Original Research Article

Corona Product of three molecular descriptors

Abstract

In this manuscript, we investigate the behavior of S index, Y and S coindices under the different corona products of graphs. We consider the corona product of two simple connected graphs A and B, and derived closed-form expressions for the S index, Y and S coindices of graphs in terms of the indices and coindices and structural properties of the constituent graphs. This analysis generalizes previous results and provides new insights into the additive and multiplicative behavior of S index, Y and S coindices under graph operations.

Keywords: S Index; Y coindex; Subdivision-vertex corona

2010 Mathematics Subject Classification: 05C07; 05C38; 05C76

1 Introduction

The general Zagreb index is a broad generalization of the classical Zagreb indices (Khalifeh, et al., 2009), widely used in mathematical chemistry and graph theory to characterize molecular structures. Defined for a simple graph A=(V,E), the general Zagreb index (Melaku Berhe Belay, & Chunxiang Wang, 2020) is defined by

$$M^{\lambda+1}(A) = \sum_{uv\in E(A)}(deg^{\lambda}(u) + deg^{\lambda}(v)), \tag{1.1}$$

where $deg(u)$ and $deg(v)$ denote the degrees of the adjacent vertices u and v. Since, this general form allows to analyze a wide spectrum of structural descriptors by choosing different functions. For example, the classical first Zagreb, F, Y and S indices (Nagarajan & Kayalvizhi, 2023) are obtained by setting n=1,2,3 and 4, respectively.The flexibility of the general Zagreb index makes it a powerful tool for capturing various topological and chemical properties of graphs, enabling more analysis than its classical counterparts.

The corona product of two graphs is a graph operation that has been extensively studied due to its structural complexity and wide range of applications in network theory, chemistry, and computer science. Given two simple graphs A and B of order n_i and size m_i respectively, where $i = 1, 2.$ The corona product $A \odot B$ is formed by taking one copy of A and $|V(A)|$ copies of B, then joining the each vertex of A to all the vertices in a corresponding copy of B. This construction results in a graph with rich combinatorial properties. In particular, studying the S index, Y and S coindices on corona products helps deepen our understanding of how such indices behave under graph transformations. This encompasses a wide class of degree-based descriptors, provides a flexible framework for analyzing the structural characteristics of corona graphs. Investigating its behavior under the corona operation not only generalizes previous results related to S index, Y and S coindices but also offers new insights into the interplay between local and global graph properties. In this manuscript, we proceed to see about the different types of corona product of graphs such as subdivision-vertex corona, subdivision-edge corona, subdivision-vertex neighborhood corona, subdivision-edge neighborhood corona and vertex edge corona

2 Subdivision-vertex corona

The subdivision vertex corona is the graph operation that combines features of both subdivision and corona products, creating a complex structure with an enhanced analytical properties. Given two simple connected graphs A and B, the subdivision vertex corona, denoted by $A \odot_s B$, is obtained by first subdividing every edge of A (i.e., inserting a new vertex into each edge), and then attaching a copy of B to each of these newly inserted subdivision vertices. Specifically, each subdivision vertex becomes adjacent to all the other vertices in its corresponding copy of B, while the original vertices of A remains unchanged. This construction increases the size and complexity of original graph significantly, making it particularly useful for exploring the topological indices in a more intricate setting. The Subdivision-vertex corona $A \odot_s B$ has order $m_1 + n_1 + n_1 n_2$ and size $2m_1 + n_1 n_2 + n_1 m_2$. Then, the degree of the vertices in $A \odot_s B$ are given by

$$\begin{aligned} deg_{A \odot_s B}(u_i) &= deg_A(u_i) + n_2 \; where \; i = 1, 2, \ldots, n_1 \\ deg_{A \odot_s B}(e_i) &= 2 \; where \; i = 1, 2, \ldots, m_1 \\ deg_{A \odot_s B}(u_j^i) &= deg_B(v_j) + 1 \; where \; i = 1, 2, \ldots, n_1 \; and \; j = 1, 2, \ldots, n_2 \end{aligned}$$

Theorem 2.1. *The Subdivision-vertex corona $A \odot_s B$ of S index is given by*

$$\begin{aligned} S(A \odot_s B) &= S(A) + n_1 n_2^5 + 5n_2 Y(A) + 10n_2^2 F(A) + 10n_2^3 M_1(A) + 10n_2^4 m_1 + 32m_1 \\ &\quad + n_1(S(B) + n_2 + 5Y(B) + 10F(B) + 10M_1(B) + 10m_2 \end{aligned}$$

Proof. By the definition of Subdivision-vertex corona $A \odot_s B$, we get that

$$\begin{aligned} S(A \odot_s B) &= \sum_{i=1}^{n_1} (deg_A(u_i) + n_2)^5 + \sum_{i=1}^{m_1} 2^5 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (deg_B(v_j) + 1)^5 \\ &= \sum_{i=1}^{n_1} ((deg_A(u_i))^5 + {n_2}^5 + 5n_2 deg_A(u_i))^4 + 10(deg_A(u_i))^3 n_2^2 + 10(deg_A(u_i))^2 n_2^3 \\ &\quad + 5n_2^4 deg_A(u_i)) + 32m_1 + n_1 \sum_{j=1}^{n_2} ((deg_B(v_j))^5 + 1^5 + 5deg_B(v_j))^4 \\ &\quad + 10(deg_B(v_j))^3 + 10(deg_B(v_j)^2) + 5deg_B(v_j)) \\ S(A \odot_s B) &= S(A) + n_1 n_2^5 + 5n_2 Y(A) + 10n_2^2 F(A) + 10n_2^3 M_1(A) + 10n_2^4 m_1 + 32m_1 \\ &\quad + n_1(S(B) + n_2 + 5Y(B) + 10F(B) + 10M_1(B) + 10m_2 \end{aligned}$$

