A Study on Dual Hyperbolic Generalized Leonardo Numbers

In this study, we define dual hyperbolic generalized Leonardo numbers in detail and focus on three specific cases: Dual hyperbolic modified Leonardo numbers, dual hyperbolic Leonardo-Lucas numbers, and dual hyperbolic Leonardo numbers.

In addition, we include numerous identities and matrices for these sequences, as well as recurrence relations, Binet's formulae, generating functions, Simpson's formula, Honsberger's identity, and several summation formulas.

Keywords: Dual hyperbolic generalized Leonardo numbers, dual hyperbolic modified Leonardo numbers, dual hyperbolic Leonardo-Lucas numbers, dual hyperbolic Leonardo numbers.

1. Introduction

The hypercomplex numbers systems, [1], are extensions of real numbers. Some commutative examples of hypercomplex number systems are complex numbers,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers, [2],

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

and dual numbers, [3],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Some non-commutative examples of hypercomplex number systems are quaternions, [4],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [5] and sedenions [6]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [7], [8], [9]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [4] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [10]. H. H. Cheng and S. Thompson [11] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [12] introduced dual hyperbolic numbers.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and addition of dual hyperbolic numbers is defined as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. But $\mathbb{H}_{\mathbb{D}}$ is not field because every dual hyperbolic numbers doesn't have an inverse. For more information on the dual hyperbolic numbers, see [12].

Now let us recall the definition of generalized Leonardo numbers. See [13] and [14]

A generalised Leonardo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-3} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

It is possible to extend the sequence $\{W_n\}_{n\geq 0}$ to negative subscripts. To do this, for $n = 1, 2, 3, \ldots$ we define

$$W_{-n} = 2W_{-(n-2)} - W_{-(n-3)}.$$

Thus, recurrence (1.1) holds for all integer n.

We can use (1.1) to obtain Binet's formula of generalised Leonardo numbers. Binet's formula of generalised Leonardo numbers can be given as

$$\begin{aligned} W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^{n+1} - z_2\beta^{n+1}}{\alpha-\beta} - z_3 \end{aligned} \tag{1.2}$$

where

$$z_1 = W_2 - (2-\alpha)W_1 + (1-\alpha)W_0, \tag{1.3}$$

$$z_2 = W_2 - (2-\beta)W_1 + (1-\beta)W_0, \tag{1.4}$$

$$z_3 = W_2 - W_1 - W_0. \tag{1.5}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 2x^2 + 1 = \left(x^2 - x - 1\right)(x-1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= \frac{1+\sqrt{5}}{2}, \\ \beta &= \frac{1-\sqrt{5}}{2}, \\ \gamma &= 1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta+\gamma &= 2, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= 0, \\ \alpha\beta\gamma &= -1, \end{aligned}$$

or

$$\alpha+\beta = 1,\ \alpha\beta = -1.$$

The first few generalised Leonardo numbers with positive and negative subscripts are given in the following Table 1.

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$2W_1 - W_2$
2	W_2	$2W_0 - W_1$
3	$2W_2 - W_0$	$4W_1 - W_0 - 2W_2$
4	$4W_2 - W_1 - 2W_0$	$4W_0 - 4W_1 + W_2$
5	$7W_2 - 2W_1 - 4W_0$	$9W_1 - 4W_0 - 4W_2$
6	$12W_2 - 4W_1 - 7W_0$	$9W_0 - 12W_1 + 4W_2$
7	$20W_2 - 7W_1 - 12W_0$	$22W_1 - 12W_0 - 9W_2$
8	$33W_2 - 12W_1 - 20W_0$	$22W_0 - 33W_1 + 12W_2$
9	$54W_2 - 20W_1 - 33W_0$	$56W_1 - 33W_0 - 22W_2$
10	$88W_2 - 33W_1 - 54W_0$	$56W_0 - 88W_1 + 33W_2$

Table 1. A few generalised Leonardo numbers

Now we define three special cases of the sequence $\{W_n\}$. Modified Leonardo sequence $\{G_n\}_{n\geq 0}$, Leonardo-Lucas sequence $\{H_n\}_{n\geq 0}$ and Leonardo sequence $\{l_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_n = 2G_{n-1} - G_{n-3}, \qquad G_0 = 0, G_1 = 1, G_2 = 2, \tag{1.6}$$

$$H_n = 2H_{n-1} - H_{n-3}, \qquad H_0 = 3, H_1 = 2, H_2 = 4, \tag{1.7}$$

$$l_n = 2l_{n-1} - l_{n-3}, \qquad l_0 = 1, l_1 = 1, l_2 = 3, \tag{1.8}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{l_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = 2G_{-(n-2)} - G_{-(n-3)}$$

$$H_{-n} = 2H_{-(n-2)} - H_{-(n-3)}$$

$$l_{-n} = 2l_{-(n-2)} - l_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.6)-(1.8) hold for all integer n.

G_n, H_n and l_n are the sequences A000071, A001612, A001595 in [15], respectively.

Next, we present the first few values of the modified Leonardo, Leonardo-Lucas and Leonardo numbers with positive and negative subscripts:

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
G_n	0	1	2	4	7	12	20	33	54	88	143	$\cdots$
G_{-n}	0	0	-1	0	-2	1	-4	4	-9	12	-22	$\cdots$
H_n	3	2	4	5	8	12	19	30	48	77	124	$\cdots$
H_{-n}	3	0	4	-3	8	-10	19	-28	48	-75	124	$\cdots$
l_n	1	1	3	5	9	15	25	41	67	109	177	$\cdots$
l_{-n}	1	-1	1	-3	3	-7	9	-17	25	-43	67	$\cdots$

Table 2.The first few values of the special third-order numbers with positive and negative subscripts.

For all integers n, modified Leonardo, Leonardo-Lucas and Leonardo numbers (using initial conditions in (1.3)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} = \frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta} - 1 \\ H_n &= \alpha^n + \beta^n + \gamma^n = \alpha^n + \beta^n + 1 \\ l_n &= \frac{2(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta} - 1 \end{aligned}$$

respectively. Note that Binet's formulas of Fibonacci and Lucas numbers, respectively, are

$$\begin{aligned} F_n &= \frac{\alpha^n}{\alpha-\beta} + \frac{\beta^n}{\beta-\alpha} = \frac{\alpha^n-\beta^n}{\alpha-\beta}, \\ l_n &= \alpha^n + \beta^n, \end{aligned}$$

and so

$$G_n = F_{n+2} - 1, \tag{1.9}$$

$$H_n = L_n + 1, \tag{1.10}$$

$$l_n = 2F_{n+1} - 1. \tag{1.11}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.1. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalised Leonardo sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1)x^2}{1 - 2x + x^3}.$$

Proof. Take $r = 2, s = 0, t = -1$ in Lemma 1 in [13]. □

The previous lemma gives the following results as particular examples.

Corollary 1.2. *Generated functions of modified Leonardo, Leonardo-Lucas and Leonardo numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-2x+x^3},\\ \sum_{n=0}^{\infty} H_n x^n &= \frac{3-4x}{1-2x+x^3},\\ \sum_{n=0}^{\infty} l_n x^n &= \frac{1-x+x^2}{1-2x+x^3},\end{aligned}$$

respectively.

2. Dual Hyperbolic generalized Leonardo Numbers

In this section, we define dual hyperbolic generalized Leonardo numbers and present some properties such as Binet's formula and generating function.

Dual hyperbolic generalized Leonardo numbers $\{\widetilde{W}_n\}_{n\geq 0} = \{\widetilde{W}_n(\widetilde{W}_0, \widetilde{W}_1, \widetilde{W}_2)\}_{n\geq 0}$ are defined by

$$\widetilde{W}_n = 2\widetilde{W}_{n-1} - \widetilde{W}_{n-3}, \tag{2.1}$$

with the initial conditions

$$\begin{aligned}\widetilde{W}_0 &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon\left(2W_2 - W_0\right),\\ \widetilde{W}_1 &= W_1 + jW_2 + \varepsilon\left(2W_2 - W_0\right) + j\varepsilon\left(4W_2 - W_1 - 2W_0\right),\\ \widetilde{W}_2 &= W_2 + j\left(2W_2 - W_0\right) + \varepsilon\left(4W_2 - W_1 - 2W_0\right) + j\varepsilon\left(7W_2 - 2W_1 - 4W_0\right)\end{aligned}$$

not all being zero. The sequences $\{\widetilde{W}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widetilde{W}_{-n} = 2\widetilde{W}_{-(n-2)} - \widetilde{W}_{-(n-3)} \tag{2.2}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (2.1) hold for all integer n.

