Total Domination Polynomials - A new approach

Abstract

For a graph $G = (V, E)$, the *open neighbourhood hypergraph* of G, denoted by $ONH(G)$, is the hypergraph with vertex set V and edge set $\{N_G(x)|x \in V\}$. A *vertex cover* in $ONH(G)$ is a set of vertices intersecting every edge of $ONH(G)$, which is equivalent to a *total dominating set* in G. Using the interplay between total dominating sets and vertex cover in hypergraphs, we determine the total domination polynomial of some classes of graphs.

Keywords: total domination, vertex cover, total domination polynomial.

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1 Introduction

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite non-empty set and $E(G)$ is a collection of unordered pairs of vertices called edges. If u and v are two vertices of a graph and if the unordered pair $\{u, v\}$ is an edge denoted by e, we say that e is an edge between u and v. We write the edge $\{u, v\}$ as uv. An edge of the form uu is known as a loop. The *open neighbourhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V|uv \in E(G)\}$. If the graph G is clear from the context, we write $N(v)$ rather than $N_G(v)$. Notations and definitions not given here can be found in Balakrishnan and Ranganathan, 2012,Berge and Minieka, 1973 or Henning and Yeo, 2008. A *hypergraph* $H = (V, E)$ is a finite nonempty set $V = V(H)$ of elements called *vertices*, together with a finite multi set $E = E(H)$ of subsets of V, called *hyper edges* or simply *edges.* The *order* and *size* of H are $|V|$ and $|E|$, respectively. A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. Every simple graph is a 2-uniform hypergraph. In a hypergraph, an edge E_i with $|E_i| = 2$, is drawn as a curve connecting its two vertices. An edge E_i with $|E_i| = 1$, is drawn as a loop as in a graph. A subset T of vertices in a hypergraph H is a *transversal*(also called *vertex cover*) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. For further information on hypergraphs refer Berge and Minieka, 1973 or Voloshin, 2009. Let $\mathcal{C}(H, i)$ be the family of vertex covering sets of H with cardinality i and let $c(H, i) = |\mathcal{C}(H, i)|$. The polynomial $\mathcal{C}(H, x) = \sum_{i=\tau(H)}^{|V(H)|} c(H, i)x^i$ is defined as *vertex cover* *polynomial* of H. For a graph $G = (V, E)$, the *ONH(G)* or H_G is the *open neighbourhood hypergraph* of G; $H_G = (V, C)$ is the hypergraph with vertex set $V(H_G) = V$ and with edge set $E(H_G) = C = \{N_G(x)|x \in V\}$, consisting of the open neighbourhoods of vertices of V in G.A *total dominating set,* abbreviated TD-set, of a graph $G = (V, E)$ with no isolated vertex is set S of vertices of G such that every vertex of G is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Let $\mathcal{D}_t(G, i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathcal{D}_t(G, i)|$. The polynomial $\mathcal{D}_t(G, x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G, i)x^i$ is defined as *total domination polynomial* of GVijayan and Kumar, 2012. Here we need the following.

Definition 1.1. A graph G in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called the root of G. Let G be a rooted graph. The graph $G^{(n)}$ obtained by identifying the roots of n copies of G is called a one-point union of the n copies of G.

Definition 1.2. An n-gon book of k pages denoted by $C_n^{2(k)}$ is the graph obtained when k copies of the cycle C_n share a common edge.

Definition 1.3. Given k natural numbers, the generalized theta graph $\theta(n_1, n_2, \ldots, n_k)$ is obtained by connecting two vertices u and v by k parallel paths of length $n_1 - 1, n_2 - 1, \ldots, n_k - 1.$

Definition 1.4. The tree T_{n_1,n_2,n_3} is a rooted tree consisting of three branches of length n_1, n_2 and n_3.