Hence, we obtained the required outcome. ☐

Theorem 2.2. *The Subdivision-vertex corona $A \odot_s B$ of Y coindex is given by*

$$\begin{aligned}\overline{Y}(A \odot_s B) &= (\Phi(n) - 4n_2 - 1)F(A) + n_1(\Phi(n) - 5)F(B) + 3n_2(\Phi(n) - 1 - 2n_2)M_1(A) \\ &\quad +3n_1(\Phi(n) - 3)M_1(B) - Y(A) - n_1Y(B) + \chi(n)\end{aligned}$$

where

$$\begin{aligned}\Phi(n) &= n_1n_2 + n_1 + m_1 \\ and\ \chi(n) &= 6(\Phi(n) - 1)n_2^2m_1 + 8(\Phi(n) - 3)m_1 + \\ &\quad (\Phi(n) - 2)(n_1n_2(n_2^2 + 1)) + 2(3(\Phi(n) - 1) - 4) \\ &\quad n_1m_2 - 8m_1n_2^3\end{aligned}$$

Proof. By, the Relationship between the topological indices and coindices (Alameri, et al., 2020), we can obtain that

$$\overline{Y}(A \odot_s B) = (n - 1)F(A \odot_s B) - Y(A \odot_s B) \tag{2.1}$$

Since, $n = m_1 + n_1 + n_1n_2$ and by (Nilanjan De, 2017; Nagarajan & Durga, 2023), we get that

$$\begin{aligned}F(A \odot_s B) &= F(A) + 3n_2M_1(A) + 6n_2m_1 + n_1n_2^3 + 8m_1 + n_1(F(B) + 3M_1(B) + 6m_2 + n_2) \\ Y(A \odot_s B) &= Y(A) + 4n_2F(A) + 6n_2^2M_1(A) + 8m_1n_2^3 + n_1n_2^4 + 16m_1 + n_1Y(B) \\ &\quad +4n_1F(B) + 6n_1M_1(B) + 8n_1m_2 + n_1n_2\end{aligned}$$

Substituting these values in the equation 2.1, we can obtain the required outcome. □

Theorem 2.3. *The Subdivision-vertex corona $A \odot_s B$ of S coindex is given by*

$$\begin{aligned}\overline{S}(A \odot_s B) &= (\Phi(n) - 5n_2 - 1)Y(A) + n_1(\Phi(n) - 6)Y(B) + 2n_2(2(\Phi(n) - 1) - 5n_2)F(A) \\ &\quad +2n_1(2(\Phi(n) - 1) - 5)F(B) + 2n_2^2(3(\Phi(n) - 1) - 5n_2)M_1(A) \\ &\quad +2n_1(3(\Phi(n) - 1) - 5)M_1(B) - S(A) - n_1S(B) + \zeta(n)\end{aligned}$$

where

$$\begin{aligned}\zeta(n) &= (\phi(n) - 1)(8m_1n_2^3 + n_1n_2^4 + 16m_1 + 8n_1m_2 + n_1n_2) \\ &\quad -n_1n_2^5 - 10n_2^4m_1 - 32m_1 - n_1n_2 - 10m_2n_1\end{aligned}$$

Proof. By, the Relationship between S index and coindex (Nagarajan & Kayalvizhi, 2023),

$$\overline{S}(A \odot_s B) = (n - 1)Y(A \odot_s B) - S(A \odot_s B) \tag{2.2}$$

The conclusion is reached through the application of Theorem 2.1 and Theorem 2.2. □

Corollary 2.4. *By using Theorem 2.1, we can obtain the following results for path graphs and cycle graphs.*

$$\begin{aligned}S(P_n \odot_s P_m) &= nm^5 + 80m^2n + 40m^3n + 10m^4n - 140m^2 - 60m^3 \\ &\quad -10m^4 + 323mn - 358n - 150m - 94 \\ S(C_n \odot_s C_m) &= nm^5 + 10nm^4 + 40nm^3 + 80nm^2 + 323nm + 64n \\ S(C_n \odot_s P_m) &= nm^5 + 10nm^4 + 40nm^3 + 80nm^2 + 323nm - 358n\end{aligned}$$

