Original Research Article

A Rotating Field Solution of the Dirac Equation

Abstract

A deterministic field theory of the Dirac equation along lines that Einstein, and possibly Dirac, may have approved of, was given by Toyoki Koga. In this paper, we work out some relevant properties of Koga's solution to the Dirac equation. In particular, we consider his claim that the solution contains a term representing a rotating field. We confirm his claim and find additional information on coordinate transformations using the Hopf map.

Keywords: Dirac equation, Klein-Gordon equation, Rotating Field

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1 Introduction

The basic structure of Quantum Mechanics was developed in the 1920s. In 1927 Pauli extended the theory of Schrödinger to deal with the magnetic moment of the electron by introducing (apparent) spin as an axiom. The following year, Dirac proposed an equation for the electron field $\psi(x, y, z, t)$ which took special relativity into account.

As for the interpretation of the equations of Quantum Mechanics, there was a near consensus. Most physicists accepted the view of Heisenberg and Bohr, which became known as the Copenhagen Interpretation. The electron was believed to be a sizeless particle whose properties like momentum and energy were to be found by applying certain operators to the field ψ. The result of applying an operator corresponded to a "measurement". Ideas like wave-particle duality, the uncertainty principle and the role of the observer were introduced.

Among the founders of Quantum Theory, a few like Einstein, Schrödinger and de Broglie did not accept these ideas initially. Most of them eventually gave up their opposition; an exception was Einstein. In Einstein's opinion, Quantum Mechanics, e.g., the solution of the Schrödinger equation, was a purely statistical theory; it applied only to the average behaviour of an ensemble. He believed that there ought to be an underlying deterministic theory which would yield Quantum Mechanics on averaging over ensembles. However, Einstein was unable to provide such a theory.

During the 1970s, Toyoki Koga published the results of his study of the foundations of Quantum Mechanics ((T. Koga , 1972), (J. Mehra , 1973), (T. Koga , 1975), (T. Koga , 1975), (T. Koga , 1976), (T. Koga , 1980), (T. Koga, 1983)). As a consequence of his efforts to remove the various inconsistencies that he found, he developed a deterministic theory of the electron. Specifically, he found solutions to the Schrödinger, Klein-Gordon and Dirac equations which represent localised fields around a singularity at the centre of the electron ((T. Koga , 1972),(J. Mehra , 1973), (T. Koga , 1975), (T. Koga , 1975)), Chapters IV and V of (T. Koga , 1980) and Chapter V of (T. Koga, 1983)). He then developed a theory of the electron field which involves the internal gravitational field of the electron ((T. Koga , 1976), Chapter VI of (T. Koga , 1980)). This implies both the Dirac theory and Maxwell's equations as limiting cases. Koga applied these ideas to various phenomena in quantum electrodynamics and particle physics (Chapters VII-IX of (T. Koga , 1980)). We are not concerned with all this here.

As a justification for his solutions, Koga showed that a de Broglie wave could be obtained by averaging over an ensemble of his solutions to the Schrödinger equation, which he called elementary fields in his books (earlier, in his journal articles, he used the term wavelets).

Thus, Koga's theory partly agrees with Einstein's ideas. However, Koga showed that the Schrödinger equation itself has a solution which can be interpreted as a localised field with deterministic motion, contradicting Einstein's view that the implication of this equation was solely statistical.

The first textbook of Quantum Mechanics was possibly "The Principles of Quantum Mechanics" by Dirac ((Dirac , 1958)), first published in 1930. The preface of this book shows that Dirac was at that time in total agreement with the Copenhagen Interpretation.

By 1970, however, Dirac had virtually made a U-turn in this matter. In that year, a symposium on twentieth century physics was held at ICTP, Trieste, Italy. In his talk "Development of the physicist's conception of nature" ((J. Mehra , 1973), Chapter 1) Dirac asserted that many physicists were uncomfortable about having indeterminacy in the basic laws of physics and he was one of them; he accepted it only tentatively as "the best that one can do in our present state of knowledge".