Theorem 1.1. *Henning and Yeo, 2008 The ONH of a connected bipartite graph consists of two components, while the ONH of a connected graph that is not bipartite is connected.*

Theorem 1.2. *Henning and Yeo, 2013 If G is a graph with no isolated vertex and H_G is the ONH of G, then $\gamma_t(G) = \tau(H_G)$.*

Theorem 1.3. *Dong et al., 2002 Let G be a graph and $L = \{x \in V(G)|\, xx \in E(G)\}$. Then $\mathcal{C}(G, x) = x^{|L|}\mathcal{C}(G - L, x)$.*

Theorem 1.4. *Dong et al., 2002 Let G be a graph with no loops and $V(G) \geq 2$. Let $u \in V(G)$ and $d = |N_G(u)|$. Then $\mathcal{C}(G, x) = x\mathcal{C}(G - u, x) + x^d\mathcal{C}(G - u - N_G(u), x)$.*

Theorem 1.5. *Dong et al., 2002 Let $G = G_1 \cup G_2$ be the union of two graphs G_1 and G_2. Then, $\mathcal{C}(G, x) = \mathcal{C}(G_1, x)\mathcal{C}(G_2, x)$.*

Theorem 1.6. *Dong et al., 2002 For the path graph P_n, where $n > 1$, we have*
$\mathcal{C}(P_n, x) = \sum_{i=0}^{n} \binom{i+1}{n-i} x^i.$

Theorem 1.7. *Dong et al., 2002 For the cycle graph C_n, where $n \geq 3$, we have*
$\mathcal{C}(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i.$

Theorem 1.8. *Latheesh kumar and Anil Kumar, 2016 The total domination polynomial of a connected bipartite graph G is the product of the vertex cover polynomials of the two components of H_G, while the total domination polynomial of a connected graph that is not bipartite is the vertex cover polynomial of H_G.*

Let P_n be the path $(1, 2, \ldots, n)$. Then $P_n^{'}$ is the graph with vertex set $V(Pn^{'}) = V(P_n)$ and edge set $E(Pn^{'}) = E(P_n) \cup \{11\}$. Let $P_n^{''}$ is the graph with vertex set $V(Pn^{''}) = V(P_n)$ and edge set $E(Pn^{'}) = E(P_n) \cup \{11, nn\}$.

Lemma 1.9. latheesh2017 *For the graph $P_n^{'}$ and $P_n^{''}$, we have*

$$\begin{aligned} \mathcal{C}(P_n^{'}, x) &= x\, \mathcal{C}(P_{n-1}, x) \\ \mathcal{C}(P_n^{''}, x) &= x^2\, \mathcal{C}(P_{n-2}, x). \end{aligned}$$

2 Main Results

Definition 2.1. Let G be a graph and A be a subset of $V(G)$. Let $\mathcal{C}^A(G,x),\ \mathcal{C}^{A^*}(G,x)$ and $\mathcal{C}_A(G,x)$ be polynomials in which the coefficient of x^i is the number of vertex covering sets of cardinality i containing at least one vertex from A, all vertices from A and no vertex from A respectively.

Lemma 2.1. *If $1, 2, \ldots, n$ are the vertices of the path P_n, then*

$$(i)\ \mathcal{C}^{\{1\}}(P_n, x) = x\, \mathcal{C}(P_{n-1}, x).$$
$$(ii)\ \mathcal{C}_{\{1\}}(P_n, x) = x\, \mathcal{C}(P_{n-1}, x).$$
$$(iii)\ \mathcal{C}^{\{1,n\}^*}(P_n, x) = x^2\, \mathcal{C}(P_{n-2}, x).$$
$$(iv)\ \mathcal{C}_{\{1,n\}}(P_n, x) = x^2\, \mathcal{C}(P_{n-4}, x).$$

Proof. **(i)** Note that S is a vertex covering set of P_n containing the vertex 1 if and only if S is a vertex covering set of the graph $P_n^{'}$ shown in figure 1.

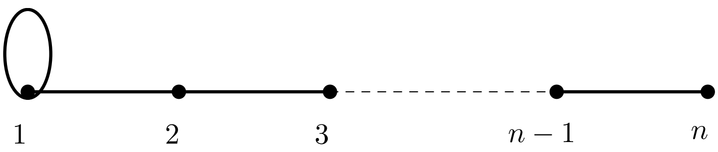

Figure 1: The graph $P_n^{'}$

Therefore the proof follows from Theorem 1.3.