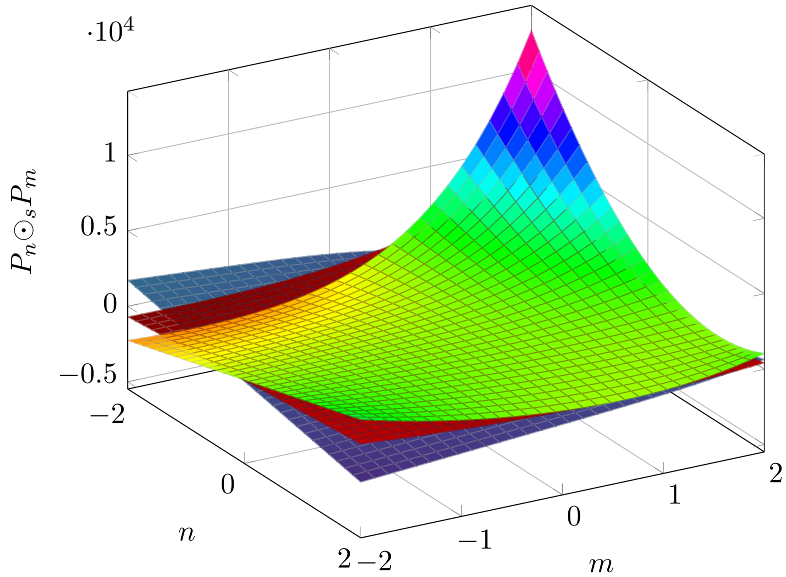

Figure 1: Comparison of $S(P_n \odot_s P_m), \overline{Y}(P_n \odot_s P_m)$ *and* $\overline{S}(P_n \odot_s P_m)$

3 Subdivision-edge corona

The subdivision-edge corona is a graph operation that merges the principles of edge subdivision and the corona product to produce a graph with enhanced structural intricacy. Given two simple graphs A and B, the subdivision-edge corona, denoted by $A \star_s B$, is constructed by first subdividing each edge of A, introducing a new vertex into every edge. Then, for the each of these subdivision of the vertices, a copy of graph B is attached by connecting subdivision vertex to every vertex in its corresponding copy of B. Hence, this operation results in a significant increase in the number of vertices and edges, which generates a graph with a layered and modular architecture.
The Subdivision-edge corona $A \star_s B$ has order $n_1 + m_1 + m_1 n_2$ and size $2m_1 + m_1 n_2 + m_1 m_2$. Then, the degree of the vertices in $A \star_s B$ are given by

$$\begin{aligned} deg_{A\star_s B}(u_i) &= deg_A(u_i) \; where \; i = 1, 2, \ldots, n_1 \\ deg_{A\star_s B}(e_i) &= 2 + n_2 \; where \; i = 1, 2, \ldots, m_1 \\ deg_{A\star_s B}(u_j^i) &= deg_B(v_j) + 1 \; where \; i = 1, 2, \ldots, m_1 \; and \; j = 1, 2, \ldots, n_2 \end{aligned}$$

Theorem 3.1. *The Subdivision-edge corona $A \star_s B$ of S index is given by*

$$S(A\star_s B) = S(A) + m_1(S(B) + n_2 + 5Y(B) + 10m_2 + 10M_1(B) + 10F(B) + (n_2+2)^5)$$

Proof. By the definition of Subdivision-edge corona $A \star_s B$, we get that

$$\begin{aligned} S(A\star_s B) &= \sum_{i=1}^{n_1} (deg_A(u_i))^5 + \sum_{i=1}^{m_1} (2+n_2)^5 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} (deg_B(v_j)+1)^5 \\ &= S(A) + m_1(n_2+2)^5 + m_1 \sum_{j=1}^{n_2} ((deg_B(v_j))^5 + 1^5 + 5deg_B(v_j)^4 \\ &\quad +10(deg_B(v_j))^3 + 10(deg_B(v_j)))^2 + 5deg_B(v_j)) \\ S(A\star_s B) &= S(A) + m_1(S(B) + n_2 + 5Y(B) + 10m_2 + 10M_1(B) + 10F(B) + (n_2+2)^5) \end{aligned}$$

Hence, this completes the proof. □

Theorem 3.2. *The Subdivision-edge corona* $A\star_s B$ *of Y coindex is given by*

$$\begin{aligned}\overline{Y}(A\star_s B) &= (\rho-1)F(A)+(n_1(\rho-1)-4m_1)F(B)+3(n_1(\rho-1)-2m_1)M_1(B)-Y(A)\\ &\quad -m_1Y(B)+n_1(\rho-1)(6m_2+n_2+(n_2+2)^3)-(m_1(n_2+2))^4-8m_1m_2-m_1n_2\end{aligned}$$

where $\rho = n_1+m_1+m_1n_2$

Proof. By, the Relationship between the topological indices and coindices, we can obtain that

$$\overline{Y}(A\star_s B) = (n-1)F(A\star_s B) - Y(A\star_s B) \tag{3.1}$$

Since, $n = n_1+m_1+m_1n_2$ and by (Nilanjan De, 2017; Nagarajan & Durga, 2023), we get that

$$\begin{aligned}F(A\star_s B) &= F(A)+m_1F(B)+3m_1M_1(B)+6m_1m_2+m_1(n_2+2)^3+m_1n_2\\ Y(A\star_s B) &= Y(A)+(m_1(n_2+2))^4+m_1Y(B)+4m_1F(B)+6m_1M_1(B)+8m_1m_2+m_1n_2\end{aligned}$$

Substituting these values in the equation 3.1, we can obtain the required outcome. □

Theorem 3.3. *The Subdivision-edge corona* $A\star_s B$ *of S coindex is given by*

$$\begin{aligned}\overline{S}(A\star_s B) &= (\rho-1)Y(A)+m_1(\rho-6)Y(B)+2m_1(2(\rho-1)-5)F(B)\\ &\quad +2m_1(3(\rho-1)-5)M_1(B)-S(A)-m_1S(B)+(\rho-1)((m_1(n_2+2))^4\\ &\quad +8m_1m_2+m_1n_2)-m_1(n_2+2)^5-10m_1m_2\end{aligned}$$

