeta_1\xi,\zeta_1\xi,\lambda t),\acute{\mathcal{M}}(\Delta_5\zeta_4\xi,\zeta_2\zeta_3\xi,\zeta_2\zeta_3\xi,(-\lambda+2)t)\end{matrix}\right\},$$

from given ($A^{3.3.2}$), we get

$$\acute{\mathcal{M}}(\zeta_1\xi,\zeta_3\zeta_2\xi,\zeta_3\zeta_2\xi,kt)$$
$$\geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\Delta_5\zeta_4\xi,\zeta_1\xi,\zeta_1\xi,t),\acute{\mathcal{M}}(\zeta_3\Delta_6\zeta_3\xi,\zeta_3\zeta_2\xi,\zeta_3\zeta_2\xi,t),\\ \acute{\mathcal{M}}(\Delta_5\zeta_4\xi,\zeta_3\Delta_6\zeta_3\xi,\zeta_3\Delta_6\zeta_3\xi,t),\\ \acute{\mathcal{M}}(\zeta_3\Delta_6\zeta_3\xi,\zeta_1\xi,\zeta_1\xi,\lambda t),\acute{\mathcal{M}}(\Delta_5\zeta_4\xi,\zeta_3\zeta_2\xi,\zeta_3\zeta_2\xi,(-\lambda+2)t)\end{matrix}\right\}.$$

By equation (3.7), one can have

$$\acute{\mathcal{M}}(\xi,\zeta_3\xi,\zeta_3\xi,kt) \geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\xi,\xi,\xi,t),\acute{\mathcal{M}}(\zeta_3\xi,\zeta_3\xi,\zeta_3\xi,t),\acute{\mathcal{M}}(\xi,\zeta_3\xi,\zeta_3\xi,t),\\ \acute{\mathcal{M}}(\zeta_3\xi,\xi,\xi,\lambda t),\acute{\mathcal{M}}(\xi,\zeta_3\xi,\zeta_3\xi,(-\lambda+2)t)\end{matrix}\right\}.$$

Considering as λ tend to 1,

$\acute{\mathcal{M}}(\xi,\zeta_3\xi,\zeta_3\xi,kt) \geq \min\{1,\acute{\mathcal{M}}(\xi,\zeta_3\xi,\zeta_3\xi,t)\}$, i.e., $\acute{\mathcal{M}}(\xi,\zeta_3\xi,\zeta_3\xi,kt) \geq \acute{\mathcal{M}}(\xi,\zeta_3\xi,\zeta_3\xi,t)$.

Form Lemma 2.11, we have

$\xi = \zeta_3\xi$ and $\xi = \Delta_6\zeta_3\xi$ i.e., $\xi = \Delta_6\xi$.

Therefore, $\xi = \zeta_3\xi = \Delta_6\xi$. (3.8)

Again, if we put $\zeta_4\xi = \varpi$ and $w = \xi$ in ($A^{3.3.4}$), we obtain

$$\acute{\mathcal{M}}(\zeta_1\zeta_4\xi,\zeta_2\xi,\zeta_2\xi,kt)$$
$$\geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\Delta_5\zeta_4\zeta_4\xi,\zeta_1\zeta_4\xi,\zeta_1\zeta_4\xi,t),\acute{\mathcal{M}}(\Delta_6\zeta_3\xi,\zeta_2\xi,\zeta_2\xi,t),\\ \acute{\mathcal{M}}(\Delta_5\zeta_4\zeta_4\xi,\Delta_6\zeta_3\xi,\Delta_6\zeta_3\xi,t),\\ \acute{\mathcal{M}}(\Delta_6\zeta_3\xi,\zeta_1\zeta_4\xi,\zeta_1\zeta_4\xi,\lambda t),\acute{\mathcal{M}}(\Delta_5\zeta_4\zeta_4\xi,\zeta_2\xi,\zeta_2\xi,(-\lambda+2)t)\end{matrix}\right\}.$$

By, given hypothesis ($A^{3.3.2}$), one can get

$$\acute{\mathcal{M}}(\zeta_4\zeta_1\xi,\zeta_2\xi,\zeta_2\xi,kt)$$
$$\geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\zeta_4\Delta_5\zeta_4\xi,\zeta_4\zeta_1\xi,\zeta_4\zeta_1\xi,t),\acute{\mathcal{M}}(\Delta_6\zeta_3\xi,\zeta_2\xi,\zeta_2\xi,t),\\ \acute{\mathcal{M}}(\zeta_4\Delta_5\zeta_4\xi,\Delta_6\zeta_3\xi,\Delta_6\zeta_3\xi,t),\\ \acute{\mathcal{M}}(\Delta_6\zeta_3\xi,\zeta_4\zeta_1\xi,\zeta_4\zeta_1\xi,\lambda t),\acute{\mathcal{M}}(\zeta_4\Delta_5\zeta_4\xi,\zeta_2\xi,\zeta_2\xi,(-\lambda+2)t)\end{matrix}\right\}.$$