(ii) If S is a vertex covering set of P_n and $S \cap \{1\} = \phi$, then $2 \in S$. So S is a vertex covering set of the graph K shown in figure 2. Therefore from Theorem 1.3 the result follows.

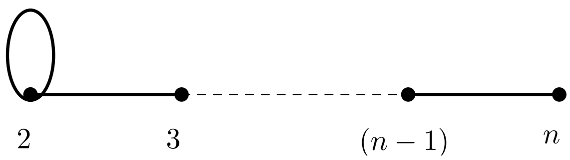

Figure 2: The graph K.

(iii) If S is a vertex covering set of P_n containing the vertices 1 and n, then S is a vertex covering set of the graph $P_n^{''}$ shown in figure 3.

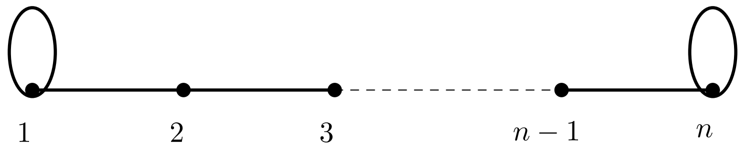

Figure 3: The graph $P_n^{''}$

Therefore the proof follows from theorem 1.3.

(iv) Let S be a vertex covering set of P_n such that $S \cap \{1, n\} = \phi$, then S is a vertex covering set of the graph shown in figure 4.

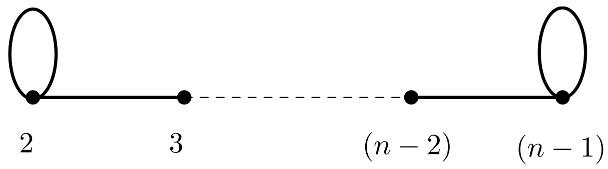

Figure 4: The graph K_1

Therefore from Theorem 1.3 the proof follows. □

Next, we find the total domination polynomial of the tree T_{n_1,n_2,n_3}.

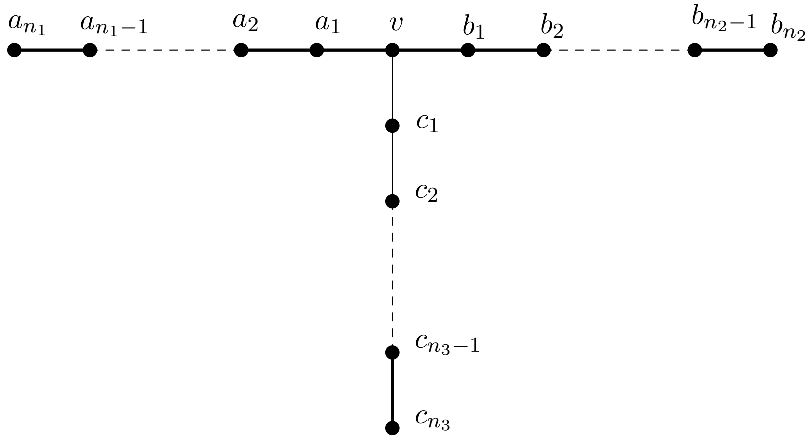

Figure 5: The tree T_{n_1,n_2,n_3}.

Theorem 2.2. *If n_1, n_2, n_3 are even and T_1, T_2 be the components of the open neighbourhood hypergraph of the tree T_{n_1,n_2,n_3}, then*

$$\mathcal{C}(T_1, x) = x\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}}, x) + x^3\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-1}, x) \text{ and}$$

$$\mathcal{C}(T_2, x) = x^4\left[(x+1)^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-2}, x) + (x+2)\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-1}, x)\right]$$

Proof. Let $X = \{x_i : i \text{ is odd}\}$ and $Y = \{y_j : j \text{ is even}\} \cup \{v\}$ be the partite sets of T_{n_1,n_2,n_3}. Let T_1 and T_2 be the components of the open neighbourhood hypergraph of T_{n_1,n_2,n_3}, such that $E(T_1) = \{N(x) : x \in X\}$ and $E(T_2) = \{N(y) : y \in Y\}$. Then T_1 can be represented as shown in figure 6.