Proof. By, the Relationship between S index and coindex,

$$\overline{S}(A\star_s B) = (n-1)Y(A\star_s B) - S(A\star_s B) \tag{3.2}$$

Consequently, the result follows from the application of Theorem 3.1 and Theorem 3.2. □

Corollary 3.4. *By using Theorem 3.1, we can obtain the following results for path graphs and cycle graphs.*

$$\begin{aligned}S(P_n\star_s P_m) &= m^5(n-1)+10m^4(n-1)+40m^3(n-1)+80m^2(n-1)\\ &\quad +323m(n-1)-358n+328\\ S(C_n\star_s C_m) &= nm^5+10nm^4+40nm^3+80nm^2+323nm+64n\\ S(C_n\star_s P_m) &= nm^5+10nm^4+40nm^3+80nm^2+323nm-358n\end{aligned}$$

4 Subdivision-vertex neighborhood corona

The subdivision vertex neighborhood corona is a complex graph operation that builds on the concepts of edge subdivision of graph and vertex neighborhood expansion to create a highly structured graph. Given two simple graphs A and B, the subdivision vertex neighborhood corona, denoted by $A \oplus_s B$, is formed by the first subdividing every edge of A, introducing a new vertex into each edge. Then, for each vertex u in A, a copy of B is attached to all the neighbors of u in A(including the new subdivision vertices adjacent to u) are connected to every vertex in the corresponding copy of B. This operation results in a densely connected structure that significantly increases the degree complexity and size of the graph.

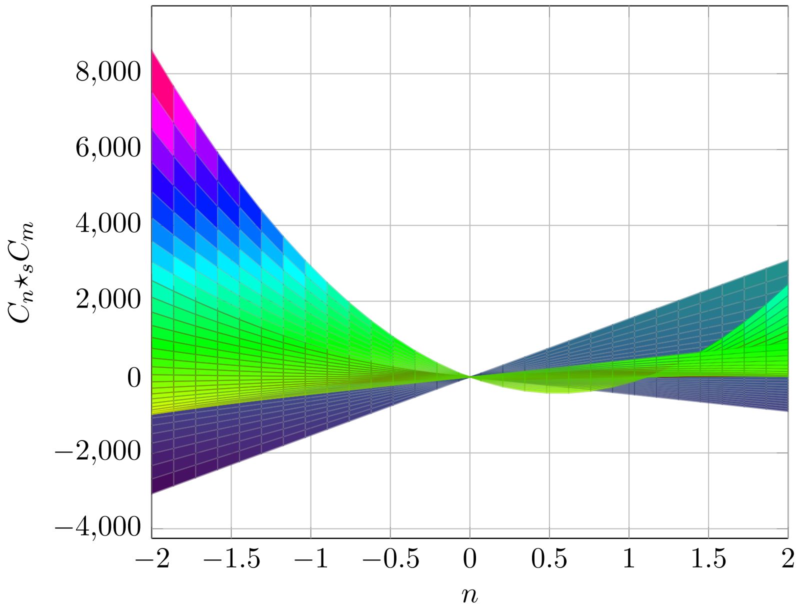

Figure 2: Comparison of $S(C_n \star_s C_m), \overline{Y}(C_n \star_s C_m)$ and $\overline{S}(C_n \star_s C_m)$

The Subdivision-vertex neighborhood corona $A \oplus_s B$ has order $n_1 + m_1 + n_1 n_2$ and size $2m_1 + 2m_1 n_2 + n_1 m_2$. Then, the degree of the vertices in $A \oplus_s B$ are given by

$$\begin{aligned}
deg_{A\oplus_s B}(u_i) &= deg_A(u_i) \text{ } where \text{ } i = 1, 2, \ldots, n_1 \\
deg_{A\oplus_s B}(e_i) &= 2 + 2n_2 \text{ } where \text{ } i = 1, 2, \ldots, m_1 \\
deg_{A\oplus_s B}(u_j^i) &= deg_A(u_i) + deg_B(v_j) \text{ } where \text{ } i = 1, 2, \ldots, n_1 \text{ } and \text{ } j = 1, 2, \ldots, n_2
\end{aligned}$$

Theorem 4.1. *The Subdivision-vertex neighborhood corona $A \oplus_s B$ of S index is given by*

$$\begin{aligned}
S(A \oplus_s B) &= S(A) + 32m_1(n_2+1)^5 + n_1 S(B) + n_2 S(A) \\
&\quad +10(m_1 Y(B) + m_2 Y(A) + F(B)M_1(A) + M_1(B)F(A))
\end{aligned}$$

Proof. By the definition of Subdivision-vertex neighborhood corona $A \oplus_s B$, we get that

$$\begin{aligned}
S(A \oplus_s B) &= \sum_{i=1}^{n_1} (deg_A(u_i))^5 + \sum_{i=1}^{m_1} (2+2n_2)^5 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} (deg_A(u_i) + deg_B(v_j))^5 \\
&= S(A) + m_1(2n_2+2)^5 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} ((deg_B(v_j))^5 + (deg_A(u_i))^5 \\
&\quad +5deg_A(u_i)(deg_B(v_j))^4 + 5deg_B(v_j)(deg_A(u_i))^4 \\
&\quad +10(deg_A(u_i))^2(deg_B(v_j))^3 + 10(deg_A(u_i))^3(deg_B(v_j))^2) \\
S(A \oplus_s B) &= S(A) + 32m_1(n_2+1)^5 + n_1 S(B) + n_2 S(A) \\
&\quad +10(m_1 Y(B) + m_2 Y(A) + F(B)M_1(A) + M_1(B)F(A))
\end{aligned}$$