From equation (2.7), we get

$$\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,kt) \geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\zeta_4\xi,\zeta_4\xi,\zeta_4\xi,t),\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,t),\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,t),\\ \acute{\mathcal{M}}(\xi,\zeta_4\xi,\zeta_4\xi,\lambda t),\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,(-\lambda+2)t)\end{matrix}\right\}.$$

$$\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,kt) \geq \min\{1,1,\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,t),\acute{\mathcal{M}}(\xi,\zeta_4\xi,\zeta_4\xi,\lambda t),\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,(-\lambda+2)t)\},$$

as λ assumes to 1,

$\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,kt) \geq \min\{1,\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,t)\}$, i.e., $\acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,kt) \geq \acute{\mathcal{M}}(\zeta_4\xi,\xi,\xi,t)$.

By, considering Lemma 2.11, we get

$\xi = \zeta_4\xi$ and $\xi = \Delta_5\zeta_4\xi$ i.e., $\xi = \zeta_4\xi$.

Thus, $\xi = \zeta_4\xi = \Delta_5\xi$. . (3.9)

Using equations (3.7), (3.8) and (3.9), one can obtain

$$\xi = \Delta_6\xi = \Delta_5\xi = \zeta_4\xi = \zeta_3\xi = \zeta_2\xi = \zeta_1\xi.$$

Hence, ξ is CFP of six self-maps $\zeta_1,\zeta_2,\zeta_3,\zeta_4,\Delta_5$ and Δ_6.

Uniqueness: To show uniqueness of FP, let $\mathfrak{u}_{\sigma}$ be another FP of six self-maps $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \Delta_5$ and Δ_6 i.e., $\zeta_1 \mathfrak{u}_{\sigma} = \zeta_2 \mathfrak{u}_{\sigma} = \zeta_3 \mathfrak{u}_{\sigma} = \zeta_4 \mathfrak{u}_{\sigma} = \Delta_5 \mathfrak{u}_{\sigma} = \Delta_6 \mathfrak{u}_{\sigma} = \mathfrak{u}_{\sigma}$. Put $\xi = \varpi$ and $\mathfrak{u}_{\sigma} = \mathit{w}$ in ($A^{3.3.4}$), one can have

$$\acute{\mathcal{M}}(\zeta_1\xi, \zeta_2\mathfrak{u}_{\sigma}, \zeta_2\mathfrak{u}_{\sigma}, kt)$$
$$\geq \min\left\{\begin{array}{c}\acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \zeta_1\xi, \zeta_1\xi, t), \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{u}_{\sigma}, \zeta_2\mathfrak{u}_{\sigma}, \zeta_2\mathfrak{u}_{\sigma}, t), \acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \Delta_6\zeta_3\mathfrak{u}_{\sigma}, \Delta_6\zeta_3\mathfrak{u}_{\sigma}, t), \\ \acute{\mathcal{M}}(\Delta_6\zeta_3\mathfrak{u}_{\sigma}, \zeta_1\xi, \zeta_1\xi, \lambda t), \acute{\mathcal{M}}(\Delta_5\zeta_4\xi, \zeta_2\mathfrak{u}_{\sigma}, \zeta_2\mathfrak{u}_{\sigma}, (-\lambda+2)t)\end{array}\right\}.$$

Letting as $\lambda \to 1$, we obtain

$$\acute{\mathcal{M}}(\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, kt) \geq \min\left\{\begin{array}{c}\acute{\mathcal{M}}(\Delta_5\xi, \xi, \xi, t), \acute{\mathcal{M}}(\Delta_6\mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, t), \acute{\mathcal{M}}(\Delta_5\xi, \Delta_6\mathfrak{u}_{\sigma}, \Delta_6\mathfrak{u}_{\sigma}, t), \\ \acute{\mathcal{M}}(\Delta_6\mathfrak{u}_{\sigma}, \xi, \xi, t), \acute{\mathcal{M}}(\Delta_5\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, t)\end{array}\right\},$$

$$\acute{\mathcal{M}}(\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, kt) \geq \min\left\{\begin{array}{c}\acute{\mathcal{M}}(\xi, \xi, \xi, t), \acute{\mathcal{M}}(\mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, t), \acute{\mathcal{M}}(\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, t), \\ \acute{\mathcal{M}}(\mathfrak{u}_{\sigma}, \xi, \xi, t), \acute{\mathcal{M}}(\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, t)\end{array}\right\}.$$