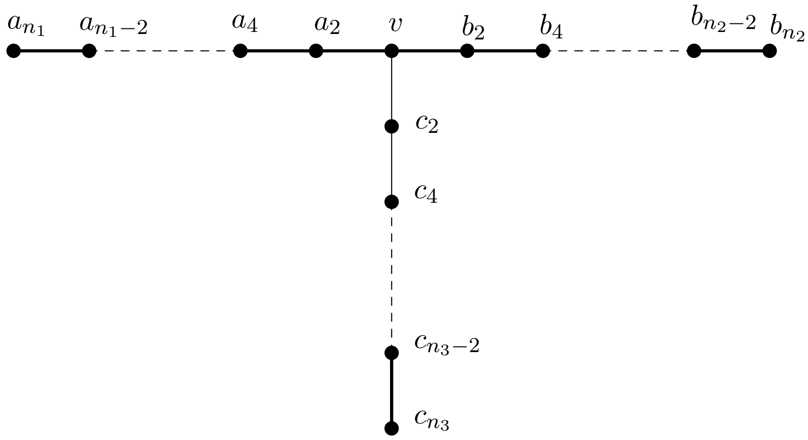

Figure 6: The graph T_1.

By Theorem 1.4, we have

$$\begin{aligned}\mathcal{C}(T_1, x) &= x\mathcal{C}(T_1 - v, x) + x^3\mathcal{C}(T_1 - v - \{a_2, b_2, c_2\}, x) \\ &= \prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i}{2}}, x) + x^3 \prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i}{2}-1}, x).\end{aligned}$$

Next, we find the vertex cover polynomial of T_2. It can be observed that $E(T_2) = \{a_1, b_1, c_1\} \cup E(T_a) \cup E(T_b) \cup E(T_c)$, where the graphs T_a, T_b and T_c are shown in figure 7. Let $A = \{a_1, b_1, c_1\}$. Note that a set S is vertex covering set of T_2 if and only if $S \cap A \neq \phi$ and S is a vertex covering set of $T_a \cup T_b \cup T_c$. From Theorem 1.5 $\mathcal{C}(T_a \cup T_b \cup T_c, x) = \mathcal{C}(T_a, x)\mathcal{C}(T_b, x)\mathcal{C}(T_c, x)$. Therefore to compute the vertex cover polynomial of T_2 we need to consider the following disjoint cases only. We compute the the vertex cover polynomials using Lemma 1.9, Lemma 2.1 and Theorem 1.3.

Case 1: If $S \cap A = A$, we get

$$\begin{aligned}\mathcal{C}(T_a, x)\mathcal{C}(T_b, x)\mathcal{C}(T_c, x) &= x^2\mathcal{C}(P_{\frac{n_1}{2}-2}, x)x^2\mathcal{C}(P_{\frac{n_2}{2}-2}, x)x^2\mathcal{C}(P_{\frac{n_3}{2}-2}, x) \\ &= x^6\mathcal{C}(P_{\frac{n_1}{2}-2}, x)\mathcal{C}(P_{\frac{n_2}{2}-2}, x)\mathcal{C}(P_{\frac{n_3}{2}-2}, x).\end{aligned}$$

Case 2: If $S \cap A = \{a_1, b_1\}$, proceeding as in Case 1, we get

$$\begin{aligned}\mathcal{C}(T_a, x)\mathcal{C}(T_b, x)\mathcal{C}(T_c, x) &= x^2\mathcal{C}(P_{\frac{n_1}{2}-2}, x)x^2\mathcal{C}(P_{\frac{n_2}{2}-2}, x)x\mathcal{C}(P_{\frac{n_3}{2}-1}, x) \\ &= x^5\mathcal{C}(P_{\frac{n_1}{2}-2}, x)\mathcal{C}(P_{\frac{n_2}{2}-2}, x)\mathcal{C}(P_{\frac{n_3}{2}-1}, x).\end{aligned}$$