Hence, we obtained the required outcome. □

Theorem 4.2. *The Subdivision-vertex neighborhood corona $A \oplus_s B$ of Y coindex is given by*

$$\begin{aligned}\overline{Y}(A\oplus_s B) &= (\eta-1)(n_2+1)-8m_2)F(A)+(n_1(\eta-1)-8m_1)F(B)\\ &+6(\eta-1)(m_1M_1(B)+m_2M_1(A))-6M_1(A)M_1(B)-(1+n_2)Y(A)\\ &-n_1Y(B)+((\eta-1)-2(n_2+1)8m_1(n_2+1)^3\end{aligned}$$

where $\eta = n_1 + m_1 + n_1 n_2$

Proof. By, the Relationship between the topological indices and coindices, we can obtain that

$$\overline{Y}(A\oplus_s B) = (n-1)F(A\oplus_s B) - Y(A\oplus_s B) \tag{4.1}$$

Since, $n = m_1 + n_1 + n_1 n_2$ and by (Nilanjan De, 2017; Nagarajan & Durga, 2023), we get that

$$\begin{aligned}F(A\oplus_s B) &= F(A)+8m_1(n_2+1)^3+n_1F(B)+6m_1M_1(B)+6m_2M_1(A)+n_2F(A)\\ Y(A\oplus_s B) &= Y(A)+16m_1(n_2+1)^4+n_1Y(B)+8m_1F(B)+6M_1(B)M_1(A)\\ &+8m_2F(A)+n_2Y(A))\end{aligned}$$

Substituting these values in the equation 4.1, we can obtain the required outcome. ☐

Theorem 4.3. *The Subdivision-vertex neighborhood corona $A \oplus_s B$ of S coindex is given by*

$$\begin{aligned}\overline{S}(A\oplus_s B) &= (\eta+n_2+10m_2-1)Y(A)+(n_1(\eta-1)-10m_1)Y(B)+2(4m_2(\eta-1)\\ &-5M_1(B))F(A)+2(4m_1(\eta-1)-5M_1(A))F(B)+(\eta-1)(16m_1(n_2+1)^4\\ &+6M_1(A)M_1(B))-32m_1(n_2+1)^5-(n_2+1)S(A)-n_1S(B)\end{aligned}$$

Proof. By, the Relationship between S index and coindex,

$$\overline{S}(A\oplus_s B) = (n-1)Y(A\oplus_s B) - S(A\oplus_s B) \tag{4.2}$$

The conclusion is reached through the application of Theorem 4.1 and Theorem 4.2. ☐

Corollary 4.4. *By using Theorem 4.1, we can obtain the following results for path graphs and cycle graphs.*

$$\begin{aligned}S(P_n\oplus_s P_m) &= 32m^5(n-1)+160m^4(n-1)+320m^3(n-1)+320m^2(n-1)\\ &+1184nm-1722m-1498n+2186\\ S(C_n\oplus_s C_m) &= 32nm^5+160nm^4+320nm^3+320nm^2+1184nm+64n\\ S(C_n\oplus_s P_m) &= 32nm^5+160nm^4+320nm^3+320nm^2+1184nm-1498n\end{aligned}$$

5 Subdivision-edge neighborhood corona

The subdivision edge neighborhood corona is a sophisticated graph operation that extends the ideas of edge subdivision and neighborhood attachment to form a richly structured graph. Given two simple graphs A and B, the subdivision edge neighborhood corona, often denoted by $A \otimes_s B$, is constructed by first subdividing each edge of A, introducing a new vertex into every edge. Then, for each subdivision vertex (which lies between two adjacent vertices in A), a copy of B is added such that the neighbors of the two original end-vertices of the corresponding edge in A are connected to all vertices in that copy of B.

The Subdivision-edge neighborhood corona $A \otimes_s B$ has order $n_1 + m_1 + m_1 n_2$ and size $2m_1 + m_1 n_2 + m_1 m_2$. Then, the degree of the vertices in $A \otimes_s B$ are given by

$$\begin{aligned}deg_{A\otimes_s B}(u_i) &= (n_2+1)deg_A(u_i)\ where\ i=1,2,\ldots,n_1\\ deg_{A\otimes_s B}(e_i) &= 2\ where\ i=1,2,\ldots,m_1\\ deg_{A\otimes_s B}(u_j^i) &= deg_B(v_j)+2\ where\ i=1,2,\ldots,m_1\ and\ j=1,2,\ldots,n_2\end{aligned}$$