In this paper we study some properties of Koga's solution to the Dirac equation. In particular, we consider his claim that the solution represents a rotating field. We confirm his claim and find additional information. This is of both physical and mathematical interest.

2 The Klein-Gordon and Dirac Equations

This paper can be considered a sequel to (K.V. Didimos et.al. , 2017) on deterministic Pauli spin theory.

We first briefly describe Koga's solution.

Suppose $\Phi(x, y, z, t)$ is a solution to the Klein-Gordon equation for a free electron with no external forces acting on it (this is the only case treated here),

$$\left(\hbar^2 \frac{\partial^2}{\partial t^2} - \hbar c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + m^2 c^4\right) \Phi = 0.$$

Here Φ can be a complex-valued scalar map or an n-tuple of such maps, for any n.

We take $n = 4$ in order to use Φ to get a solution Ψ to the Dirac equation.

Then the Klein-Gordon equation can be written as

$$D_0 D_1 \Phi = 0 \text{ or } D_1 D_0 \Phi = 0$$

where the operators D_0 and D_1 commute:

$$D_0 = i\hbar\beta \frac{\partial}{\partial t} + i\hbar c\beta \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z}\right) - mc^2,$$

$$D_1 = i\hbar\beta \frac{\partial}{\partial t} + i\hbar c\beta \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z}\right) + mc^2.$$

Here $\beta,\ \alpha_1,\ \alpha_2$ and α_3 are certain 4×4 matrices; following Koga, we choose them to be the matrices which were first given by Dirac. The Dirac equation is nothing but

$$D_0\Psi = 0$$

and hence a solution to it is given by $\Psi = D_1\Phi$.

3 Solutions to the Dirac equation

Suppose we consider a free electron at rest in our inertial frame with its centre at the origin of our coordinate system. This case suffices for our purposes; there is no loss of generality. There exists a (complex) scalar solution to the Klein-Gordon equation, given by

$$\varphi = a \exp{(iS/\hbar)}$$

where

$$\begin{aligned} a &= \exp{(-\kappa r)}/r, \\ S &= -Ect \\ \text{with} & \\ r &= |\mathbf{r}|, \\ E^2 &= m^2c^2 - \hbar^2\kappa^2 \end{aligned}$$

where κ is a positive constant, $\mathbf{r}$ = position vector, cE = energy, (the value of κ is not given by the theory and is to be chosen to make the result conform to reality).

This solution was given by Koga in (T. Koga, 1983). It is very similar to his solution to the Schrödinger equation. He then took, as a 4-dimensional solution to the Klein-Gordon equation, $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T$ with

$$\varphi_j = a \exp{(iS/\hbar)} A_j$$

where A_j are arbitrary complex constants.

Koga wrote down a 4-dimensional solution to the Dirac equation, $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, by evaluating $D_1\Phi$ with $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T$. He then tried to demonstrate that this solution, with arbitrary A_1, A_2, A_3, A_4, represents a rotating field, similar to a spinning top. He was not very successful, probably because he did not assign specific, suitable values to the constants A_j as we shall do in this paper, but kept them arbitrary.

Pandey and Chakravarti ((S.K. Pandey et.al. , 2009)) translated the complex scalar solution of the Klein-Gordon equation $\psi = a \exp{(iS/\hbar)}$ into Geometric Algebra and got a solution to the Dirac equation, but did not interpret the terms properly. One purpose of this paper is to correct the error in their paper. It turns out that although Geometric Algebra was initially very helpful in finding a solution, it did not clearly display some other solutions and the relation between them; the conventional approach makes things clear.

