Original Research Article

Associative Algebras Satisfying Quadratic Equations

Abstract

In this work, we classify Quadratic Algebras that are generated by a set $G = \{x_1, x_2, \ldots, x_n\}$, over a field K that satisfy a polynomial identity of the form $X^2 = aX + b$, where $a, b \in K$ and X can vary over the elements of A or over the elements of the multiplicative semigroup S generated by G.

Keywords: Associative algebra, Polynomial identity, Nilpotency index, Nagata-Higman theorem

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1 Introduction

Associative algebras play a crucial role in algebraic structures and have numerous applications in various fileds of mathematics and physics. When a field F has characteristic 0 and A is an F-algebra, we define A aas a nil-algebra if there exists a positive integer n such that $a^n = 0$ for every $ainA$. In this case, n is called the nil-index of A. An algebra A is said to be nilpotent of index m if any product of m elements of A is zero.

In [1], Nagata proved that is A is a nil-algebra with nil-index n, then A is nilpotent of index $N(n)$ for some $N(n) \in \mathbb{N}$. Higman [2] provide a refined result, showing that $n^2 \leq N(n) \leq 2^n - 1$. Razmyslov [3] improved Higman's upper bound to $N(n) \geq n^2$, and Kuzmin [4] enhanced the lower bound to $N(n) \geq \frac{n(n+1)}{2}$. Woo Lee [5] demonstrated that a nil-algebra A with nil-index 3 is nilpotent of index 6.

This paper focuses on classifying and characterizing associative algebras generated by a set $G = \{x_1, x_2, \cdots, x_m\}$ over a field K that satisfy a quadratic polynomial identity of the form $X^2 = aX + b$, where $a, b \in K$ and X can vary over the elements of A or a multiplicative semigroup S generated by G.

2 The case $X^2 = k$

Lemma 2.1. *Let S be the set of words in $\{x_1, x_2, \cdots, x_m\}$ and A an Associative Algebra over a field K containing S satisfying $X^2 = k$, $\forall X \in S$, for some $k \in K$. Then $k \in \{0, 1\}$.*

Proof. Since $k = (x^2)^2 = k^2$, then $k \in \{0, 1\}$. □

2.1 The case $X^2 = 0$

Theorem 2.2. *Let $A = \langle x_1, x_2, x_3, \dots, x_m \rangle$ over a field of characteristic $p \neq 2$, satisfying $X^2 = 0$, $\forall X \in A$, then A is nilpotent of index at most 3 and the dimension of A is at most $\frac{m(m+1)}{2}$.*

Proof. Let $x, y, z \in A$, then

$$(x+y)^2 = x^2 + xy + yx + y^2 = 0 \Rightarrow xy = -yx \tag{2.1}$$

and

$$(x+yz)^2 = x^2 + xyz + yzx + (yz)^2 = xyz + yzx = 0 \Rightarrow xyz = -yzx \tag{2.2}$$

Now observe that

$$xyz \overset{(2.1)}{=} -(yxz) \overset{(2.1)}{=} -(y(-zx)) = yzx \tag{2.3}$$

By (2.3) and (2.2), we obtain that

$$2xyz = 0. \tag{2.4}$$

As A is an Algebra of characteristic $p \neq 2$ and $2xyz = 0$, it follows that $xyz = 0$. Therefore A is nilpotent of index at most 3.

By (2.1), (2.2) and (2.3),it follows that the basis of A is subset of

$$\left(\bigcup_{k=1}^{m} \{x_k\} \right) \bigcup \left(\bigcup_{1 \leq i < j \leq m} \{x_i x_j\} \right)$$

which has $\frac{m(m+1)}{2}$ elements. Hence, the dimension of A is at most $\frac{m(m+1)}{2}$.

□

Example 2.3. *Consider $A = \langle x, y \rangle$, where every element of the Associative Algebra satisfies the condition $X^2 = 0$.*

Thus,

$$(x+y)^2 = x^2 + y^2 + xy + yx = 0 \Rightarrow xy = -yx$$

$$(x+xy)^2 = x^2 + (xy)^2 + x^2 y + xyx = 0 \Rightarrow xyx = 0$$

Indeed, we obtain that words of length greater than 2 in the semigroup S generated by x and y are equal to zero.

In this case, any element $w \in A$ can be written as

$$w = ax + by + cxy,$$

where $a, b, c \in K$.