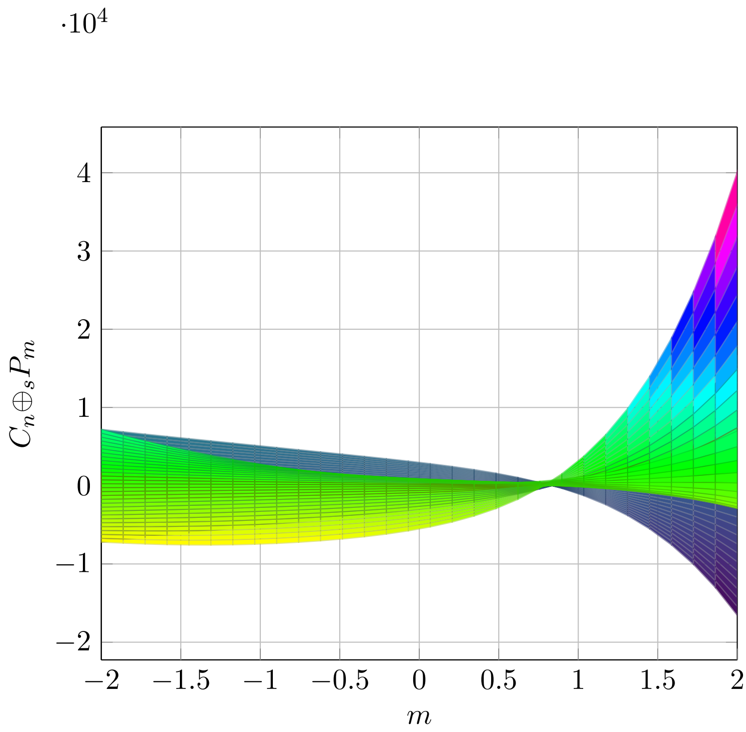

Figure 3: Comparison of $S(C_n \oplus_s P_m), \overline{Y}(C_n \oplus_s P_m)$ and $\overline{S}(C_n \oplus_s P_m)$

Theorem 5.1. *The Subdivision-edge neighborhood corona $A \otimes_s B$ of S index is given by*

$$\begin{aligned} S(A \otimes_s B) &= (n_2+1)^5 S(A) + 32m_1 + n_1 S(B) + 32n_1 n_2 \\ &\quad +10(n_1 Y(B) + 40F(B)n_1 + 80M_1(B)n_1 + 160m_2 n_1 \end{aligned}$$

Proof. By the definition of Subdivision-edge neighborhood corona $A \otimes_s B$, we get that

$$\begin{aligned} S(A \otimes_s B) &= \sum_{i=1}^{n_1} (n_2+1)^5 (deg_A(u_i))^5 + \sum_{i=1}^{m_1} 2^5 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (2 + deg_B(v_j))^5 \\ &= (n_2+1)^5 S(A) + 32m_1 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ((deg_B(v_j))^5 + 2^5 + 10deg_B(v_j)^4 \\ &\quad +40(deg_B(v_j))^3 + 80(deg_B(v_j))^2 + 80deg_B(v_j) \\ S(A \otimes_s B) &= (n_2+1)^5 S(A) + 32m_1 + n_1 S(B) + 32n_1 n_2 \\ &\quad +10(n_1 Y(B) + 40F(B)n_1 + 80M_1(B)n_1 + 160m_2 n_1 \end{aligned}$$

Hence, we obtained the required outcome. □

Theorem 5.2. *The Subdivision-edge neighborhood corona $A \otimes_s B$ of Y coindex is given by*

$$\begin{aligned} \overline{Y}(A \otimes_s B) &= (n_2+1)^3 (\mu - 1)F(A) + n_1(\mu - 9)F(B) + 6n_1(\mu - 5)M_1(B) + 8(\mu - 1)n_1 \\ &\quad (3m_2 + n_2 + 1) - 16(m_1 + n_1 m_2 + n_1 n_2) - (n_2 + 10)^4 Y(A) - n_1 Y(B) \end{aligned}$$

where $\mu = n_1 + m_1 + m_1 n_2$

Proof. By, the Relationship between the topological indices and coindices, we can obtain that

$$\overline{Y}(A\otimes_s B) = (n-1)F(A\otimes_s B) - Y(A\otimes_s B) \tag{5.1}$$

Since, $n = n_1 + m_1 + m_1 n_2$ and by (Nilanjan De, 2017; Nagarajan & Durga, 2023), we get that

$$\begin{aligned} F(A\oplus_s B) &= (n_2+1)^3 F(A) + n_1(F(B) + 6M_1(B) + 24m_2 + 8n_2 + 8) \\ Y(A\oplus_s B) &= (n_2+1)^4 Y(A) + 16m_1 + n_1 Y(B) + 8n_1 F(B) + 24M_1(B)n_1 \\ &\quad +64n_1 m_2 + 16n_1 n_2 \end{aligned}$$

Substituting these values in the equation 5.1, we can obtain the required outcome. □

Theorem 5.3. *The Subdivision-edge neighborhood corona $A\otimes_s B$ of S coindex is given by*

$$\begin{aligned} \overline{S}(A\otimes_s B) &= (n_2+1)^4((\mu-1)Y(A) - (n_2+1)S(A)) - n_1 S(B) + n_1(\mu-11)Y(B) \\ &\quad +8n_1(\mu-6)F(B) + 8n_1(3(\mu-1)-10)M_1(B) + 16(\mu-1)(m_1 \\ &\quad +4n_1 m_2 + n_1 n_2) - 32(m_1 + n_1 n_2 + 5n_1 m_2) \end{aligned}$$