Case 3: Similarly if $S \cap A = \{a_1, c_1\}$, we get

$$\begin{aligned}\mathcal{C}(T_a, x)\mathcal{C}(T_b, x)\mathcal{C}(T_c, x) &= x^2\mathcal{C}(P_{\frac{n_1}{2}-2}, x)x\mathcal{C}(P_{\frac{n_2}{2}-1}, x)x^2\mathcal{C}(P_{\frac{n_3}{2}-2}, x) \\ &= x^5\mathcal{C}(P_{\frac{n_1}{2}-2}, x)\mathcal{C}(P_{\frac{n_2}{2}-1}, x)\mathcal{C}(P_{\frac{n_3}{2}-2}, x).\end{aligned}$$

Case 4: If $S \cap A = \{b_1, c_1\}$, proceeding as in Case 3, we get

$$\begin{aligned} \mathcal{C}(T_a,x)\mathcal{C}(T_b,x)\mathcal{C}(T_c,x) &= x\mathcal{C}(P_{\frac{n_1}{2}-1},x)x^2\mathcal{C}(P_{\frac{n_2}{2}-2},x)x^2\mathcal{C}(P_{\frac{n_3}{2}-2},x) \\ &= x^5\mathcal{C}(P_{\frac{n_1}{2}-1},x)\mathcal{C}(P_{\frac{n_2}{2}-2},x)\mathcal{C}(P_{\frac{n_3}{2}-2},x). \end{aligned}$$

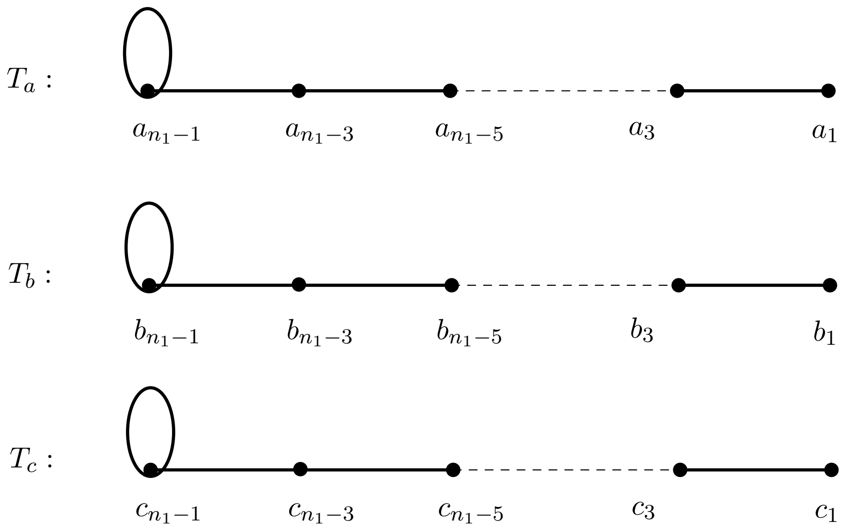

Figure 7: The Graphs T_a, T_b and T_c.

Case 5: If $S \cap A = \{a_1\}$, we get

$$\begin{aligned} \mathcal{C}(T_a,x)\mathcal{C}(T_b,x)\mathcal{C}(T_c,x) &= x^2\mathcal{C}(P_{\frac{n_1}{2}-2},x)x\mathcal{C}(P_{\frac{n_2}{2}-1},x)x\mathcal{C}(P_{\frac{n_3}{2}-1},x) \\ &= x^4\mathcal{C}(P_{\frac{n_1}{2}-2},x)\mathcal{C}(P_{\frac{n_2}{2}-1},x)\mathcal{C}(P_{\frac{n_3}{2}-1},x). \end{aligned}$$

Similarly, we get the results of Case 6 and 7 as given below.

Case 6: If $S \cap A = \{b_1\}$, we get

$$\begin{aligned} \mathcal{C}(T_a,x)\mathcal{C}(T_b,x)\mathcal{C}(T_c,x) &= x\mathcal{C}(P_{\frac{n_1}{2}-1},x)x^2\mathcal{C}(P_{\frac{n_2}{2}-2},x)x\mathcal{C}(P_{\frac{n_3}{2}-1},x) \\ &= x^4\mathcal{C}(P_{\frac{n_1}{2}-1},x)\mathcal{C}(P_{\frac{n_2}{2}-2},x)\mathcal{C}(P_{\frac{n_3}{2}-1},x). \end{aligned}$$