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem 2.4. *Let $A = \langle x_1, x_2, x_3, \ldots, x_m \rangle$ over a field of characteristic 2, satisfying $X^2 = 0, \forall X \in A$, then A is nilpotent of index at most $m+1$ and the dimension of this Algebra is less than or equal to $2^m - 1$.*

Proof. By (2.1), we obtain that a basis of A over a field F of characteristic 2, satisfying $X^2 = 0$, $\forall X \in A$, is formed by elements of the form $x_1^{\xi_1} x_2^{\xi_2} \ldots x_m^{\xi_m}$, where $\xi_1, \ldots, \xi_m \in \{0, 1\}$ and not all ξ_i are zero. Therefore, $\dim(A) \leq 2^m - 1$. □

2.2 The case $X^2 = 1$

Theorem 2.5. *Let S be the set of words in $\{x_1, x_2, \ldots, x_m\}$ and A an Associative Algebra over a field K generated by S and satisfying $X^2 = 1$, $\forall X \in S$, then A is an Abelian Algebra and $\dim(A) = 2^m$.*

Proof. First, observe that $yx = x^2 yxy^2 = x(xy)^2 y = xy, \forall x, y \in \{x_1, \cdots, x_m\}$.

Therefore, the generators of A are of the form $x_1^{\xi_1} x_2^{\xi_2} \cdots x_m^{\xi_m}$, where $\xi_1, \cdots, \xi_m \in \{0, 1\}$.

Thus, $dim(A) = 2^m$. □

Example 2.6. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K generated by S satisfying $X^2 = 1$, $\forall X \in S$, then there exists a basis $\{1, x, y, xy\}$ where any element $w \in A$ can be written as*

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy,$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K$.

If $w_1 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$ and $w_2 = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$, then

$$\begin{array}{rcl} w_1 w_2 & = & (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4) \\ & + & (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_4 + \alpha_4\beta_3)x \\ & + & (\alpha_1\beta_3 + \alpha_2\beta_4 + \alpha_3\beta_1 + \alpha_4\beta_2)y \\ & + & (\alpha_1\beta_4 + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_4\beta_1)xy \end{array}$$

In this case, we can take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

3 The case $X^2 = kX$

Lemma 3.1. *Let S be the set of words in $\{x_1, x_2, \ldots, x_m\}$ and A an Associative Algebra over a field K containing S satisfying $X^2 = kX$, $\forall X \in S$, for some $k \in K$. Then $k \in \{0, 1\}$.*

Proof. $k^2 x = k(kx) = kx^2 = (x^2)^2 = (kx)^2 = k^2(x)^2 = k^3 x \Rightarrow k^2(k-1)x = 0 \Rightarrow k \in \{0, 1\}$. □

3.1 The case $X^2 = X$

Theorem 3.2. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K generated by S satisfying $X^2 = X$, $\forall X \in A$, then*

- *A is Abelian;*
- *A has characteristic 2;*
- *$dim(A) \leq 3$.*

Proof. Let $w \in A$. Thus, $(2w)^2 = 4w$ and $(2w)^2 = 2w$, which implies $2w = 0$. Therefore, A has characteristic 2.

Since
$$(x + y)^2 = x + y \tag{3.1}$$
and
$$(x + y)^2 = x^2 + y^2 + xy + yx = x + y + xy + yx \tag{3.2}$$
then
$$xy = yx. \tag{3.3}$$

Therefore, A is an Abelian algebra.

By (3.3) e by definition of A, it follows that a basis for A is a subset of $\{x, y, xy\}$. Thus, $\dim(A) \leq 3$.

□

Example 3.3. *The Algebra over $\mathbb{Z}_2$ generated by x and y, where $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ satisfies the hypotheses of Theorem 3.2 and has dimension 3 as a vector space.'*

Theorem 3.4. *If S is the set of words in $\{x_1, x_2, \ldots, x_m\}$ and A is an Associative Algebra over a field K generated by S satisfying $X^2 = X$, $\forall X \in A$, then*

- *$\mathcal{A}$ is Abelian;*
- *A has characteristic 2;*
- *$dim(A) \leq 2^m - 1$;*

Proof. By Theorem 3.2, it follows that A is Abelian and A has characteristic 2. As A is Abelian and by definition of A, it follows that a base of A is a subset of