We will assume, with no loss of generality, that the electron is at rest in our inertial frame:

$$\mathbf{u} = 0.$$

Then $\mathbf{r}' = \mathbf{r}$ and $E^2 = m^2c^2 - \hbar^2\kappa^2$. Koga's solution to the Dirac equation is as follows. He defined

$$\mathbf{R} = (\mathbf{r} - \mathbf{u}t)\left(\frac{1}{|\mathbf{r} - \mathbf{u}t|^2} + \frac{\kappa}{|\mathbf{r} - \mathbf{u}t|\,(1 - u^2/c^2)^{1/2}}\right)$$

which reduces, when $\mathbf{u} = 0$, to

$$\mathbf{R} \;=\; \mathbf{r}\left(\frac{1}{r^2} + \frac{\kappa}{r}\right), \text{ where } \quad r = |\mathbf{r}|.$$

Now the solution Ψ, with arbitrary constants A_j, has components

$$\begin{aligned}
\psi_1 &= a\exp(iS/\hbar)\left[A_1\,(Ec+mc^2) - A_4\,i\hbar c\,(R_x - iR_y) - A_3\;i\hbar c\;R_z\right],\\
\psi_2 &= a\exp(iS/\hbar)\left[A_2\,(Ec+mc^2) - A_3\,i\hbar c\,(R_x + iR_y) + A_4\;i\hbar c\;R_z\right],\\
\psi_3 &= a\exp(iS/\hbar)\left[A_3\,(Ec-mc^2) + A_2\,i\hbar c\,(R_x - iR_y) + A_1\;i\hbar c\;R_z\right],\\
\psi_4 &= a\exp(iS/\hbar)\left[A_4\,(Ec-mc^2) + A_1\,i\hbar c\,(R_x + iR_y) - A_2\;i\hbar c\;R_z\right].
\end{aligned}$$

These expressions were obtained by putting **u** $= 0$ in the solution given by Koga.

4 The choice of the coefficients

Not all such solutions can be expected to be physically realistic; Koga mentioned that the arbitrary coefficients A_j need to be appropriately chosen. But he did not make any choice and tried to prove, using arbitrary A_j, that the solution represents a spinning field. Note that Ec stands for energy.

We propose four possible solutions, each obtained by taking one A_j to be 1 and the others to be 0.

Taking $A_1 = 1,\ A_2 = A_3 = A_4 = 0$ gives the solution corresponding to that obtained by Pandey and Chakravarti:

$$\Psi = a\exp(iS/\hbar)\begin{pmatrix} Ec+mc^2 \\ 0 \\ i\hbar c R_z \\ i\hbar c(R_x + iR_y) \end{pmatrix}.$$

This can be written as the sum of four column vectors, the second term being the zero vector:

$$\begin{aligned}
\Psi &= a\exp(iS/\hbar)\begin{pmatrix} Ec+mc^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}\\
&+ a\exp(iS/\hbar)\begin{pmatrix} 0 \\ 0 \\ i\hbar c R_z \\ 0 \end{pmatrix}\\
&+ a\exp(iS/\hbar)\begin{pmatrix} 0 \\ 0 \\ 0 \\ i\hbar c(R_x + iR_y) \end{pmatrix}.
\end{aligned}$$

The three nonzero vectors can be interpreted as follows: since $Ec + mc^2$ is constant, the first vector is a solution to the Klein-Gordon equation which represents a field without spin. The second vector is a field which has rotational symmetry about the z-axis. Finally, the last vector represents a spinning field with angular velocity

$$\omega = Ec/\hbar.$$

This can be verified by observing that if we take a rotating frame with the same origin, z-axis and angular velocity ω, the field is a constant field in this frame. We do this in the next section. Both the second and third nonzero vectors are needed to describe the effect of rotation of axes.

At this point it can be mentioned that Pandey and Chakravarti S.K. Pandey et.al. (2009) made the error of mixing the last two components. In the present approach, it is impossible to do this.

Similarly, if we take $A_2 = 1$ and $A_1 = A_3 = A_4 = 0$ we get

$$\begin{aligned} \psi_1 &= 0, \\ \psi_2 &= a\exp(iS/\hbar)(Ec + mc^2), \\ \psi_3 &= a\exp(iS/\hbar)i\hbar c(R_x - iR_y), \\ \psi_4 &= a\exp(iS/\hbar)(-i\hbar cR_z). \end{aligned}$$

Again there are three nonzero column vectors. They can be interpreted exactly as in the previous case (but in a different order) except that the direction of rotation is reversed because $R_x - iR_y$ is the complex conjugate of $R_x + iR_y$ (their arguments are the negatives of each other). If the first solution is defined to be spin up, then the second is spin down.