Case 7: If $S \cap A = \{c_1\}$, we get

$$\begin{aligned} \mathcal{C}(T_a,x)\mathcal{C}(T_b,x)\mathcal{C}(T_c,x) &= x\mathcal{C}(P_{\frac{n_1}{2}-1},x)x\mathcal{C}(P_{\frac{n_2}{2}-1},x)x\mathcal{C}(P_{\frac{n_3}{2}-2},x) \\ &= x^4\mathcal{C}(P_{\frac{n_1}{2}-1},x)\mathcal{C}(P_{\frac{n_2}{2}-1},x)\mathcal{C}(P_{\frac{n_3}{2}-2},x). \end{aligned}$$

Therefore adding the expressions in the above cases, we obtain

$$\mathcal{C}(T_2,x) = x^4\left[(x+1)^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-2},x) + (x+2)\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i}{2}-1},x)\right]$$

This completes the proof. □

Theorem 2.3. *If n_1, n_2, n_3 are even and T_1, T_2 be the components of the open neighbourhood hypergraph of the tree T_{n_1,n_2,n_3}, then*

$$D_t(T_{n_1,n_2,n_3}, x) = \mathcal{C}(T_1, x)\mathcal{C}(T_2, x).$$

Proof. The proof follows immediately from Theorem 1.8. □

Corollary 2.4. *If $n_1 = n_2 = n_3 = 2n$, then $D_t(T_{n_1,n_2,n_3}, x) = x^7(x+1)^2$ $[\mathcal{C}(P_{n-1}, x)\mathcal{C}(P_{n-2}, x)]^3 + x^7(x+2)\,[\mathcal{C}(P_{n-1}, x)]^6 + x^5(x+1)^2\,[\mathcal{C}(P_n, x)\mathcal{C}(P_{n-2}, x)]^3$ $+\, x^5(x+2)\,[\mathcal{C}(P_n, x)\mathcal{C}(P_{n-1}, x)]^3$.*

Proof. The proof follows from Theorem 2.2 and 2.3. □

Theorem 2.5. *If n_1, n_2, n_3 are odd and T_1, T_2 are the components of the open neighbourhood hypergraph of the tree T_{n_1,n_2,n_3}, then we have*

$$\mathcal{C}(T_1, x) = x^4\left[\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-1}{2}-1}, x) + x^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-1}{2}-2}, x)\right];$$

$$\mathcal{C}(T_2, x) = x(x+1)^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-1}{2}}, x) + x(x+2)\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i+1}{2}}, x).$$

Proof. Proceeding as in Theorem 2.2, we can represent T_1 as shown in figure 8.

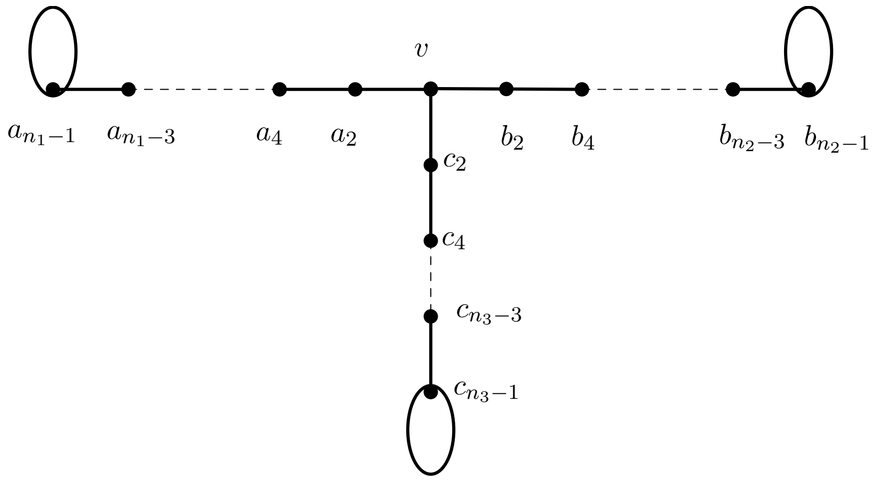

Figure 8: The Graph T_1.