Note that for all integers n, we get

$$\widetilde{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{2.3}$$

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The first few dual hyperbolic generalized Leonardo numbers with positive and negative subscripts are presented in the following tables.

n	$\widetilde{W}_n$
0	$W_0 + jW_1 + \varepsilon W_2 + j\varepsilon\,(2W_2 - W_0)$
1	$W_1 + jW_2 + \varepsilon\,(2W_2 - W_0) + j\varepsilon\,(4W_2 - W_1 - 2W_0)$
2	$W_2 + j\,(2W_2 - W_0) + \varepsilon\,(4W_2 - W_1 - 2W_0) + j\varepsilon\,(7W_2 - 2W_1 - 4W_0)$
3	$2W_2 - W_0 + j\,(4W_2 - W_1 - 2W_0) + \varepsilon\,(7W_2 - 2W_1 - 4W_0) + j\varepsilon\,(12W_2 - 4W_1 - 7W_0)$
4	$4W_2 - W_1 - 2W_0 + j\,(7W_2 - 2W_1 - 4W_0) + \varepsilon\,(12W_2 - 4W_1 - 7W_0) + j\varepsilon\,(20W_2 - 7W_1 - 12W_0)$
5	$7W_2 - 2W_1 - 4W_0 + j\,(12W_2 - 4W_1 - 7W_0) + \varepsilon\,(20W_2 - 7W_1 - 12W_0) + j\varepsilon\,(33W_2 - 12W_1 - 20W_0)$
6	$12W_2 - 4W_1 - 7W_0 + j\,(20W_2 - 7W_1 - 12W_0) + \varepsilon\,(33W_2 - 12W_1 - 20W_0) + j\varepsilon\,(54W_2 - 20W_1 - 33W_0)$
7	$20W_2 - 7W_1 - 12W_0 + j\,(33W_2 - 12W_1 - 20W_0) + \varepsilon\,(54W_2 - 20W_1 - 33W_0) + j\varepsilon\,(88W_2 - 33W_1 - 54W_0)$
8	$33W_2 - 12W_1 - 20W_0 + j\,(54W_2 - 20W_1 - 33W_0) + \varepsilon\,(88W_2 - 33W_1 - 54W_0) + j\varepsilon\,(143W_2 - 54W_1 - 88W_0)$
$\vdots$	$\vdots$

Table 3.The first few dual hyperbolic generalized Leonardo numbers with positive subscript.

n	$\widetilde{W}_{-n}$
1	$2W_1 - W_2 + jW_0 + \varepsilon W_1 + j\varepsilon W_2$
2	$2W_0 - W_1 + j\,(2W_1 - W_2) + \varepsilon W_0 + j\varepsilon W_1$
3	$4W_1 - W_0 - 2W_2 + j\,(2W_0 - W_1) + \varepsilon\,(2W_1 - W_2) + j\varepsilon W_0$
4	$4W_0 - 4W_1 + W_2 + j\,(4W_1 - W_0 - 2W_2) + \varepsilon\,(2W_0 - W_1) + j\varepsilon\,(2W_1 - W_2)$
5	$9W_1 - 4W_0 - 4W_2 + j\,(4W_0 - 4W_1 + W_2) + \varepsilon\,(4W_1 - W_0 - 2W_2) + j\varepsilon\,(2W_0 - W_1)$
6	$9W_0 - 12W_1 + 4W_2 + j\,(9W_1 - 4W_0 - 4W_2) + \varepsilon\,(4W_0 - 4W_1 + W_2) + j\varepsilon\,(4W_1 - W_0 - 2W_2)$
7	$22W_1 - 12W_0 - 9W_2 + j\,(9W_0 - 12W_1 + 4W_2) + \varepsilon\,(9W_1 - 4W_0 - 4W_2) + j\varepsilon\,(4W_0 - 4W_1 + W_2)$
8	$22W_0 - 33W_1 + 12W_2 + j\,(22W_1 - 12W_0 - 9W_2) + \varepsilon\,(9W_0 - 12W_1 + 4W_2) + j\varepsilon\,(9W_1 - 4W_0 - 4W_2)$
$\vdots$	$\vdots$

Table 4.The first few dual hyperbolic generalized Leonardo numbers with negative subscript.

Dual hyperbolic modified Leonardo numbers, $\widetilde{W}_n(j+2\varepsilon+4j\varepsilon, 1+2j+4\varepsilon+7j\varepsilon, 2+4j+7\varepsilon+12j\varepsilon) = \widetilde{G}_n$, are defined by

$$\widetilde{G}_n = 2\widetilde{G}_{n-1} - \widetilde{G}_{n-3} \tag{2.4}$$

with the initial conditions

$$\widetilde{G}_0 = j + 2\varepsilon + 4j\varepsilon, \widetilde{G}_1 = 1 + 2j + 4\varepsilon + 7j\varepsilon, \widetilde{G}_2 = 2 + 4j + 7\varepsilon + 12j\varepsilon.$$

Dual hyperbolic Leonardo-Lucas numbers, $\widetilde{W}_n(2j+4\varepsilon+5j\varepsilon+3, 2+4j+5\varepsilon+8j\varepsilon, 4+5j+8\varepsilon+12j\varepsilon) = \widetilde{H}_n$, are defined by

$$\widetilde{H}_n = 2\widetilde{H}_{n-1} - \widetilde{H}_{n-3} \tag{2.5}$$

with the initial conditions

$$\widetilde{H}_0 = 3 + 2j + 4\varepsilon + 5j\varepsilon, \widetilde{H}_1 = 2 + 4j + 5\varepsilon + 8j\varepsilon, \widetilde{H}_2 = 4 + 5j + 8\varepsilon + 12j\varepsilon.$$

and dual hyperbolic Leonardo numbers , $\widetilde{W}_n(1 + j + 3\varepsilon + 5j\varepsilon, 1 + 3j + 5\varepsilon + 9j\varepsilon, 3 + 5j + 9\varepsilon + 15j\varepsilon) = \widetilde{l}_n$, are defined by

$$\widetilde{l}_n = 2\widetilde{l}_{n-1} - \widetilde{l}_{n-3} \tag{2.6}$$

with the initial conditions

$$\widetilde{l}_0 = 1 + j + 3\varepsilon + 5j\varepsilon, \widetilde{l}_1 = 1 + 3j + 5\varepsilon + 9j\varepsilon, \widetilde{l}_2 = 3 + 5j + 9\varepsilon + 15j\varepsilon.$$

The sequences $\{\widetilde{G}_n\}_{n\geq 0}$, $\{\widetilde{H}_n\}_{n\geq 0}$ and $\{\widetilde{l}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining,

$$\begin{aligned}
\widetilde{G}_{-n} &= 2\widetilde{G}_{-(n-2)} - \widetilde{G}_{-(n-3)}, \\
\widetilde{H}_{-n} &= 2\widetilde{H}_{-(n-2)} - \widetilde{H}_{-(n-3)}, \\
\widetilde{l}_{-n} &= 2\widetilde{l}_{-(n-2)} - \widetilde{l}_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (2.4)-(2.6) hold for all integer n.

Note that for all integers n, we have

$$\begin{aligned}
\widetilde{G}_n &= G_n + jG_{n+1} + \varepsilon G_{n+2} + j\varepsilon G_{n+3}, \\
\widetilde{H}_n &= H_n + jH_{n+1} + \varepsilon H_{n+2} + j\varepsilon H_{n+3}, \\
\widetilde{l}_n &= l_n + jl_{n+1} + \varepsilon l_{n+2} + j\varepsilon l_{n+3}.
\end{aligned}$$

The first few values of dual hyperbolic modified Leonardo numbers, dual hyperbolic Leonardo-Lucas numbers and dual hyperbolic Leonardo numbers with positive and negative subscript are given in the following Tables.

n	$\widetilde{G}_n$	$\widetilde{G}_{-n}$
0	$j + 2\varepsilon + 4j\varepsilon$	$j + 2\varepsilon + 4j\varepsilon$
1	$1 + 2j + 4\varepsilon + 7j\varepsilon$	$\varepsilon + 2j\varepsilon$
2	$2 + 4j + 7\varepsilon + 12j\varepsilon$	$-1 + j\varepsilon$
3	$4 + 7j + 12\varepsilon + 20j\varepsilon$	$-j$
4	$7 + 12j + 20\varepsilon + 33j\varepsilon$	$-2 - \varepsilon$
5	$12 + 20j + 33\varepsilon + 54j\varepsilon$	$1 - 2j - j\varepsilon$
6	$20 + 33j + 54\varepsilon + 88j\varepsilon$	$-4 + j - 2\varepsilon$
7	$33 + 54j + 88\varepsilon + 143j\varepsilon$	$4 - 4j + \varepsilon - 2j\varepsilon$
8	$54 + 88j + 143\varepsilon + 232j\varepsilon$	$-9 + 4j - 4\varepsilon + j\varepsilon$
$\vdots$	$\vdots$	$\vdots$

Table 5. Generalized Modified Leonardo numbers with positive and negative subscripts.

n	$\widetilde{H}_n$	$\widetilde{H}_{-n}$
0	$3+2j+4\varepsilon+5j\varepsilon$	$3+2j+4\varepsilon+5j\varepsilon$
1	$2+4j+5\varepsilon+8j\varepsilon$	$3j+2\varepsilon+4j\varepsilon$
2	$4+5j+8\varepsilon+12j\varepsilon$	$4+3\varepsilon+2j\varepsilon$
3	$5+8j+12\varepsilon+19j\varepsilon$	$-3+4j+3j\varepsilon$
4	$8+12j+19\varepsilon+30j\varepsilon$	$8+4\varepsilon-3j$
5	$12+19j+30\varepsilon+48j\varepsilon$	$-10+8j-3\varepsilon+4j\varepsilon$
6	$19+30j+48\varepsilon+77j\varepsilon$	$19+8\varepsilon-10j-3j\varepsilon$
7	$30+48j+77\varepsilon+124j\varepsilon$	$-28+19j-10\varepsilon+8j\varepsilon$
8	$48+77j+124\varepsilon+200j\varepsilon$	$48+19\varepsilon-28j-10j\varepsilon$
$\vdots$	$\vdots$	$\vdots$

Table 6. Special cases of generalized Leonardo-Lucas numbers with positive and negative subscripts.

n	$\widetilde{l}_n$	$\widetilde{l}_{-n}$
0	$1+j+3\varepsilon+5j\varepsilon$	$1+j+3\varepsilon+5j\varepsilon$
1	$1+3j+5\varepsilon+9j\varepsilon$	$-1+j+\varepsilon+3j\varepsilon$
2	$3+5j+9\varepsilon+15j\varepsilon$	$1-j+\varepsilon+j\varepsilon$
3	$5+9j+15\varepsilon+25j\varepsilon$	$-3+j-\varepsilon+j\varepsilon$
4	$9+15j+25\varepsilon+41j\varepsilon$	$3-3j+\varepsilon-j\varepsilon$
5	$15+25j+41\varepsilon+67j\varepsilon$	$-7+3j-3\varepsilon+j\varepsilon$
6	$25+41j+67\varepsilon+109j\varepsilon$	$9-7j+3\varepsilon-3j\varepsilon$
7	$41+67j+109\varepsilon+177j\varepsilon$	$-17+9j-7\varepsilon+3j\varepsilon$
8	$67+109j+177\varepsilon+287j\varepsilon$	$25-17j+9\varepsilon-7j\varepsilon$
$\vdots$	$\vdots$	$\vdots$

Table 7. Special cases of generalized Leonardo numbers with positive and negative subscripts.