Two similar choices can be made, $A_3 = 1$ and $A_4 = 1$. The solutions are similar to the ones described above. But the Klein-Gordon term contains $Ec - mc^2$ instead of $Ec + mc^2$. Later on, the first two solutions are denoted by Ψ_f and Ψ_s restpectively.

These four solutions have some common features. Each has three nonzero components: a constant component, a component independent of x and y with the same magnitude in all four, and a component giving a rotating field with angular velocity $|Ec/\hbar|$. The first two solutions have a constant component of the same magnitude, $|Ec + mc^2|$. Similarly for the last two. It was mentioned earlier that E satisfies $E^2 = m^2c^2 - \hbar^2\kappa^2$. If we assume the positive root for E in the first two solutions and the negative root in the last two, the constant component has the same value in all four. All this suggests that these solutions really represent the electron. The last two can be considered negative energy solutions. We will not say any more about them.

If we delete the third and fourth components of these vectors, we get analogues of solutions of the Pauli equation.

5 Verification of the rotating field

It suffices to consider the first solution $A_1 = 1,\ A_2 = A_3 = A_4 = 0$. Let $F = F(x, y, z, t)$ be a vector field. Suppose a point with coordinates (x, y, z, t) in our inertial frame has coordinates (x', y', z, t) in a rotating frame with the same z-axis and angular velocity ω. Let $x + iy = \rho e^{i\theta}$ where $\rho = |x + iy|$. Then $x' + iy' = \rho e^{i(\theta - \omega t)}$. The condition

$$F(x, y, z, t) = F(x', y', z, 0) \text{ for all points}$$

is equivalent to the statement that the field F is rotating with an angular velocity ω relative to the inertial frame.

Now suppose $F = \psi_4$ and the angular velocity is $\omega = Ec/\hbar$. Then, using $S = -Ect$ and $x' + iy' = \exp(-i\omega t)(x + iy)$ we get

$$\begin{aligned} \psi_4(x, y, z, t) &= a\exp\left(iS/\hbar\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ i\hbar c(R_x + iR_y) \end{pmatrix} \\ &= a\exp\left(-i\omega t\right)(i\hbar c)\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ x + iy \end{pmatrix} \end{aligned}$$

and

$$\psi_4(x', y', z, 0) = a(i\hbar c)\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ x' + iy' \end{pmatrix}$$
$$= \psi_4(x, y, z, t).$$

This shows that ψ_4 is a rotating field.

6 Rotation of axes

As we are studying an electron in its rest frame, the only possible coordinate transformation is a rotation of axes. Suppose $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ with $n_1^2 + n_2^2 + n_3^2 = 1$. We can describe the components of $\mathbf{n}$ using spherical coordinates:

$$n_1 = \sin\theta\cos\phi,$$
$$n_2 = \sin\theta\sin\phi,$$
$$n_3 = \cos\theta$$

where θ is unique if we assume $0 \leq \theta \leq \pi$, and ϕ is unique modulo 2π.

We define the Hopf map f. The following definition and several equivalent ones are given in Socolovsky. M (2001); this one is the most convenient for us.

A general point $P \in S^3$ can be described as

$$P = e^{i\xi} c\cos(\theta/2) \tag{6.1}$$
$$e^{i\phi}\sin(\theta/2)$$
(6.2)

where $0 \leq \theta \leq \pi$. Here θ is unique and ϕ can also be made unique by putting suitable bounds on it; ξ is arbitrary. Let

$$f(P) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in S^2. \tag{6.3}$$

We see that

(i) f is continuous, independent of ξ and depends only on the ratio of the components of P, $e^{i\phi}\tan(\theta/2)$,

(ii) every point of S^2 is $f(P)$ for some $P \in S^3$,

(iii) $fc10 = (0, 0, 1)$ (the north pole of S^2),

(iv) $fc01 = (0, 0, -1)$ (the south pole of S^2), and

(v) $f(P)$ uniquely determines P (except for the value of ξ).