Proof. By, the Relationship between S index and coindex,

$$\overline{S}(A\otimes_s B) = (n-1)Y(A\otimes_s B) - S(A\otimes_s B) \tag{5.2}$$

The conclusion is reached through the application of Theorem 5.1 and Theorem 5.2. □

Corollary 5.4. *By using Theorem 5.1, we can obtain the following results for path graphs and cycle graphs.*

$$\begin{aligned} S(P_n\otimes_s P_m) &= 32nm^5 - 62m^5 + 160nm^4 - 310m^4 + 320nm^3 - 620m^3 + 320nm^2 - 620m^2 \\ &\quad +1184nm - 1338n - 310m - 94 \\ S(C_n\otimes_s C_m) &= 32nm^5 + 160nm^4 + 320nm^3 + 320nm^2 + 1184nm + 64n \\ S(C_n\otimes_s P_m) &= 32nm^5 + 160nm^4 + 320nm^3 + 320nm^2 + 1184nm - 1338n \end{aligned}$$

6 Vertex-edge corona

The vertex-edge corona is a graph operation that combines the structural properties of vertices and edges to create a more intricate graph topology. Given two simple graphs A and B, the vertex-edge corona, often denoted by $A\veebar B$ or using a similar notation depending on context, is formed by attaching a copy of H to each vertex and each edge of the graph A. Specifically, for every vertex u in A, a copy of B is added and connected to u; similarly, for each edge e=uv in A, another copy of B is added and connected to both endpoints u and v. This operation significantly increases the size and connectivity of the original graph, resulting in a hybrid structure that models systems with both point-based and relationship-based extensions.
The Vertex-edge corona $A\veebar_s B$ has order $n_1 + m_1 n_2 + n_1 n_2$ and size $n_1(m_2+n_2) + m_1(m_2 + 2n_2 + 1)$. Then, the degree of the vertices in $A\veebar_s B$ are given by

$$\begin{aligned} deg_{A\veebar_s B}(u_i) &= (n_2+1)deg_A(u_i) + n_2 \ where \ u_i \in V(A) \\ deg_{A\veebar_s B}(v_{ij}) &= deg_B(v_j) + 2 \ where \ v_{ij} \in V_{ie}(B) \\ deg_{A\veebar_s B}(w_{ij}) &= deg_B(w_j) + 1 \ where \ w_{ij} \in V_{ie}(B) \end{aligned}$$

Theorem 6.1. *The Vertex-edge corona $A\veebar_s B$ of S index is given by*

$$\begin{aligned} S(A\veebar_s B) &= (n_2+1)^5 S(A) + (n_1+m_1)S(B) + 5(n_2+1)^4 n_2 Y(A) + 5(n_1+2m_1)Y(B) \\ &\quad +10(n_2+1)^3 n_2^2 F(A) + 10(n_1+8m_1)F(B) + 10(n_2+1)^2 n_2^3 M_1(A) + 32m_1 n_2 \\ &\quad +10(n_1+8m_1)M_1(B) + 10(n_2+1)m_1 n_2^4 + 160m_1 m_2 + 10n_1 m_2 + n_1 n_2^5 + n_1 n_2 \end{aligned}$$

Proof. By the definition of Vertex-edge corona $A\veebar_s B$, we get that

$$S(A\veebar_s B) = \sum_{u_i\in V(A)}((n_2+1)deg_A(u_i)+n_2)^5 + \sum_{e_i\in E(A)}\sum_{v_{ij}\in V(B)}(deg_B(v_{ij})+2)^5 + \sum_{v_i\in V(A)}\sum_{w_{ij}\in V(B)}(deg_B(w_j)+1)^5$$

By simplifying the above summations, we can obtain the following result

$$\begin{aligned} S(A\veebar_s B) &= (n_2+1)^5 S(A) + (n_1+m_1)S(B) + 5(n_2+1)^4 n_2 Y(A) + 5(n_1+2m_1)Y(B) \\ &\quad +10(n_2+1)^3 n_2^2 F(A) + 10(n_1+8m_1)F(B) + 10(n_2+1)^2 n_2^3 M_1(A) + 32m_1n_2 \\ &\quad +10(n_1+8m_1)M_1(B) + 10(n_2+1)m_1n_2^4 + 160m_1m_2 + 10n_1m_2 + n_1n_2^5 + n_1n_2 \end{aligned}$$

As a result, the claim is established. □

Theorem 6.2. *The Vertex-edge corona $A\veebar_s B$ of Y coindex is given by*

$$\begin{aligned} \overline{Y}(A\veebar_s B) &= (n_2+1)^3(\varsigma-1)F(A) - (5n_2+1)Y(A)) - m_1Y(B) + (n_1+m_1)(\varsigma-1) - 8m_1) \\ &\quad F(B) + 3n_2(n_2+1)^2 M_1(A)(\varsigma-1-2n_2) + 3M_1(B)((n_1+2m_1)(\varsigma-1) - 8m_1) \\ &\quad (\varsigma-1)(n_2^3(6m_1+n_1) + (6m_2+n_2)n_1 + 2m_1(12m_2+4n_2+3n_2^2)) \\ &\quad -8(m_1(n_2+1)n_2^3 + n_1m_2) - n_1n_2(1+n_2^3) \end{aligned}$$

where $\varsigma = n_1 + m_1n_2 + n_1n_2$

Proof. By, the Relationship between the topological indices and coindices, we can obtain that

$$\overline{Y}(A\veebar_s B) = (n-1)F(A\veebar_s B) - Y(A\veebar_s B) \tag{6.1}$$