We fix the following notations:

$$\begin{aligned} \widetilde{\alpha} &= 1+j\alpha+\varepsilon\alpha^2+j\varepsilon\left(2\alpha^2-1\right), & (2.7)\\ \widetilde{\beta} &= 1+j\beta+\varepsilon\beta^2+j\varepsilon\left(2\beta^2-1\right), \\ \widetilde{\gamma} &= 1+j\gamma+\varepsilon\gamma^2+j\varepsilon\left(2\gamma^2-1\right), \\ \widetilde{1} &= 1+j+\varepsilon+j\varepsilon. & (2.8) \end{aligned}$$

where

$$\begin{aligned}
\widetilde{\alpha}^2 &= \left(1+\alpha^2\right)+2\alpha j+\left(10\alpha^2-2\alpha-4\right)\varepsilon+j\varepsilon\left(8\alpha^2-4\right), \\
\widetilde{\beta}^2 &= \left(1+\beta^2\right)+2\beta j+\varepsilon\left(10\beta^2-2\beta-4\right)+j\varepsilon\left(8\beta^2-4\right), \\
\widetilde{\gamma}^2 &= \widetilde{1}^2=2+2j+4\varepsilon+4j\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\alpha}\widetilde{\beta} &= j+3j\varepsilon, \\
\widetilde{\alpha}\widetilde{\gamma} &= \widetilde{\alpha}\widetilde{1}=(1+\alpha)+j(\alpha+1)+\varepsilon\left(3\alpha^2+\alpha\right)+j\varepsilon\left(3\alpha^2+\alpha\right), \\
\widetilde{\beta}\widetilde{\gamma} &= \widetilde{\beta}\widetilde{1}=(1+\beta)+j(1+\beta)+\varepsilon\left(3\beta^2+\beta\right)+j\varepsilon\left(3\beta^2+\beta\right)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\alpha}\widetilde{\beta}^2 &= \beta+j+\varepsilon\left(2\beta^2+3\beta-1\right)+j\varepsilon\left(\beta^2+3\right) \\
\widetilde{\beta}\widetilde{\alpha}^2 &= \alpha+j+\varepsilon\left(2\alpha^2+3\alpha-1\right)+j\varepsilon\left(\alpha^2+3\right) \\
\widetilde{\alpha}\widetilde{\gamma}^2 &= \widetilde{\alpha}\widetilde{1}^2=2\alpha+2+\varepsilon\left(6\alpha^2+4\alpha+2\right)+j(2\alpha+2)+j\varepsilon\left(6\alpha^2+4\alpha+2\right) \\
\widetilde{\beta}\widetilde{\gamma}^2 &= \widetilde{\beta}\widetilde{1}^2=2\beta+2+\varepsilon\left(6\beta^2+4\beta+2\right)+j(2\beta+2)+j\varepsilon\left(6\beta^2+4\beta+2\right)
\end{aligned}$$

Next, we present Binet's formula for dual hyperbolic generalized Leonardo numbers.

THEOREM 2.1. *Binet's formula of dual hyperbolic generalized Leonardo numbers can be presented as follows:*

$$\begin{aligned}
\widetilde{W}_n &= \frac{z_1\widetilde{\alpha}\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{z_2\widetilde{\beta}\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{z_3\widetilde{\gamma}\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\
&= \frac{z_1\widetilde{\alpha}\alpha^{n+1}-z_2\widetilde{\beta}\beta^{n+1}}{\alpha-\beta}-z_3\widetilde{1}
\end{aligned}$$

where z_1, z_2 and z_3 are given as

$$\begin{aligned}
z_1 &= W_2-(2-\alpha)W_1+(1-\alpha)W_0 \\
z_2 &= W_2-(2-\beta)W_1+(1-\beta)W_0, \\
z_3 &= W_2-W_1-W_0.
\end{aligned}$$

Proof. Since $\widetilde{W}_n=W_n+jW_{n+1}+\varepsilon W_{n+2}+j\varepsilon W_{n+3}$ and

$$W_n=\frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},$$

we have

$$\begin{aligned}
\widetilde{W}_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}\left(1+j\alpha+\varepsilon\alpha^2+j\varepsilon\alpha^3\right) \\
&\quad+\frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}\left(1+j\beta+\varepsilon\beta^2+j\varepsilon\beta^3\right)+\frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\left(1+j\gamma+\varepsilon\gamma^2+j\varepsilon\gamma^3\right)
\end{aligned}$$

using the notations given in (2.7) we have the result. □

Now we define three particular cases of the sequence $\{\widetilde{W}_n\}$ as follows: the dual hyperbolic modified Leonardo sequence $\{\widetilde{G}_n\}_{n\geq 0}$, the dual hyperbolic Leonardo-Lucas sequence $\{\widetilde{H}_n\}_{n\geq 0}$, and the dual hyperbolic Leonardo sequence $\{\widetilde{l}_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations,

$$\widetilde{G}_n = \frac{\widetilde{\alpha}\alpha^{n+2} - \widetilde{\beta}\beta^{n+2}}{\alpha - \beta} - \widetilde{1}, \tag{2.9}$$

$$\widetilde{H}_n = \frac{(3-\alpha)\,\widetilde{\alpha}\alpha^{n+1} - (3-\beta)\,\widetilde{\beta}\beta^{n+1}}{\alpha-\beta} + \widetilde{1} = \widetilde{\alpha}\alpha^n + \widetilde{\beta}\beta^n + \widetilde{1},$$

$$\widetilde{l}_n = \frac{2\widetilde{\alpha}\alpha^{n+1} - 2\widetilde{\beta}\beta^{n+1}}{\alpha - \beta} - \widetilde{1}. \tag{2.10}$$

The next Theorem presents the generating function of dual hyperbolic generalized Leonardo numbers.

THEOREM 2.2. *Let* $f_{\widetilde{W}_n}(x) = \sum_{n=0}^{\infty} \widetilde{W}_n x^n$ *denote the generating function of dual hyperbolic generalized Leonardo numbers. Then,*

$$f_{\widetilde{W}_n}(x) = \frac{\widetilde{W}_0 + (\widetilde{W}_1 - 2\widetilde{W}_0)x + (\widetilde{W}_2 - 2\widetilde{W}_1)x^2}{1 - 2x + x^3}. \tag{2.11}$$

Proof. Using the definition of dual hyperbolic generalised Leonardo numbers, and substracting $2xf_{\widetilde{W}_n}(x)$ and $-x^3 f_{\widetilde{W}_n}(x)$ from $f_{\widetilde{W}_n}(x)$ we obtain

$$\begin{aligned}
(1-2x-x^3)f_{\widetilde{W}_n}(x) &= \sum_{n=0}^{\infty}\widetilde{W}_n x^n - 2x\sum_{n=0}^{\infty}\widetilde{W}_n x^n - x^3\sum_{n=0}^{\infty}\widetilde{W}_n x^n, \\
&= \sum_{n=0}^{\infty}\widetilde{W}_n x^n - 2\sum_{n=0}^{\infty}\widetilde{W}_n x^{n+1} - \sum_{n=0}^{\infty}\widetilde{W}_n x^{n+3}, \\
&= \sum_{n=0}^{\infty}\widetilde{W}_n x^n - 2\sum_{n=1}^{\infty}\widetilde{W}_{n-1} x^{n} - \sum_{n=3}^{\infty}\widetilde{W}_{n-3} x^{n}, \\
&= (\widetilde{W}_0 + \widetilde{W}_1 x + \widetilde{W}_2 x^2) - 2(\widetilde{W}_0 x + \widetilde{W}_1 x^2) + \sum_{n=3}^{\infty}(\widetilde{W}_n - 2\widetilde{W}_{n-1} - \widetilde{W}_{n-3})x^n, \\
&= (\widetilde{W}_0 + \widetilde{W}_1 x + \widetilde{W}_2 x^2) - 2(\widetilde{W}_0 x + \widetilde{W}_1 x^2), \\
&= \widetilde{W}_0 + (\widetilde{W}_1 - 2\widetilde{W}_0)x + (\widetilde{W}_2 - 2\widetilde{W}_1)x^2,
\end{aligned}$$

and dividing boths sides with $1 - 2x - x^3$ above equation, we get (2.11). □

Theorem 2.2 gives following results as special cases,

$$\begin{aligned}
f_{\widetilde{G}_n}(x) &= \sum_{n=0}^{\infty}\widetilde{G}_n x^n = \frac{(j+2\varepsilon+4j\varepsilon) + (1-j\varepsilon)\,x - (\varepsilon+2j\varepsilon)\,x^2}{1-2x+x^3}, \\
f_{\widetilde{H}_n}(x) &= \sum_{n=0}^{\infty}\widetilde{H}_n x^n = \frac{(3+2j+4\varepsilon+5j\varepsilon) - (4+3\varepsilon+2j\varepsilon)\,x - (3j+2\varepsilon+4j\varepsilon)\,x^2}{1-2x+x^3}, \\
f_{\widetilde{l}_n}(x) &= \sum_{n=0}^{\infty}\widetilde{l}_n x^n = \frac{(1+j+3\varepsilon+5j\varepsilon) - (1-j+\varepsilon+j\varepsilon)\,x + (1-j-\varepsilon-3j\varepsilon)\,x^2}{1-2x+x^3}.
\end{aligned}$$

3. Some Identities About Recurrence Relations of Dual hyperbolic generalized Leonardo Numbers

In this section, we present some identities on dual hyperbolic modified Leonardo, dual hyperbolic Leonardo-Lucas and dual hyperbolic Leonardo numbers.