As a consequence of (i), we can extend the domain of f to all the points of $\mathbb{C}^2$ except the origin. There is also an S^2-valued map $f(\psi)$ where $\psi = c\psi_1\psi_2$ is a solution of the Pauli equation (K.V. Didimos et.al. , 2017). It should be noted that $f(\psi)$ depends on t alone; for a free electron, $f(\psi)$ is constant. Thus, f is a candidate for the direction map. But is f compatible with the Pauli spin theory? In other words, for a free electron, do $f(\psi)$ and the spin angular momentum vector $\mathbf{s}$ have the same direction in $\mathbb{R}^3$? By properties (iii) and (iv), $f(\psi)$ gives the spin direction of a spin-up or spin-down electron.

Then we have

$$\mathbf{n} \;=\; f\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$$

and

$$\mathbf{-n} \;=\; f\begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}.$$

Let

$$\Psi_{\mathbf{n}} = \cos(\theta/2)\Psi_f + e^{i\phi}\sin(\theta/2)\Psi_s.$$

In this section we shall explain how $\Psi_{\mathbf{n}}$ can be transformed into a Dirac field with spin-up axis $\mathbf{n}$.

The idea is to consider what happens when we rotate the $\mathbf{n}$-axis to make it coincide with the z-axis. Corresponding to this, there is a unitary linear operator T on $\mathbb{C}^2$, i.e., a 2×2 unitary matrix, which takes

$\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

We apply T to the first two components of $\Psi_{\mathbf{n}}$ and, separately, to the last two components. This amounts to multiplying the column vector $\Psi_{\mathbf{n}}$ by the 4×4 matrix

$$M = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

.

We will show that after rewriting appropriately, the result is an expression identical to Ψ_f. The rewriting consists of replacing the components of $\mathbf{R}$ by expressions in the new components. This can be done as soon as the new coordinate axes have been chosen.

In the operator D_0 which defines the Dirac equation, we have to change the matrices α_j and β in such a way as to preserve the commutativity relations. So we replace α_j with $M\alpha_j M^{-1}$ and β with $M\beta M^{-1}$. This yields a new Dirac equation of which $M\Psi_{\mathbf{n}}$ is a solution.

Now we analyse the effect of applying T. The first two components of $\Psi_{\mathbf{n}}$ form a constant multiple of $\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$. It therefore suffices to consider the effect on the last two components of $\Psi_{\mathbf{n}}$.

There is a useful simplification. The desired change of axis can be obtained by composing two computationally far simpler rotations as follows.

The vector $\mathbf{n}$ can first be taken into the xz-plane by a rotation through the angle ϕ about the z-axis. This amounts to assuming that $\theta = 0$. In this case, R_z is unchanged and the polar forms for $R_x + iR_y$ and $R_x - iR_y$ help in finding the new components of $\mathbf{R}$.

Assume that this has been done. Then, by a rotation about the y-axis through the angle θ, we can take $\mathbf{n}$ (which is now in the xz-plane) to the z-axis. In this case we have $\phi = 0$ and R_y doesn't change. It helps to use the polar forms for $R_x + iR_z$ and $R_x - iR_z$.

This shows that it is enough to consider only the spin up solution Ψ_f.

7 Conclusion

We have given a complete description of what Koga tried to obtain.

The Hopf map plays the same role in the non-relativistic and relativistic cases, but the Schrödinger equation has to be replaced by Klein-Gordon equation.

It would be of interest to obtain the relation between the solutions given here and Dirac's spinor solution. We expect that the latter is an ensemble average of the former.

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Competing Interests

Author has declared that no competing interests exist.

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