Then, $\acute{\mathcal{M}}(\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, kt) \geq \min\{1, \acute{\mathcal{M}}(\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, t)\}$ i.e., $\acute{\mathcal{M}}(\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, kt) \geq \acute{\mathcal{M}}(\xi, \mathfrak{u}_{\sigma}, \mathfrak{u}_{\sigma}, t)$.
Hence, $\xi = \mathfrak{u}_{\sigma}$.
Thus, we established the uniqueness of CFP ξ.

Example 3.4: Let $\breve{\mathfrak{U}} = [-3,3]$ be a complete in $\mathcal{M}$-FMS and two self-maps $\tilde{\wp}, \acute{T}: \breve{\mathfrak{U}} \to \breve{\mathfrak{U}}$ be defined as: $\tilde{\wp}(\varpi) = \begin{cases} 6 & \text{if } \varpi = \frac{1}{3} \\ \varpi & \text{if } \varpi \in [-3,2] - \left\{\frac{1}{3}\right\} \\ \frac{(4-\varpi)}{6} & \text{if } \varpi \in (2,3] \end{cases}$ and $\acute{T}(\varpi) = \begin{cases} \frac{1}{3} & \text{if } \varpi \in [-3,2] \\ \frac{\varpi}{6} & \text{if } \varpi \in (2,3] \end{cases}$.

Now, consider a sequence $\mathfrak{p}_m = 2 + \frac{1}{6m}$ from $\breve{\mathfrak{U}}$, for each non-negative integer m. Letting as, m tends to ∞, both $\tilde{\wp}\mathfrak{p}_m$ and $\acute{T}\mathfrak{p}_m$ converges to $\frac{1}{3}$ i.e., $\lim_{m\to\infty} \tilde{\wp}\mathfrak{p}_m = \lim_{m\to\infty} \acute{T}\mathfrak{p}_m = \frac{1}{3}$. Since, $\tilde{\wp}\left(\frac{1}{3}\right) = 6$ and $\acute{T}\left(\frac{1}{3}\right) = \frac{1}{3}$, thus, one can obtain

$$\lim_{m\to\infty} \tilde{\wp}\tilde{\wp}\mathfrak{p}_m = \lim_{m\to\infty} \tilde{\wp}\tilde{\wp}\left(2 + \frac{1}{6m}\right) = \lim_{m\to\infty} \tilde{\wp}\left(\frac{1}{3} - \frac{1}{36m}\right) = \frac{1}{3} = \acute{T}\left(\frac{1}{3}\right),$$
$$\lim_{m\to\infty} \acute{T}\acute{T}\mathfrak{p}_m = \lim_{m\to\infty} \acute{T}\acute{T}\left(2 + \frac{1}{6m}\right) = \lim_{m\to\infty} \acute{T}\left(\frac{1}{3} + \frac{1}{36m}\right) = \frac{1}{3} \neq \tilde{\wp}\left(\frac{1}{3}\right) = 6,$$
$$\lim_{m\to\infty} \tilde{\wp}\acute{T}\mathfrak{p}_m = \lim_{m\to\infty} \tilde{\wp}\acute{T}\left(2 + \frac{1}{6m}\right) = \lim_{m\to\infty} \tilde{\wp}\left(\frac{1}{3} + \frac{1}{36m}\right) = \lim_{m\to\infty} \left(\frac{1}{3} + \frac{1}{36m}\right) = \frac{1}{3},$$
$$\lim_{m\to\infty} \acute{T}\tilde{\wp}\mathfrak{p}_m = \lim_{m\to\infty} \acute{T}\tilde{\wp}\left(2 + \frac{1}{6m}\right) = \lim_{m\to\infty} \acute{T}\left(\frac{1}{3} - \frac{1}{36m}\right) = \frac{1}{3}.$$

Hence, the maps not compatible of type (K) in $\breve{\mathfrak{U}}$.