THEOREM 3.1. *The following equations hold for all integer n*

$$\widetilde{H}_n = 3\widetilde{G}_{n+1} - 4\widetilde{G}_n, \tag{3.1}$$

$$5\widetilde{G}_n = 9\widetilde{H}_{n+2} - 6\widetilde{H}_{n+1} - 8\widetilde{H}_n \tag{3.2}$$

$$2\widetilde{G}_n = -\widetilde{l}_{n+2} + 2\widetilde{l}_{n+1} + \widetilde{l}_n, \tag{3.3}$$

$$\widetilde{l}_n = -\widetilde{G}_{n+2} + 3\widetilde{G}_{n+1} - \widetilde{G}_n, \tag{3.4}$$

$$2\widetilde{H}_n = 4\widetilde{l}_{n+2} - 5\widetilde{l}_{n+1} - \widetilde{l}_n, \tag{3.5}$$

$$5\widetilde{l}_n = 11\widetilde{H}_{n+2} - 9\widetilde{H}_{n+1} - 7\widetilde{H}_n. \tag{3.6}$$

Proof. To prove identity (3.1), we can write $GH_n = aGG_{n+2} + bGG_{n+1} + cGG_n$ and put $n = 0$. Then we get,

$$\begin{aligned} GH_0 &= aGG_2 + bGG_1 + cGG_0, \\ 3 + 2j + 4\varepsilon + 5j\varepsilon &= a(2 + 4j + 7\varepsilon + 12j\varepsilon) + b(1 + 2j + 4\varepsilon + 7j\varepsilon) + c(j + 2\varepsilon + 4j\varepsilon), \end{aligned}$$

and

$$\begin{aligned} 3 &= 2a + b \\ 2 &= 4a + 2b + c \\ 4 &= 7a + 4b + 2c \\ 5 &= 12a + 7b + 4c. \end{aligned}$$

Hence, we obtain $a = 0, b = 3, c = -4$. The other identities can be found similarly. □

LEMMA 3.2. *(See [16])We assume that* $f_{a_n}(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

The generating functions of the even and odd-indexed dual hyperbolic generalized Leonardo sequences are provided by the following theorem.

THEOREM 3.3. *The generating functions of the sequence* $\widetilde{W}_{2n}$ *and* $\widetilde{W}_{2n+1}$ *are provided by*

$$f_{\widetilde{W}_{2n}}(x) = \sum_{n=0}^{\infty} \widetilde{W}_{2n} x^n = \frac{\widetilde{W}_0 + \left(-4\widetilde{W}_0 + \widetilde{W}_2\right) x + \left(2\widetilde{W}_0 - \widetilde{W}_1\right) x^2}{-x^3 + 4x^2 - 4x + 1}, \tag{3.7}$$

$$f_{\widetilde{W}_{2n+1}}(x) = \sum_{n=0}^{\infty} \widetilde{W}_{2n+1} x^n = \frac{\widetilde{W}_1 - \left(\widetilde{W}_0 + 4\widetilde{W}_1 - 2\widetilde{W}_2\right) x + \left(2\widetilde{W}_1 - \widetilde{W}_2\right) x^2}{-x^3 + 4x^2 - 4x + 1}. \tag{3.8}$$

Proof. We only prove (3.7). From Theorem 2.2 we can obtain following identities:

$$f_{\widetilde{W}_{2n}}(\sqrt{x}) = \frac{\widetilde{W}_0 + \left(\widetilde{W}_1 - 2\widetilde{W}_0\right)\sqrt{x} + \left(\widetilde{W}_2 - 2\widetilde{W}_1\right) x}{1 - 2\sqrt{x} + x^{\frac{3}{2}}},$$

$$f_{\widetilde{W}_{2n}}(-\sqrt{x}) = \frac{\widetilde{W}_0 - \left(\widetilde{W}_1 - 2\widetilde{W}_0\right)\sqrt{x} + (\widetilde{W}_2 - 2\widetilde{W}_1) x}{1 + 2\sqrt{x} - x^{\frac{3}{2}}}.$$

Then, using Lemma 3.2 identity (3.7) can be proved . The other identity can be proved similarly. □

From Theorem 3.3, we get the following corollary.

COROLLARY 3.4.

(a):

$$f_{\widetilde{G}_{2n}}(x) = \frac{j + 2\varepsilon + 4j\varepsilon + (2 - \varepsilon - 4j\varepsilon)x + (-1 + j\varepsilon)x^2}{-x^3 + 4x^2 - 4x + 1},$$

$$f_{\widetilde{G}_{2n+1}}(x) = \frac{1 + 2j + 4\varepsilon + 7j\varepsilon + (-j - 4\varepsilon - 8j\varepsilon)x + (\varepsilon + 2j\varepsilon)x^2}{-x^3 + 4x^2 - 4x + 1},$$

(b):

$$f_{\widetilde{H}_{2n}}(x) = \frac{3 + 2j + 4\varepsilon + 5j\varepsilon - (8 + 3j + 8\varepsilon + 8j\varepsilon)x + (4 + 3\varepsilon + 2j\varepsilon)x^2}{-x^3 + 4x^2 - 4x + 1},$$

$$f_{\widetilde{H}_{2n+1}}(x) = \frac{2 + 4j + 5\varepsilon + 8j\varepsilon + (-3 - 8j - 8\varepsilon - 13j\varepsilon)x + (3j + 2\varepsilon + 4j\varepsilon)x^2}{-x^3 + 4x^2 - 4x + 1},$$

(c):

$$f_{\widetilde{l}_{2n}}(x) = \frac{1 + j + 3\varepsilon + 5j\varepsilon + (-1 + j - 3\varepsilon - 5j\varepsilon)x + (1 - j + \varepsilon + j\varepsilon)x^2}{-x^3 + 4x^2 - 4x + 1},$$

$$f_{\widetilde{l}_{2n+1}}(x) = \frac{1 + 3j + 5\varepsilon + 9j\varepsilon + (1 - 3j - 5\varepsilon - 11j\varepsilon)x + (-1 + j + \varepsilon + 3j\varepsilon)x^2}{-x^3 + 4x^2 - 4x + 1}.$$

From Corollary 3.4 we can obtain the following corollary which presents the identities on dual hyperbolic generalised Leonardo sequences.

COROLLARY 3.5.

(a): $(-1 + j\varepsilon)\widetilde{l}_{2n-4} + (2 - \varepsilon - 4j\varepsilon)\widetilde{l}_{2n-2} + (j + 2\varepsilon + 4j\varepsilon)\widetilde{l}_{2n}$

$$= (1 - j + \varepsilon + j\varepsilon)\widetilde{G}_{2n-4} + (-1 + j - 3\varepsilon - 5j\varepsilon)\widetilde{G}_{2n-2} + (1 + j + 3\varepsilon + 5j\varepsilon)\widetilde{G}_{2n}$$

(b): $(-1+j\varepsilon)\,\widetilde{H}_{2n-4}+(2-\varepsilon-4j\varepsilon)\,\widetilde{H}_{2n-2}+(j+2\varepsilon+4j\varepsilon)\,\widetilde{H}_{2n}$

$$=(4+3\varepsilon+2j\varepsilon)\,\widetilde{G}_{2n-4}-(8+3j+8\varepsilon+8j\varepsilon)\,\widetilde{G}_{2n-2}+(3+2j+4\varepsilon+5j\varepsilon)\,\widetilde{G}_{2n}$$

(c): $(4+3\varepsilon+2j\varepsilon)\,\widetilde{l}_{2n-4}-(8+3j+8\varepsilon+8j\varepsilon)\,\widetilde{l}_{2n-2}+(3+2j+4\varepsilon+5j\varepsilon)\,\widetilde{l}_{2n}$

$$=(1-j+\varepsilon+j\varepsilon)\,\widetilde{H}_{2n-4}+(-1+j-3\varepsilon-5j\varepsilon)\,\widetilde{H}_{2n-2}+(1+j+3\varepsilon+5j\varepsilon)\,\widetilde{H}_{2n}$$

(d): $(\varepsilon+2j\varepsilon)\,\widetilde{H}_{2n-3}+(-j-4\varepsilon-8j\varepsilon)\,\widetilde{H}_{2n-1}+(1+2j+4\varepsilon+7j\varepsilon)\,\widetilde{H}_{2n+1}$

$$=(3j+2\varepsilon+4j\varepsilon)\,\widetilde{G}_{2n-3}+(-3-8j-8\varepsilon-13j\varepsilon)\,\widetilde{G}_{2n-1}+(2+4j+5\varepsilon+8j\varepsilon)\,\widetilde{G}_{2n+1}$$

(e): $(-1+j+\varepsilon+3j\varepsilon)\,\widetilde{G}_{2n-3}+(1-3j-5\varepsilon-11j\varepsilon)\,\widetilde{G}_{2n-1}+(1+3j+5\varepsilon+9j\varepsilon)\,\widetilde{G}_{2n+1}$

$$=(\varepsilon+2j\varepsilon)\,\widetilde{l}_{2n-3}+(-j-4\varepsilon-8j\varepsilon)\,\widetilde{l}_{2n-1}+(1+2j+4\varepsilon+7j\varepsilon)\,\widetilde{l}_{2n+1}$$

(f): $(3j+2\varepsilon+4j\varepsilon)\,\widetilde{l}_{2n-3}+(-3-8j-8\varepsilon-13j\varepsilon)\,\widetilde{l}_{2n-1}+(2+4j+5\varepsilon+8j\varepsilon)\,\widetilde{l}_{2n+1}$

$$=(-1+j+\varepsilon+3j\varepsilon)\,\widetilde{H}_{2n-3}+(1-3j-5\varepsilon-11j\varepsilon)\,\widetilde{H}_{2n-1}+(1+3j+5\varepsilon+9j\varepsilon)\,\widetilde{H}_{2n+1}$$