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m} \mid \alpha_1, \ldots, \alpha_m \in \{0, 1\}, \alpha_1^2 + \cdots + \alpha_m^2 \neq 0\}$$

Thus, A has dimension at most $2^m - 1$. □

Theorem 3.5. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K generated by S satisfying $X^2 = X$, $\forall X \in S$, then $\dim(A) \leq 6$.*

Proof. It suffices to observe that the set $\{x, y, xy, yx, xyx, yxy\}$ generates the Algebra A. Note that if $A = \langle x, y \rangle$, where

$$x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then A satisfies the conditions of the proposition, and $\beta = \{x, y, xy, yx, xyx, yxy\}$ is a basis for A with $\dim(A) = 6$. □

4 The case $X^2 = aX + b$, with $a, b \in K^*$

Theorem 4.1. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K with characteristic 2 generated by S satisfying $X^2 = X + 1$, $\forall X \in S$, then A is the field $GF(4)$ with four elements $\{0, 1, x, 1 + x\}$.*

Proof. We have

$$(x^2y)^2 = x^2y + 1 = (x + 1)y + 1 = xy + y + 1 \tag{4.1}$$

and

$$\begin{aligned} (x^2y)^2 &= ((x + 1)y)^2 \\ &= (xy + y)^2 \\ &= (xy)^2 + y^2 + xy^2 + yxy \\ &= xy + 1 + y + 1 + x(y + 1) + yxy \\ &= 2xy + 2 + y + x + yxy = y + x + yxy \end{aligned} \tag{4.2}$$

By (4.1) and (4.2), we have

$$yxy = xy + x + 1 \tag{4.3}$$

$$y^2xy = yxy + yx + y = xy + x + 1 + yx + y \tag{4.4}$$

$$y^2xy = (y + 1)xy = yxy + xy = 2xy + x + 1 = x + 1 \tag{4.5}$$

By (4.4) and (4.5), we have

$$xy = yx + y \tag{4.6}$$

Thus, by simmetry,

$$yx = xy + x \tag{4.7}$$

By (4.6) and (4.7), we have that $x = y$. □

Theorem 4.2. *If S is the set of words in $\{x, y\}$ and A is an Associative Algebra over a field K with characteristic greater than 2 generated by S satisfying $X^2 = aX + b$, $\forall X \in S$ with $a, b \in K^*$, then* *(i)* $b = -1$ *e* $a = 2$ *or* $a = -1$;

(ii) $\{1, x, y, xy\}$ *is a basis for* A *as a vector space over* K*;*

(iii) $dim(\mathcal{A}) = 4$.

Proof. (i) Since $X^2 = aX + b$, then

$$\begin{aligned} (X^2)^2 &= (aX+b)^2 \\ &= a^2X^2 + 2abX + b^2 \\ &= a^2(aX+b) + 2abX + b^2 \\ &= (a^3+2ab)X + a^2b + b^2 \end{aligned} \tag{4.8}$$

and

$$(X^2)^2 = aX^2 + b = a(aX+b) + b = a^2X + ab + b \tag{4.9}$$

Therefore, we obtain

$$\left\{ \begin{array}{l} a^2 = a^3 + 2ab \\ ab + b = a^2b + b^2 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} a = a^2 + 2b \\ a + 1 = a^2 + b \end{array} \right. \Rightarrow \left\{ \begin{array}{l} b = -1 \\ a^2 - a - 2 = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} b = -1 \\ a = 2 \text{ or } a = -1 \end{array} \right. \tag{4.10}$$

(ii) and (iii) For $X^2 = -X - 1$, we have

$$\begin{aligned} (x^2y)^2 &= -x^2y - 1 \\ &= -(-x-1)y - 1 \\ &= xy + y - 1 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} (x^2y)^2 &= [(-x-1)y]^2 \\ &= (xy+y)^2 \\ &= (xy)^2 + y^2 + xy^2 + yxy \\ &= -xy - 1 - y - 1 + x(-y-1) + yxy \\ &= -2xy - 2 - y - x + yxy \end{aligned} \tag{4.12}$$