Corollary 3.5: Consider $(\breve{\mathfrak{U}}, \acute{\mathcal{M}}, \mathfrak{S})$ be a complete $\mathcal{M}$-FMS. If $\zeta_1, \zeta_2, \zeta_3$ and ζ_4 are self-maps on $\breve{\mathfrak{U}}$ s.t. they satisfies:
($3^{3.5.1}$) $\zeta_1(\breve{\mathfrak{U}}) \subset \zeta_3(\breve{\mathfrak{U}})$, $\zeta_2(\breve{\mathfrak{U}}) \subset \zeta_4(\breve{\mathfrak{U}})$;
($A^{3.5.2}$) (ζ_1, ζ_4), (ζ_2, ζ_3) is compatible of type (K) where one of them is contionus;
($A^{3.5.3}$) for all $\varpi, \mathit{w}, \xi \in \breve{\mathfrak{U}}$ and $0 < \lambda < 2$, $\exists\ 0 < k < 1$ s.t.:

$$\acute{\mathcal{M}}(\zeta_1\varpi, \zeta_2\mathit{w}, \zeta_2\mathit{w}, kt)$$
$$\geq \min\left\{\begin{array}{c}\acute{\mathcal{M}}(\zeta_3\varpi, \zeta_1\varpi, \zeta_1\varpi, t), \acute{\mathcal{M}}(\zeta_4\mathit{w}, \zeta_2\mathit{w}, \zeta_2\mathit{w}, t), \acute{\mathcal{M}}(\zeta_3\varpi, \zeta_4\mathit{w}, \zeta_4\mathit{w}, t), \\ \acute{\mathcal{M}}(\zeta_4\mathit{w}, \zeta_1\varpi, \zeta_1\varpi, \lambda t), \acute{\mathcal{M}}(\zeta_3\varpi, \zeta_2\mathit{w}, \zeta_2\mathit{w}, (-\lambda+2)t)\end{array}\right\}.$$

Then, self-maps $\zeta_1, \zeta_2, \zeta_3$ and ζ_4 have unique CFP in $\breve{\mathfrak{U}}$.
Proof: If we consider $\Delta_5 = \Delta_6 = I$ in Theorem 3.3, one can easily do the proof.

Corollary 3.6: Consider $(\breve{\mathfrak{U}}, \acute{\mathcal{M}}, \mathfrak{S})$ be a complete $\mathcal{M}$-FMS. If ζ_1, ζ_2 and ζ_3 are three self-maps on $\breve{\mathfrak{U}}$ s.t. they satisfies:
($A^{3.6.1}$) $\zeta_1(\breve{\mathfrak{U}}) \subset \zeta_2(\breve{\mathfrak{U}}) \cap \zeta_3(\breve{\mathfrak{U}})$;
($A^{3.6.2}$) (ζ_1, ζ_2), (ζ_1, ζ_3) is compatible of type (K), where ζ_1 is contionus;
($A^{3.6.3}$) for every $\varpi, \mathit{w}, \xi \in \breve{\mathfrak{U}}$ and $0 < \lambda < 2$, $\exists\ 0 < k < 1$ s.t.:

$$\acute{\mathcal{M}}(\zeta_1\varpi,\zeta_1 w,\zeta_1 w,kt)$$
$$\geq \min\left\{\begin{matrix}\acute{\mathcal{M}}(\zeta_2\varpi,\zeta_1\varpi,\zeta_1\varpi,t),\acute{\mathcal{M}}(\zeta_3 w,\zeta_1 w,\zeta_1 w,t),\acute{\mathcal{M}}(\zeta_2\varpi,\zeta_3 w,\zeta_3 w,t),\\ \acute{\mathcal{M}}(\zeta_4 w,\zeta_1\varpi,\zeta_1\varpi,\lambda t),\acute{\mathcal{M}}(\zeta_2\varpi,\zeta_1 w,\zeta_1 w,(-\lambda+2)t)\end{matrix}\right\}.$$

Then, self-maps ζ_1, ζ_2 and ζ_3 have unique CFP in $\mathfrak{A}$.

Proof: By considering $\zeta_3 = \zeta_4 = I$ in Corollary 2.2, one can have the proof.

4. CONCLUSION

In the manuscript, we established CFP theorems in M-FMS for self-maps by using compatible of type (K) with some examples, since FP theory has many applications in various branches of mathematics and generalized-FMS. These results extend and generalized some FP theorems existing in the literature.

ABBREVIATIONS

FMS: Fuzzy metric space; FPT: Fixed point theory; CFP: Common Fixed point; s.t.: Such that.

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