Proof. From Corollary 3.4 we obtain

$$\begin{aligned}&\left(j+2\varepsilon+4j\varepsilon+(2-\varepsilon-4j\varepsilon)\,x+(-1+j\varepsilon)\,x^{2}\right)f_{\widetilde{l}_{2n}}(x)\\&\qquad=\left(1+j+3\varepsilon+5j\varepsilon+(-1+j-3\varepsilon-5j\varepsilon)\,x+(1-j+\varepsilon+j\varepsilon)\,x^{2}\right)f_{\widetilde{G}_{2n}}(x).\end{aligned}$$

The L.H.S. (left hand side) is equal to

$$\begin{aligned}L.H.S.&=\left(j+2\varepsilon+4j\varepsilon+(2-\varepsilon-4j\varepsilon)\,x+(-1+j\varepsilon)\,x^{2}\right)\sum_{n=0}^{\infty}\widetilde{l}_{2n}x^{n}\\&=(-1+j\varepsilon)\sum_{n=0}^{\infty}\widetilde{l}_{2n}x^{n+2}+(2-\varepsilon-4j\varepsilon)\sum_{n=0}^{\infty}\widetilde{l}_{2n}x^{n+1}+(j+2\varepsilon+4j\varepsilon)\sum_{n=0}^{\infty}\widetilde{l}_{2n}x^{n}\\&=(-1+j\varepsilon)\sum_{n=2}^{\infty}\widetilde{l}_{2n-4}x^{n}+(2-\varepsilon-4j\varepsilon)\sum_{n=1}^{\infty}\widetilde{l}_{2n-2}x^{n}+(j+2\varepsilon+4j\varepsilon)\sum_{n=0}^{\infty}\widetilde{l}_{2n}x^{n}\end{aligned}$$

$$\begin{aligned}&=(j+2\varepsilon+4j\varepsilon)\left(\widetilde{l}_{0}+\widetilde{l}_{2}x\right)+(2-\varepsilon-4j\varepsilon)\,\widetilde{l}_{0}x\\&\qquad+\sum_{n=2}^{\infty}((-1+j\varepsilon)\,\widetilde{l}_{2n-4}+(2-\varepsilon-4j\varepsilon)\,\widetilde{l}_{2n-2}+(j+2\varepsilon+4j\varepsilon)\,\widetilde{l}_{2n})x^{n}\end{aligned}$$

$$\begin{aligned}&=(1+j+11\varepsilon+9j\varepsilon)+(7+5j+42\varepsilon+36j\varepsilon)\,x\\&\qquad+\sum_{n=2}^{\infty}((-1+j\varepsilon)\,\widetilde{l}_{2n-4}+(2-\varepsilon-4j\varepsilon)\,\widetilde{l}_{2n-2}+(j+2\varepsilon+4j\varepsilon)\,\widetilde{l}_{2n})x^{n}\end{aligned}$$

whereas the R.H.S. (right hand side) is equal to

$$\begin{aligned} R.H.S. &= \left(1+j+3\varepsilon+5j\varepsilon+(-1+j-3\varepsilon-5j\varepsilon)x+(1-j+\varepsilon+j\varepsilon)x^2\right)\sum_{n=0}^{\infty}\widetilde{G}_{2n}x^n \\ &= (1+j+3\varepsilon+5j\varepsilon)\sum_{n=0}^{\infty}\widetilde{G}_{2n}x^n+(-1+j-3\varepsilon-5j\varepsilon)\sum_{n=0}^{\infty}\widetilde{G}_{2n}x^{n+1}+(1-j+\varepsilon+j\varepsilon)\sum_{n=0}^{\infty}\widetilde{G}_{2n}x^{n+2} \\ &= (1+j+3\varepsilon+5j\varepsilon)\sum_{n=0}^{\infty}\widetilde{G}_{2n}x^n+(-1+j-3\varepsilon-5j\varepsilon)\sum_{n=1}^{\infty}\widetilde{G}_{2n-2}x^{n}+(1-j+\varepsilon+j\varepsilon)\sum_{n=2}^{\infty}\widetilde{G}_{2n-4}x^{n} \end{aligned}$$

$$\begin{aligned} &= (1+j+3\varepsilon+5j\varepsilon)\left(\widetilde{G}_0+\widetilde{G}_2x\right)+(-1+j-3\varepsilon-5j\varepsilon)\widetilde{G}_0x \\ &\quad+\sum_{n=2}^{\infty}\left((1-j+\varepsilon+j\varepsilon)\widetilde{G}_{2n-4}+(-1+j-3\varepsilon-5j\varepsilon)\widetilde{G}_{2n-2}+(1+j+3\varepsilon+5j\varepsilon)\widetilde{G}_{2n}\right)x^n \end{aligned}$$

$$\begin{aligned} &= (1+j+11\varepsilon+9j\varepsilon)+(7+5j+42\varepsilon+36j\varepsilon)x \\ &\quad+\sum_{n=2}^{\infty}\left((1-j+\varepsilon+j\varepsilon)\widetilde{G}_{2n-4}+(-1+j-3\varepsilon-5j\varepsilon)\widetilde{G}_{2n-2}+(1+j+3\varepsilon+5j\varepsilon)\widetilde{G}_{2n}\right)x^n \end{aligned}$$

Comparing the coefficients and the proof of the first identity (a) is done. We can prove other identities similarly. □

We can get an identitiy related to dual hyperbolic generalised Leonardo numbers and numbers given below.

Theorem 3.6. *For all integers m, n the following identity holds:*

$$\widetilde{W}_{m+n}=\widetilde{W}_nG_{m+1}-\widetilde{W}_{n-1}G_{m-1}-\widetilde{W}_{n-2}G_m.$$

Proof. First, we assume that $m, n \geqslant 0$. We prove Theorem 3.6 by mathematical induction on m. If $m=0$ we get

$$\begin{aligned} \widetilde{W}_n &= \widetilde{W}_nG_1-\widetilde{W}_{n-1}G_{-1}-\widetilde{W}_{n-2}G_0 \\ &= \widetilde{W}_n \end{aligned}$$

since $G_1=1, G_0=0, G_{-1}=0$.If $m=1$ we get

$$\begin{aligned} \widetilde{W}_{n+1} &= \widetilde{W}_nG_2-\widetilde{W}_{n-1}G_0-\widetilde{W}_{n-2}G_1 \\ &= 2\widetilde{W}_n-\widetilde{W}_{n-2}. \end{aligned}$$

since $G_2 = 2, G_1 = 1, G_0 = 0$. We assume that the identity given holds for $m \leq k$. For $m = k+1$, we get

$$\begin{aligned}
\widetilde{W}_{(k+1)+n} &= 2\widetilde{W}_{n+k} - \widetilde{W}_{n+k-2} \\
&= 2(\widetilde{W}_n G_{k+1} - \widetilde{W}_{n-1}G_{k-1} - \widetilde{W}_{n-2}G_k) - (\widetilde{W}_n G_{k-1} - \widetilde{W}_{n-1}G_{k-3} - \widetilde{W}_{n-2}G_{k-2}) \\
&= \widetilde{W}_n\,(2G_{k+1} - G_{k-1}) - \widetilde{W}_{n-1}(2G_{k-1} - G_{k-3}) - \widetilde{W}_{n-2}\,(2G_k - G_{k-2}) \\
&= \widetilde{W}_n G_{k+2} - \widetilde{W}_{n-1}G_k - \widetilde{W}_{n-2}G_{k+1} \\
&= \widetilde{W}_n G_{(k+1)+1} - \widetilde{W}_{n-1}G_{(k+1)-1} - \widetilde{W}_{n-2}G_{(k+1)}
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 3.6. The case $m, n < 0$ can be proved similarly. □

For all integers m, n taking $\widetilde{W}_n = \widetilde{G}_n$ or $\widetilde{W}_n = \widetilde{H}_n$ or $\widetilde{W}_n = \widetilde{l}_n$, respectively, we get,

$$\begin{aligned}
\widetilde{G}_{m+n} &= \widetilde{G}_n G_{m+1} - \widetilde{G}_{n-1}G_{m-1} - \widetilde{G}_{n-2}G_m, \\
\widetilde{H}_{m+n} &= \widetilde{H}_n G_{m+1} - \widetilde{H}_{n-1}G_{m-1} - \widetilde{H}_{n-2}G_m, \\
\widetilde{l}_{m+n} &= \widetilde{l}_n G_{m+1} - \widetilde{l}_{n-1}G_{m-1} + \widetilde{l}_{n-2}G_m.
\end{aligned}$$

4. Simpson's Formula

In this section, we present Simpson's formula of generalized Leonardo numbers. This is a special cases of [17, Theorem 4.1]. We give the proof by calculating determinant and using Binet's formula of generalized Leonardo numbers.

Theorem 4.1 (Simpson's formula of generalized Gaussian Leonardo numbers). *For all integers n, we can write following equality*

$$\begin{aligned}
\begin{vmatrix} \widetilde{W}_{n+2} & \widetilde{W}_{n+1} & \widetilde{W}_n \\ \widetilde{W}_{n+1} & \widetilde{W}_n & \widetilde{W}_{n-1} \\ \widetilde{W}_n & \widetilde{W}_{n-1} & \widetilde{W}_{n-2} \end{vmatrix} &= (-1)^n \begin{vmatrix} \widetilde{W}_2 & \widetilde{W}_1 & \widetilde{W}_0 \\ \widetilde{W}_1 & \widetilde{W}_0 & \widetilde{W}_{-1} \\ \widetilde{W}_0 & \widetilde{W}_{-1} & \widetilde{W}_{-2} \end{vmatrix} \\
&= (-1)^n \left[-\widetilde{W}_0^3 + 2\widetilde{W}_0\widetilde{W}_1\widetilde{W}_{-1} + \widetilde{W}_2\widetilde{W}_{-2}\widetilde{W}_0 - \widetilde{W}_{-2}\widetilde{W}_1^2 - \widetilde{W}_2\widetilde{W}_{-1}^2 \right]
\end{aligned}$$

Proof. Using Theorem 2.1 we obtain the proof.