By (4.11) and (4.12), we obtain

$$yxy = 3xy + 2y + x + 1 \tag{4.13}$$

Now,

$$\begin{aligned} xyxy &= 3x^2y + 2xy + x^2 + x \\ &= 3(-x-1)y + 2xy + (-x-1) + x \\ &= -3xy - 3y + 2xy - 1 = -xy - 3y - 1 \end{aligned} \tag{4.14}$$

and

$$xyxy = -xy - 1 \tag{4.15}$$

By (4.14) and (4.15),

$$3y = 0 \tag{4.16}$$

By simmetry, we can prove that $3x = 0$. Therefore, K has characteristic 3.
Since K has characteristic 3, we obtain that

$$yxy = 2y + x + 1 \tag{4.17}$$

Thus,

$$y^2xy = y(2y+x+1) = 2y^2 + yx + y = -2y - 2 + yx + y = -2 + yx + 2y \tag{4.18}$$

and

$$y^2xy = (-y-1)xy = -yxy - xy = -(2y+x+1) - xy = -2y - x - 1 - xy \quad (4.19)$$

By (4.18) and (4.19), we obtain that

$$yx + xy = -4y - x + 1 = 2y + 2x + 1 \quad (4.20)$$

For $X^2 = 2X - 1$.

$$\begin{aligned}(x^2y)^2 &= 2x^2y - 1 \\ &= 2(2x-1)y - 1 \\ &= 4xy - 2y - 1\end{aligned} \quad (4.21)$$

and

$$\begin{aligned}(x^2y)^2 &= [(2x-1)y]^2 \\ &= (2xy - y)^2 \\ &= 4(xy)^2 + y^2 - 2xy^2 - 2yxy \\ &= 8xy - 4 + 2y - 1 - 2x(2y-1) - 2yxy \\ &= 4xy - 5 + 2y + 2x - 2yxy\end{aligned} \quad (4.22)$$

By (4.21) and (4.22), we obtain that

$$yxy = 2y + x - 2 \quad (4.23)$$

Now,

$$\begin{aligned}y^2xy &= 2y^2 + yx - 2y \\ &= 4y - 2 + yx - 2y \\ &= 2y - 2 + yx\end{aligned} \quad (4.24)$$

and

$$\begin{aligned}y^2xy &= (2y-1)xy \\ &= 2yxy - xy \\ &= 2(2y + x - 2) - xy \\ &= 4y + 2x - 4 - xy\end{aligned} \quad (4.25)$$

By (4.24) and (4.25),

$$yx + xy = 2y + 2x - 2 \quad (4.26)$$

Equations (4.11) to (4.25) show that $\{1, x, y, xy\}$ is a base of A as a vector space over K and ,therefore, $\dim(A) = 4$.

Note that if A is an Algebra satisfying the above conditions, then every element $w \in A$ can be written as

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy,$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K$.

Moreover, if $w_1 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$ and $w_2 = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$, then

$$\begin{aligned}w_1 w_2 \;=\; & (\alpha_1\beta_1 - \alpha_2\beta_2 - 2\alpha_3\beta_2 - \alpha_3\beta_3 - 2\alpha_3\beta_4 - 2\alpha_4\beta_2 - \alpha_4\beta_4) \\ + \; & (\alpha_1\beta_2 + \alpha_2\beta_1 + 2\alpha_3\beta_2 + 2\alpha_2\beta_2 + \alpha_3\beta_4 + 2\alpha_4\beta_2 - \alpha_4\beta_3)x \\ + \; & (\alpha_1\beta_3 - \alpha_2\beta_4 + 2\alpha_3\beta_2 + \alpha_3\beta_1 + 2\alpha_3\beta_3 + 2\alpha_3\beta_4 + \alpha_4\beta_2)y \\ + \; & (\alpha_1\beta_4 + \alpha_2\beta_3 + 2\alpha_2\beta_4 - \alpha_3\beta_2 + \alpha_4\beta_1 + 2\alpha_4\beta_3 + 2\alpha_4\beta_4)xy\end{aligned}$$

In this case, we can take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -2 & 2 & 2 & -1 \\ -1 & 0 & 2 & 0 \\ -2 & 1 & 2 & 0 \end{pmatrix}$$

□

Conclusion

In this work, we classify associative algebras that are generated by a finite set of elements and satisfy specific quadratic polynomial identities. The results presented not only generalize existing contributions to the theory of nil-algebras but also provide new insights into the structure and dimensional characteristics of associative algebras under various conditions. Computational validation was performed using the **GAP** software package [6], which confirmed the theoretical results.

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