Let $T_1^* = T_1 - \{a_{n_1-1}, b_{n_2-1}, c_{n_3-1}\}, T_1^{**} = T_1 - \{v, a_{n_1-1}, b_{n_2-1}, c_{n_3-1}\}$, and $T_1^{***} = T_1 - \{v, a_2, b_2, c_2, a_{n_1-1}, b_{n_2-1}, c_{n_3-1}\}$. Then from Theorem 1.3 and 1.4, we get,

$$\begin{aligned}\mathcal{C}(T_1, x) &= x^3\mathcal{C}\left(T_1^*, x\right)\\ &= x^3\left[x\mathcal{C}\left(T_1^{**}, x\right) + x^3\mathcal{C}\left(T_1^{***}, x\right)\right]\\ &= x^4\left[\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-1}{2}-1}, x) + x^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-1}{2}-2}, x)\right].\end{aligned}$$

Let $P_{\frac{n_1+1}{2}} = (a_1, a_3, a_5, \ldots, a_{n_1-2}, a_{n_1})$, $P_{\frac{n_2+1}{2}} = (b_1, b_3, b_5, \ldots, b_{n_2-2}, b_{n_2})$ and $P_{\frac{n_3+1}{2}} = (c_1, c_3, c_5, \ldots, c_{n_3-2}, c_{n_3})$ be three paths. Then the edge set of the graph T_2 is $E(T_2) = \{a_1, b_1, c_1\} \cup E(P_{\frac{n_1+1}{2}}) \cup E(P_{\frac{n_2+1}{2}}) \cup E(P_{\frac{n_3+1}{2}})$. Let $A = \{a_1, b_1, c_1\}$. A set S is vertex covering set of T_2 if and only if $S \cap A \neq \phi$ and S is a vertex covering set of $P_{\frac{n_1+1}{2}} \cup P_{\frac{n_2+1}{2}} \cup P_{\frac{n_3+1}{2}}$. From Theorem 1.5 $\mathcal{C}(P_{\frac{n_1+1}{2}} \cup P_{\frac{n_2+1}{2}} \cup P_{\frac{n_3+1}{2}}, x) = \prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x)$ Therefore to compute $\mathcal{C}(T_2, x)$ it is enough to consider the following disjoint cases only.

Case 1: If $S \cap A = A$, using Lemma 1.9 and Theorem 1.3, we get

$$\begin{aligned}\prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x) &= x\mathcal{C}(P_{\frac{n_1-1}{2}}, x)x\mathcal{C}(P_{\frac{n_2-1}{2}}, x)x\mathcal{C}(P_{\frac{n_3-1}{2}}, x) \\ &= x^3 \prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i-1}{2}}, x).\end{aligned}$$

Case 2: If $S \cap A = \{a_1, b_1\}$, proceeding as in Case 1, we get

$$\begin{aligned}\prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x) &= x\mathcal{C}(P_{\frac{n_1-1}{2}}, x)x\mathcal{C}(P_{\frac{n_2-1}{2}}, x)\mathcal{C}(P_{\frac{n_3+1}{2}}, x) \\ &= x^2\mathcal{C}(P_{\frac{n_1-1}{2}}, x)\mathcal{C}(P_{\frac{n_2-1}{2}}, x)\mathcal{C}(P_{\frac{n_3+1}{2}}, x).\end{aligned}$$

Case 3: Similarly if $S \cap A = \{a_1, c_1\}$, we get

$$\begin{aligned}\prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x) &= x\mathcal{C}(P_{\frac{n_1-1}{2}}, x)\mathcal{C}(P_{\frac{n_2+1}{2}}, x)x\mathcal{C}(P_{\frac{n_3+1}{2}}, x) \\ &= x^2\mathcal{C}(P_{\frac{n_1-1}{2}}, x)\mathcal{C}(P_{\frac{n_2+1}{2}}, x)\mathcal{C}(P_{\frac{n_3+1}{2}}, x).\end{aligned}$$