Since, $n = n_1 + m_1n_2 + n_1n_2$ and by (Nilanjan De, 2017; Nagarajan & Durga, 2023), we get that

$$\begin{aligned} F(A\veebar_s B) &= (n_2+1)^3 F(A) + (n_1+m_1)(F(B) + 3(n_1+2m_1)M_1(B) + 3n_2(n_2+1)^2 M_1(A) \\ &\quad +6n_2^2 m_1(n_2+1) + n_1n_2^3 + 6n_1m_2 + 8n_2m_1 + 24m_1m_2 + n_1n_2) \\ Y(A\veebar_s B) &= (n_2+1)^4 Y(A) + 4(n_2+1)^3 Y(A)n_2 + 6(n_2+1)^2 M_1(A)n_2^2 + 8m_1(n_2+1)n_2^3 \\ &\quad +n_1n_2^4 + m_1Y(B) + 8m_1F(B) + 24M_1(B)m_1 + 8n_1m_2 + n_1n_2) \end{aligned}$$

Substituting these values in the equation 6.1, we can obtain the required outcome. □

Theorem 6.3. *The Vertex-edge corona $A\veebar_s B$ of S coindex is given by*

$$\begin{aligned} \overline{S}(A\veebar_s B) &= Y(A)(n_2+1)^3(4n_2 + (n_2+1)(\varsigma - 5n_2 - 1)) + Y(B)(m_1(\varsigma-1) - 5(n_1+2m_1)) \\ &\quad -10(n_2+1)^3 n_2^2 F(A) + 2F(B)(4m_1(\varsigma-1) - 5(n_1+4m_1)) + 2n_2^2(n_2+1)^2 \\ &\quad M_1(A)(3(\varsigma-1) - 5n_2) - (n_2+1)^5 S(A) - (n_1+m_1)S(B) + 2M_1(B) \\ &\quad (12m_1(\varsigma-1) - 5(n_1+8m_1)) + (\varsigma-1)(n_2^3(8m_1(n_2+1) + n_1n_2) + n_1(8m_2+n_2)) \\ &\quad -n_2^4(10(n_2+1)m_1 + n_1n_2) - n_1(10m_2+n_2) - 32m_1(5m_2+n_2) \end{aligned}$$

Proof. By, the Relationship between S index and coindex,

$$\overline{S}(A\veebar_s B) = (n-1)Y(A\veebar_s B) - S(A\veebar_s B) \tag{6.2}$$

The conclusion is reached through the application of Theorem 6.1 and Theorem 6.2. □

Corollary 6.4. *By using Theorem 6.1, we can obtain the following results for path graphs and cycle graphs.*

$$\begin{aligned} S(P_n \veebar_s P_m) &= 243nm^5 - 422m^5 + 810nm^4 - 1460m^4 + 1080nm^3 - 2000m^3 + 720nm^2 \\ &\quad -1360m^2 + 1507nm - 1952n - 1484m + 1500 \\ S(C_n \veebar_s C_m) &= 243nm^5 + 810nm^4 + 1080nm^3 + 720nm^2 + 1507nm + 32n \\ S(C_n \veebar_s P_m) &= 243nm^5 + 810nm^4 + 1080nm^3 + 720nm^2 + 1507nm - 1952n \end{aligned}$$

7 CONCLUSION

In this study, we explored the behavior of the S-index, Y and S coindices in under the corona product of two simple connected graphs. By analyzing the structural changes introduced through the corona operation, we derived explicit expressions for the S-index, Y and S coindices of the resulting graph in terms of the properties of the original graphs. These results generalize existing findings and offer deeper insights into how graph operations impact degree-based indices. For future work, it is possible to compute these different corona products for D-index and some other topological indices and coindices.

References

Khalifeh, M.H., Yousefi-Azari, H., & Ashrafi, A.R. (2009). The first and second Zagreb indices of some graph operations. *Discrete Applied Mathematics*, 157(4): 804–811.

Melaku Berhe Belay, & Chunxiang Wang (2020). The first general Zagreb coindex of graph operations. *Applied Mathematics and Nonlinear Sciences*, 5(2): 109—120.

Nagarajan, S., Kayalvizhi, G., & Priyadharsini, G. (2021). S-Index of Different Graph Operations. *Asian Research Journal of Mathematics*, 17(12): 43–52. https://doi.org/10.9734/arjom/2021/v17i1230347

Alameri, A., Mahmoud Al-Rumaima, & Mohammed Almazah, (2020). Y-coindex of graph operations and its applications of molecular descriptors. *Journal of Molecular Structure*, 1221.

Nilanjan De, (2017). Computing F-Index of Different Corona Products of Graphs. *Bulletin of Mathematical Sciences and Applications*, 19: 24–30.

Nagarajan, S. & Durga, M. (2023). Computing Y-index of Different Corona Products of Graphs. *Asian Research Journal of Mathematics*, 19(10): 67–74.

Nagarajan, S., & Kayalvizhi, G. (2023). S-coindex of some graph operations and its properties and conjugated polymers. *Global Journal of Engineering and Technology Advances*, 16(2): 140—149. https://doi.org/10.30574/gjeta.2023.16.2.0151