From the Theorem 4.1 we get the following corollary.

Corollary 4.2. *For all integers n, we get the following identities:*

(a):
$$\begin{aligned}
\begin{vmatrix} \widetilde{G}_{n+2} & \widetilde{G}_{n+1} & \widetilde{G}_n \\ \widetilde{G}_{n+1} & \widetilde{G}_n & \widetilde{G}_{n-1} \\ \widetilde{G}_n & \widetilde{G}_{n-1} & \widetilde{G}_{n-2} \end{vmatrix} &= (-1)^n \left[-\widetilde{G}_0^3 + 2\widetilde{G}_0\widetilde{G}_1\widetilde{G}_{-1} + \widetilde{G}_2\widetilde{G}_{-2}\widetilde{G}_0 - \widetilde{G}_{-2}\widetilde{G}_1^2 - \widetilde{G}_2\widetilde{G}_{-1}^2 \right] \\
&= (-1)^n\,(1 + j + 4\varepsilon + 4j\varepsilon)
\end{aligned}$$

(b): $$\begin{vmatrix} \widetilde{H}_{n+2} & \widetilde{H}_{n+1} & \widetilde{H}_{n} \\ \widetilde{H}_{n+1} & \widetilde{H}_{n} & \widetilde{H}_{n-1} \\ \widetilde{H}_{n} & \widetilde{H}_{n-1} & \widetilde{H}_{n-2} \end{vmatrix} = (-1)^n \left[-\widetilde{H}_0^3 + 2\widetilde{H}_0\widetilde{H}_1\widetilde{H}_{-1} + \widetilde{H}_2\widetilde{H}_{-2}\widetilde{H}_0 - \widetilde{H}_{-2}\widetilde{H}_1^2 - \widetilde{H}_2\widetilde{H}_{-1}^2 \right]$$
$$= 5(-1)^n (1 + j + 4\varepsilon + 4j\varepsilon)$$

(c): $$\begin{vmatrix} \widetilde{l}_{n+2} & \widetilde{l}_{n+1} & \widetilde{l}_{n} \\ \widetilde{l}_{n+1} & \widetilde{l}_{n} & \widetilde{l}_{n-1} \\ \widetilde{l}_{n} & \widetilde{l}_{n-1} & \widetilde{l}_{n-2} \end{vmatrix} = (-1)^n \left[-\widetilde{l}_0^3 + 2\widetilde{l}_0\widetilde{l}_1\widetilde{l}_{-1} + \widetilde{l}_2\widetilde{l}_{-2}\widetilde{l}_0 - \widetilde{l}_{-2}\widetilde{l}_1^2 - \widetilde{l}_2\widetilde{l}_{-1}^2 \right]$$
$$= 4(-1)^{n+1} (1 + j + 4\varepsilon + 4j\varepsilon)$$

5. Sum Formulas

In this section, we identify some sum formulas of generalized Leonardo numbers.

THEOREM 5.1. *For all integers $n \geq 0$, we have sum formulas given below*

(a): $\sum_{k=0}^{n} \widetilde{W}_k = (n+3)\widetilde{W}_n - (n+2)\widetilde{W}_{n+2} + (n+3)\widetilde{W}_{n+1} + 2\widetilde{W}_2 - 3\widetilde{W}_1 - 2\widetilde{W}_0,$

(b): $\sum_{k=0}^{n} \widetilde{W}_{2k} = (n+1)\widetilde{W}_{2n} + (n+2)\widetilde{W}_{2n+1} - (n+1)\widetilde{W}_{2n+2} + \widetilde{W}_2 - 2\widetilde{W}_1,$

(c): $\sum_{k=0}^{n} \widetilde{W}_{2k+1} = (n+1)\widetilde{W}_{2n+1} + (n+1)\widetilde{W}_{2n} - n\widetilde{W}_{2n+2} - \widetilde{W}_0.$

Proof. From (2.3) we can write the following sum formulas.

$$\sum_{k=0}^{n} \widetilde{W}_k = \sum_{k=0}^{n} (W_k + jW_{k+1} + \varepsilon W_{k+2} + j\varepsilon W_{k+3}) \tag{5.1}$$
$$\sum_{k=0}^{n} \widetilde{W}_{2k} = \sum_{k=0}^{n} (W_{2k} + jW_{2k+1} + \varepsilon W_{2k+2} + j\varepsilon W_{2k+3}), \tag{5.2}$$
$$\sum_{k=0}^{n} \widetilde{W}_{2k+1} = \sum_{k=0}^{n} (W_{2k+1} + jW_{2k+2} + \varepsilon W_{2k+3} + j\varepsilon W_{2k+4}). \tag{5.3}$$

Using sum formulas in [18] we can write

(a): $\sum_{k=0}^{n} W_k = (n+3)W_n + (n+3)W_{n+1} - (n+2)W_{n+2} + 2W_2 - 3W_1 - 2W_0,$

$\sum_{k=0}^{n} W_{k+1} = (n+3)W_{n+1} + (n+3)W_{n+2} - (n+2)W_{n+3} + 2W_3 - 3W_2 - 2W_1,$

$\sum_{k=0}^{n} W_{k+2} = (n+3)W_{n+2} + (n+3)W_{n+3} - (n+2)W_{n+4} + 2W_4 - 3W_3 - 2W_2,$

$\sum_{k=0}^{n} W_{k+3} = (n+3)W_{n+3} + (n+3)W_{n+4} - (n+2)W_{n+5} + 2W_5 - 3W_4 - 2W_3.$

Using the above equalities in (5.1) we prove **(a):** .

(b): $\sum_{k=0}^{n} W_{2k} = (n+1)W_{2n} + (n+2)W_{2n+1} - (n+1)W_{2n+2} + W_2 - 2W_1,$

$\sum_{k=0}^{n} W_{2k+1} = (n+1)W_{2n+1} + (n+2)W_{2n+2} - (n+1)W_{2n+3} + W_3 - 2W_2,$

$$\sum_{k=0}^{n} W_{2k+2} = (n+1)\,W_{2n+2} + (n+2)W_{2n+3} - (n+1)\,W_{2n+4} + W_4 - 2W_3,$$
$$\sum_{k=0}^{n} W_{2k+3} = (n+1)\,W_{2n+3} + (n+2)W_{2n+4} - (n+1)\,W_{2n+5} + W_5 - 2W_4.$$

Using the above equalities in (5.2) we prove **(b):** .

(c): $\sum_{k=0}^{n} W_{2k+1} = (n+1)\,W_{2n} + (n+1)\,W_{2n+1} - nW_{2n+2} - W_0,$

$$\sum_{k=0}^{n} W_{2k+2} = (n+1)\,W_{2n+1} + (n+1)\,W_{2n+2} - nW_{2n+3} - W_1,$$
$$\sum_{k=0}^{n} W_{2k+3} = (n+1)\,W_{2n+2} + (n+1)\,W_{2n+3} - nW_{2n+4} - W_2,$$
$$\sum_{k=0}^{n} W_{2k+4} = (n+1)\,W_{2n+3} + (n+1)\,W_{2n+4} - nW_{2n+5} - W_3.$$

Using the above equalities in (5.3) we prove **(c):**. □

From the previous theorem, we can give the following three corollaries regarding the sum formulas of the dual hyperbolic modified Leonardo, dual hyperbolic Leonardo-Lucas and dual hyperbolic Leonardo numbers.

Corollary 5.2.

(a): $\sum_{k=0}^{n} \widetilde{G}_k = (n+3)\,\widetilde{G}_n - (n+2)\widetilde{G}_{n+2} + (n+3)\,\widetilde{G}_{n+1} - (-1+2\varepsilon+5j\varepsilon)\,,$

(b): $\sum_{k=0}^{n} \widetilde{H}_k = (n+3)\,\widetilde{H}_n - (n+2)\widetilde{H}_{n+2} + (n+3)\,\widetilde{H}_{n+1} - (4+6j+7\varepsilon+10j\varepsilon)\,,$

(c): $\sum_{k=0}^{n} \widetilde{l}_k = (n+3)\,\widetilde{l}_n - (n+2)\widetilde{l}_{n+2} + (n+3)\,\widetilde{l}_{n+1} - (-1+j+3\varepsilon+7j\varepsilon)\,.$

Corollary 5.3.

(a): $\sum_{k=0}^{n} \widetilde{G}_{2k} = (n+1)\,\widetilde{G}_{2n} + (n+2)\widetilde{G}_{2n+1} - (n+1)\,\widetilde{G}_{2n+2} - (\varepsilon+2j\varepsilon)\,,$

(b): $\sum_{k=0}^{n} \widetilde{H}_{2k} = (n+1)\,\widetilde{H}_{2n} + (n+2)\widetilde{H}_{2n+1} - (n+1)\,\widetilde{H}_{2n+2} - (3j+2\varepsilon+4j\varepsilon)\,,$

(c): $\sum_{k=0}^{n} \widetilde{l}_{2k} = (n+1)\,\widetilde{l}_{2n} + (n+2)\widetilde{l}_{2n+1} - (n+1)\,\widetilde{l}_{2n+2} - (-1+j+\varepsilon+3j\varepsilon)\,.$

Corollary 5.4.