Case 4: If $S \cap A = \{b_1, c_1\}$, proceeding as in Case 3, we get

$$\begin{aligned}\prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x) &= \mathcal{C}(P_{\frac{n_1+1}{2}}, x)x\mathcal{C}(P_{\frac{n_2-1}{2}}, x)x\mathcal{C}(P_{\frac{n_3-1}{2}}, x) \\ &= x^2\mathcal{C}(P_{\frac{n_1+1}{2}}, x)\mathcal{C}(P_{\frac{n_2-1}{2}}, x)\mathcal{C}(P_{\frac{n_3-1}{2}}, x).\end{aligned}$$

Case 5: If $S \cap A = \{a_1\}$, we get

$$\begin{aligned}\prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x) &= x\mathcal{C}(P_{\frac{n_1-1}{2}}, x)\mathcal{C}(P_{\frac{n_2+1}{2}}, x)\mathcal{C}(P_{\frac{n_3+1}{2}}, x) \\ &= x\mathcal{C}(P_{\frac{n_1-1}{2}}, x)\mathcal{C}(P_{\frac{n_2+1}{2}}, x)\mathcal{C}(P_{\frac{n_3+1}{2}}, x).\end{aligned}$$

Similarly, we get the results of Case 6 and 7 as given below.

Case 6: If $S \cap A = \{b_1\}$, we get

$$\begin{aligned}\prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x) &= \mathcal{C}(P_{\frac{n_1+1}{2}}, x)x\mathcal{C}(P_{\frac{n_2-1}{2}}, x)\mathcal{C}(P_{\frac{n_3+1}{2}}, x) \\ &= x\mathcal{C}(P_{\frac{n_1+1}{2}}, x)\mathcal{C}(P_{\frac{n_2-1}{2}}, x)\mathcal{C}(P_{\frac{n_3+1}{2}}, x).\end{aligned}$$

Case 7: If $S \cap A = \{c_1\}$, we get

$$\begin{aligned}\prod_{i=1}^{3} \mathcal{C}(P_{\frac{n_i+1}{2}}, x) &= \mathcal{C}(P_{\frac{n_1+1}{2}}, x)\mathcal{C}(P_{\frac{n_2+1}{2}}, x)x\mathcal{C}(P_{\frac{n_3-1}{2}}, x) \\ &= x\mathcal{C}(P_{\frac{n_1+1}{2}}, x)\mathcal{C}(P_{\frac{n_2+1}{2}}, x)\mathcal{C}(P_{\frac{n_3-1}{2}}, x).\end{aligned}$$

Therefore, $\mathcal{C}(T_2, x)$ is obtained by adding the vertex cover polynomials in the above cases. So $\mathcal{C}(T_2, x) = x(x+1)^2\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i-1}{2}}, x) + x(x+2)\prod_{i=1}^{3}\mathcal{C}(P_{\frac{n_i+1}{2}}, x)$. □

Theorem 2.6. *If n_1, n_2, n_3 are odd and T_1, T_2 be the components of the open neighbourhood hypergraph of the tree T_{n_1,n_2,n_3}, then*

$$D_t(T_{n_1,n_2,n_3}, x) = \mathcal{C}(T_1, x)\mathcal{C}(T_2, x).$$

Proof. The proof follows immediately from Theorem 1.8. □

Corollary 2.7. *If $n_1 = n_2 = n_3 = 2n+1$, then the TD- Polynomial of the tree T_{n_1,n_2,n_3} is* $D_t(T_{n_1,n_2,n_3}, x) = x^5(x+1)^2\left[\mathcal{C}(P_n, x)\mathcal{C}(P_{n-1}, x)\right]^3 + x^5(x+2)\left[\mathcal{C}(P_{n+1}, x)\mathcal{C}(P_{n-1}, x)\right]^3 + x^7(x+1)^2\left[\mathcal{C}(P_n, x)\mathcal{C}(P_{n-2}, x)\right]^3 + x^7(x+2)\left[\mathcal{C}(P_{n+1}, x)\mathcal{C}(P_{n-2}, x)\right]^3$.

Proof. The proof follows from Theorem 2.5 and 2.6. □

3 CONCLUSIONS

In this paper the relation between total domination sets and vertex covering sets is used to determine the total domination polynomial of differend classes of graphs.

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