(a): $\sum_{k=0}^{n} \widetilde{G}_{2k+1} = (n+1)\,\widetilde{G}_{2n+1} + (n+1)\,\widetilde{G}_{2n} - n\widetilde{G}_{2n+2} - (j+2\varepsilon+4j\varepsilon)\,,$

(b): $\sum_{k=0}^{n} \widetilde{H}_{2k+1} = (n+1)\,\widetilde{H}_{2n+1} + (n+1)\,\widetilde{H}_{2n} - n\widetilde{H}_{2n+2} - (3+2j+4\varepsilon+5j\varepsilon)\,,$

(c): $\sum_{k=0}^{n} \widetilde{l}_{2k+1} = (n+1)\,\widetilde{l}_{2n+1} + (n+1)\,\widetilde{l}_{2n} - n\widetilde{l}_{2n+2} - (1+j+3\varepsilon+5j\varepsilon)\,.$

5.1. Sums of the squares and other sum formulas. We now give the sums of squares and consecutive multiplications of dual hyperbolic generalised Leonardo numbers below.

THEOREM 5.5. *For all integers $n \geq 0$, we have sum formulas given below*

$$\textbf{(a):}\ \sum_{k=0}^{n} \widetilde{W}_k^2 = \left(n+\tfrac{7}{2}\right)\widetilde{W}_{n+3}^2 + \left(n+\tfrac{9}{2}\right)\widetilde{W}_{n+2}^2 + \left(n+\tfrac{7}{2}\right)\widetilde{W}_{n+1}^2 - 2\left(n+4\right)\widetilde{W}_{n+2}\widetilde{W}_{n+3} + 2(n+5)\widetilde{W}_{n+1}\widetilde{W}_{n+2} - 2(n+4)\widetilde{W}_{n+1}\widetilde{W}_{n+3} + 6\widetilde{W}_1\widetilde{W}_2 - 8\widetilde{W}_0\widetilde{W}_1 + 6\widetilde{W}_0\widetilde{W}_2 - \tfrac{5}{2}\widetilde{W}_2^2 - \tfrac{7}{2}\widetilde{W}_1^2 - \tfrac{5}{2}\widetilde{W}_0^2, \quad (5.4)$$

$$\textbf{(b):}\ \sum_{k=0}^{n} \widetilde{W}_k\widetilde{W}_{k+1} = (n+4)\widetilde{W}_{n+1}^2 + (n+5)\widetilde{W}_{n+2}^2 + (n+3)\widetilde{W}_{n+3}^2 + \left(2n+\tfrac{15}{2}\right)\widetilde{W}_{n+1}\widetilde{W}_{n+2} - \left(2n+\tfrac{13}{2}\right)\widetilde{W}_{n+1}\widetilde{W}_{n+3} - \left(2n+\tfrac{15}{2}\right)\widetilde{W}_{n+2}\widetilde{W}_{n+3} - 3\widetilde{W}_0^2 - 4\widetilde{W}_1^2 - 2\widetilde{W}_2^2 - \tfrac{11}{2}\widetilde{W}_0\widetilde{W}_1 + \tfrac{9}{2}\widetilde{W}_0\widetilde{W}_2 + \tfrac{11}{2}\widetilde{W}_1\widetilde{W}_2,$$

$$\textbf{(c):}\sum_{k=0}^{n} \widetilde{W}_k\widetilde{W}_{k+2} = (n+3)\widetilde{W}_{n+1}^2 + (n+4)\widetilde{W}_{n+2}^2 + (n+2)\widetilde{W}_{n+3}^2 + \left(2n+\tfrac{15}{2}\right)\widetilde{W}_{n+1}\widetilde{W}_{n+2} - \left(2n+\tfrac{11}{2}\right)\widetilde{W}_{n+1}\widetilde{W}_{n+3} - \left(2n+\tfrac{11}{2}\right)\widetilde{W}_{n+2}\widetilde{W}_{n+3} - 2\widetilde{W}_0^2 - 3\widetilde{W}_1^2 - \widetilde{W}_2^2 - \tfrac{11}{2}\widetilde{W}_1\widetilde{W}_0 + \tfrac{7}{2}\widetilde{W}_0\widetilde{W}_2 + \tfrac{7}{2}\widetilde{W}_1\widetilde{W}_2.$$

As a result of Theorem 5.5, we can give the following corollary.

COROLLARY 5.6.

$$\textbf{(a):}\ \sum_{k=0}^{n} \widetilde{G}_k^2 = \left(n+\tfrac{7}{2}\right)\widetilde{G}_{n+3}^2 + \left(n+\tfrac{9}{2}\right)\widetilde{G}_{n+2}^2 + \left(n+\tfrac{7}{2}\right)\widetilde{G}_{n+1}^2 - 2\left(n+4\right)\widetilde{G}_{n+2}\widetilde{G}_{2n+3} + 2(n+5)\widetilde{G}_{n+1}\widetilde{G}_{n+2} - 2(n+4)\widetilde{G}_{n+1}\widetilde{G}_{n+3} - \left(2+2j-2\varepsilon-3j\varepsilon\right),$$

$$\sum_{k=0}^{n} \widetilde{G}_k\widetilde{G}_{k+1} = (n+4)\widetilde{G}_{n+1}^2 + (n+5)\widetilde{G}_{n+2}^2 + (n+3)\widetilde{G}_{n+3}^2 + \left(2n+\tfrac{15}{2}\right)\widetilde{G}_{n+1}\widetilde{G}_{n+2} - \left(2n+\tfrac{13}{2}\right)\widetilde{G}_{n+1}\widetilde{G}_{n+3} - \left(2n+\tfrac{15}{2}\right)\widetilde{G}_{n+2}\widetilde{G}_{n+3} - \left(1+\frac{1}{2}j-\varepsilon-\frac{11}{2}j\varepsilon\right),$$

$$\sum_{k=0}^{n} \widetilde{G}_k\widetilde{G}_{k+2} = (n+3)\widetilde{G}_{n+1}^2 + (n+4)\widetilde{G}_{n+2}^2 + (n+2)\widetilde{G}_{n+3}^2 + \left(2n+\tfrac{15}{2}\right)\widetilde{G}_{n+1}\widetilde{G}_{n+2} - \left(2n+\tfrac{11}{2}\right)\widetilde{G}_{n+1}\widetilde{G}_{n+3} - \left(2n+\tfrac{11}{2}\right)\widetilde{G}_{n+2}\widetilde{G}_{n+3} + \left(1+\frac{3}{2}j+5\varepsilon+\frac{17}{2}j\varepsilon\right),$$

$$\textbf{(b):}\ \sum_{k=0}^{n} \widetilde{H}_k^2 = \left(n+\tfrac{7}{2}\right)\widetilde{H}_{n+3}^2 + \left(n+\tfrac{9}{2}\right)\widetilde{H}_{n+2}^2 + \left(n+\tfrac{7}{2}\right)\widetilde{H}_{n+1}^2 - 2\left(n+4\right)\widetilde{H}_{n+2}\widetilde{H}_{2n+3} + 2(n+5)\widetilde{H}_{n+1}\widetilde{H}_{n+2} - 2(n+4)\widetilde{H}_{n+1}\widetilde{H}_{n+3} - \left(17+20j+58\varepsilon+57j\varepsilon\right),$$

$$\sum_{k=0}^{n}\widetilde{H}_k\widetilde{H}_{k+1}=(n+4)\,\widetilde{H}_{n+1}^2+(n+5)\,\widetilde{H}_{n+2}^2+(n+3)\,\widetilde{H}_{n+3}^2+\left(2n+\tfrac{15}{2}\right)\widetilde{H}_{n+1}\widetilde{H}_{n+2}$$
$$-\left(2n+\tfrac{13}{2}\right)\widetilde{H}_{n+1}\widetilde{H}_{n+3}-\left(2n+\tfrac{15}{2}\right)\widetilde{H}_{n+2}\widetilde{H}_{n+3}-\left(25+\frac{43}{2}j+78\varepsilon+\frac{145}{2}j\varepsilon\right),$$

$$\sum_{k=0}^{n}\widetilde{H}_k\widetilde{H}_{k+2}=(n+3)\,\widetilde{H}_{n+1}^2+(n+4)\,\widetilde{H}_{n+2}^2+(n+2)\,\widetilde{H}_{n+3}^2+\left(2n+\tfrac{15}{2}\right)\widetilde{H}_{n+1}\widetilde{H}_{n+2}$$
$$-\left(2n+\tfrac{11}{2}\right)\widetilde{H}_{n+1}\widetilde{H}_{n+3}-\left(2n+\tfrac{11}{2}\right)\widetilde{H}_{n+2}\widetilde{H}_{n+3}-\left(29+\frac{57}{2}j+103\varepsilon+\frac{193}{2}j\varepsilon\right),$$

(c): $$\sum_{k=0}^{n}\widetilde{l}_k^2=\left(n+\tfrac{7}{2}\right)\widetilde{l}_{n+3}^2+\left(n+\tfrac{9}{2}\right)\widetilde{l}_{n+2}^2+\left(n+\tfrac{7}{2}\right)\widetilde{l}_{n+1}^2-2\,(n+4)\,\widetilde{l}_{n+2}\widetilde{l}_{2n+3}$$
$$+2(n+5)\widetilde{l}_{n+1}\widetilde{l}_{n+2}-2(n+4)\widetilde{l}_{n+1}\widetilde{l}_{n+3}-(1+j-2\varepsilon-6j\varepsilon),$$

$$\sum_{k=0}^{n}\widetilde{l}_k\widetilde{l}_{k+1}=(n+4)\,\widetilde{l}_{n+1}^2+(n+5)\,\widetilde{l}_{n+2}^2+(n+3)\,\widetilde{l}_{n+3}^2+\left(2n+\tfrac{15}{2}\right)\widetilde{l}_{n+1}\widetilde{l}_{n+2}$$
$$-\left(2n+\tfrac{13}{2}\right)\widetilde{l}_{n+1}\widetilde{l}_{n+3}-\left(2n+\tfrac{15}{2}\right)\widetilde{l}_{n+2}\widetilde{l}_{n+3}-(1-j-6j\varepsilon).$$

$$\sum_{k=0}^{n}\widetilde{l}_k\widetilde{l}_{k+2}=(n+3)\,\widetilde{l}_{n+1}^2+(n+4)\,\widetilde{l}_{n+2}^2+(n+2)\,\widetilde{l}_{n+3}^2+\left(2n+\tfrac{15}{2}\right)\widetilde{l}_{n+1}\widetilde{l}_{n+2}$$
$$-\left(2n+\tfrac{11}{2}\right)\widetilde{l}_{n+1}\widetilde{l}_{n+3}-\left(2n+\tfrac{11}{2}\right)\widetilde{l}_{n+2}\widetilde{l}_{n+3}+(21+15j+126\varepsilon+104j\varepsilon),$$

6. Matrix Formulation of $\widetilde{W}_n$

Consider the triangular sequence $\{W_n\}$ defined by the third-order recurrence relation following

$$W_n=2W_{n-1}-W_{n-3}$$

with the initial conditions

$$W_0=0,\ W_1=1,\ W_2=3.$$

We define the square matrix A of order 3 as

$$A=\begin{pmatrix}2&0&-1\\1&0&0\\0&1&0\end{pmatrix}$$

such that $\det A=-1$. Then we give the following lemma.

Lemma 6.1. *For $n\geq 0$ the following identity is true*

$$\begin{pmatrix}\widetilde{W}_{n+2}\\\widetilde{W}_{n+1}\\\widetilde{W}_n\end{pmatrix}=\begin{pmatrix}2&0&-1\\1&0&0\\0&1&0\end{pmatrix}^n\begin{pmatrix}\widetilde{W}_2\\\widetilde{W}_1\\\widetilde{W}_0\end{pmatrix}. \tag{6.1}$$

Proof. The identity (6.1) can be proved by mathematical induction on n. If $n=0$ we obtain

$$\begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_n \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n=k$. Thus, the following identity is true.:

$$\begin{pmatrix} \widetilde{W}_{k+2} \\ \widetilde{W}_{k+1} \\ \widetilde{W}_k \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix}.$$

For $n=k+1$, we get

$$\begin{aligned} \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix} &= \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{W}_{k+2} \\ \widetilde{W}_{k+1} \\ \widetilde{W}_k \end{pmatrix} \\ &= \begin{pmatrix} 2\widetilde{W}_{k+2} - \widetilde{W}_k \\ \widetilde{W}_{k+2} \\ \widetilde{W}_{k+1} \end{pmatrix} = \begin{pmatrix} \widetilde{W}_{k+3} \\ \widetilde{W}_{k+2} \\ \widetilde{W}_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}.$$

For the proof see [19].

We define

$$N_{\widetilde{W}} = \begin{pmatrix} \widetilde{W}_2 & \widetilde{W}_1 & \widetilde{W}_0 \\ \widetilde{W}_1 & \widetilde{W}_0 & \widetilde{W}_{-1} \\ \widetilde{W}_0 & \widetilde{W}_{-1} & \widetilde{W}_{-2} \end{pmatrix}, \tag{6.2}$$

$$E_{\widetilde{W}} = \begin{pmatrix} \widetilde{W}_{n+2} & \widetilde{W}_{n+1} & \widetilde{W}_n \\ \widetilde{W}_{n+1} & \widetilde{W}_n & \widetilde{W}_{n-1} \\ \widetilde{W}_n & \widetilde{W}_{n-1} & \widetilde{W}_{n-2} \end{pmatrix}. \tag{6.3}$$

Now, we have the following theorem with $N_{\widetilde{W}}$ and $E_{\widetilde{W}}$

THEOREM 6.2. *Using* $N_{\widetilde{W}}$ *and* $E_{\widetilde{W}}$, *we get*

$$A^n N_{\widetilde{W}} = E_{\widetilde{W}}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{\widetilde{W}} &= \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix} \begin{pmatrix} \widetilde{W}_2 & \widetilde{W}_1 & \widetilde{W}_0 \\ \widetilde{W}_1 & \widetilde{W}_0 & \widetilde{W}_{-1} \\ \widetilde{W}_0 & \widetilde{W}_{-1} & \widetilde{W}_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= \widetilde{W}_2 G_{n+1} - \widetilde{W}_1 G_{n-1} - \widetilde{W}_0 G_n, \\ a_{12} &= \widetilde{W}_1 G_{n+1} - \widetilde{W}_0 G_{n-1} - \widetilde{W}_{-1} G_n, \\ a_{13} &= \widetilde{W}_0 G_{n+1} - \widetilde{W}_{-1} G_{n-1} - \widetilde{W}_{-2} G_n, \\ a_{21} &= \widetilde{W}_2 G_n - \widetilde{W}_1 G_{n-2} - \widetilde{W}_0 G_{n-1}, \\ a_{22} &= \widetilde{W}_1 G_n - \widetilde{W}_0 G_{n-2} - \widetilde{W}_{-1} G_{n-1}, \\ a_{23} &= \widetilde{W}_0 G_n - \widetilde{W}_{-1} G_{n-2} - \widetilde{W}_{-2} G_{n-1}, \\ a_{31} &= \widetilde{W}_2 G_{n-1} - \widetilde{W}_1 G_{n-3} - \widetilde{W}_0 G_{n-2}, \\ a_{32} &= \widetilde{W}_1 G_{n-1} - \widetilde{W}_0 G_{n-3} - \widetilde{W}_{-1} G_{n-2}, \\ a_{33} &= \widetilde{W}_0 G_{n-1} - \widetilde{W}_{-1} G_{n-3} - \widetilde{W}_{-2} G_{n-2}, \end{aligned}$$

Using the Theorem 3.6 we see that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \widetilde{W}_{n+2} & \widetilde{W}_{n+1} & \widetilde{W}_n \\ \widetilde{W}_{n+1} & \widetilde{W}_n & \widetilde{W}_{n-1} \\ \widetilde{W}_n & \widetilde{W}_{n-1} & \widetilde{W}_{n-2} \end{pmatrix}.$$

Hence, the proof is done. □

By taking, $\widetilde{W}_n = \widetilde{G}_n$ with $\widetilde{G}_0$, $\widetilde{G}_1$, $\widetilde{G}_2$ in (6.2) and (6.3), $\widetilde{W}_n = \widetilde{H}_n$ with $\widetilde{H}_0$, $\widetilde{H}_1$, $\widetilde{H}_2$ in (6.2) and (6.3), $\widetilde{W}_n = \widetilde{l}_n$ with $\widetilde{l}_0$, $\widetilde{l}_1$, $\widetilde{l}_2$ in (6.2) and (6.3) respectively, we get:

$$N_{\widetilde{G}} = \begin{pmatrix} 2+4j+7\varepsilon+12j\varepsilon & 1+2j+4\varepsilon+7j\varepsilon & j+2\varepsilon+4j\varepsilon \\ 1+2j+4\varepsilon+7j\varepsilon & j+2\varepsilon+4j\varepsilon & \varepsilon+2j\varepsilon \\ j+2\varepsilon+4j\varepsilon & \varepsilon+2j\varepsilon & -1+j\varepsilon \end{pmatrix}, \quad E_{\widetilde{G}} = \begin{pmatrix} \widetilde{G}_{n+2} & \widetilde{G}_{n+1} & \widetilde{G}_n \\ \widetilde{G}_{n+1} & \widetilde{G}_n & \widetilde{G}_{n-1} \\ \widetilde{G}_n & \widetilde{G}_{n-1} & \widetilde{G}_{n-2} \end{pmatrix}$$

$$N_{\widetilde{H}} = \begin{pmatrix} 5j+8\varepsilon+12j\varepsilon+4 & 4j+5\varepsilon+8j\varepsilon+2 & 2j+4\varepsilon+5j\varepsilon+3 \\ 4j+5\varepsilon+8j\varepsilon+2 & 2j+4\varepsilon+5j\varepsilon+3 & 3j+2\varepsilon+4j\varepsilon \\ 2j+4\varepsilon+5j\varepsilon+3 & 3j+2\varepsilon+4j\varepsilon & 4+3\varepsilon+2j\varepsilon \end{pmatrix}, \quad E_{\widetilde{H}} = \begin{pmatrix} \widetilde{H}_{n+2} & \widetilde{H}_{n+1} & \widetilde{H}_n \\ \widetilde{H}_{n+1} & \widetilde{H}_n & \widetilde{H}_{n-1} \\ \widetilde{H}_n & \widetilde{H}_{n-1} & \widetilde{H}_{n-2} \end{pmatrix}$$

$$N_{\widetilde{l}} = \begin{pmatrix} 5j+9\varepsilon+15j\varepsilon+3 & 3j+5\varepsilon+9j\varepsilon+1 & j+3\varepsilon+5j\varepsilon+1 \\ 3j+5\varepsilon+9j\varepsilon+1 & j+3\varepsilon+5j\varepsilon+1 & -1+j+\varepsilon+3j\varepsilon \\ j+3\varepsilon+5j\varepsilon+1 & -1+j+\varepsilon+3j\varepsilon & 1-j+\varepsilon+j\varepsilon \end{pmatrix}, \quad E_{\widetilde{l}} = \begin{pmatrix} \widetilde{l}_{n+2} & \widetilde{l}_{n+1} & \widetilde{l}_n \\ \widetilde{l}_{n+1} & \widetilde{l}_n & \widetilde{l}_{n-1} \\ \widetilde{l}_n & \widetilde{l}_{n-1} & \widetilde{l}_{n-2} \end{pmatrix}.$$

From Theorem 6.2, we can write the following corollary.

Corollary 6.3. *The following identities are holds:*

(a): $A^n N_{\widetilde{G}} = E_{\widetilde{G}}$.
(b): $A^n N_{\widetilde{H}} = E_{\widetilde{H}}$.
(b): $A^n N_{\widetilde{l}} = E_{\widetilde{l}}$.

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