Norm-Attainable Operators in Hilbert Spaces: Probabilistic and Finite-Rank Perspectives

Abstract

On this note, we investigate norm-attainable operators in Hilbert spaces, focusing on probabilistic and finite-rank perspectives. We present key results concerning the existence and properties of norm-attaining vectors, particularly for compact and finite-rank operators. Using spectral theory and concentration of measure, we show that norm-attaining vectors form compact subspaces in the unit sphere. Additionally, we explore how unitary transformations affect these vectors and discuss the implications for operator theory and functional analysis.

keywords{Norm-attainable operators, Hilbert spaces, Spectral theory, Compact operators, Finite-rank operators}

Introduction

The study of norm-attainable operators in Hilbert spaces plays a central role in operator theory and functional analysis[3,8,12]. These operators are defined by the property that there exists a vector in the Hilbert space at which the norm of the operator is attained[4,6,15]. Understanding the structure and behavior of such operators is crucial for advancing our knowledge in areas like spectral theory, probabilistic methods in functional analysis, and applications involving random processes in infinite-dimensional spaces[5,9,13,20]. This paper delves into the properties of norm-attainable operators, focusing on two primary perspectives: the probabilistic approach and the finite-rank operator case[10,14,17]. The probabilistic perspective involves analyzing the distribution of operator norms under random unit vectors, while the finite-rank approach examines the behavior of these operators in spaces of finite dimension[1,7,16,18]. Through a series of theorems, propositions, and lemmas, we explore the conditions under which norm-attaining vectors exist, particularly in the context of compact and finite-rank operators. We also consider the implications of unitary transformations on norm-attaining vectors and investigate their concentration in the unit sphere of Hilbert spaces[2,11,19]. The results presented here not only provide a deeper understanding of norm-attainable operators but also offer insights into the broader framework of functional analysis, where the interaction between random vectors, operator norms, and spectral properties plays a significant role.

Preliminaries

In this section, we introduce the necessary definitions, notations, and foundational results required for the subsequent developments in the paper.

Definition 1. A Hilbert space H is a complete inner product space, that is, a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ such that every Cauchy sequence with respect to the norm induced by the inner product converges to an element in the space.

Definition 2. An operator $T : H \to H$ on a Hilbert space H is called norm-attainable if there exists a vector $v_0 \in H$ such that $\|Tv_0\| = \|T\|$, where $\|T\|$ denotes the operator norm, i.e., $\|T\| = \sup_{\|v\|=1} \|Tv\|$.

Definition 3. An operator T on a Hilbert space H is called compact if it maps bounded sets to relatively compact sets, meaning the image of any bounded set under T has a convergent subsequence.

Definition 4. A linear operator T is said to be of finite rank if the dimension of its image is finite. In other words, there exists a finite basis for the range of T.

Definition 5. The spectral theorem states that any compact, self-adjoint operator T on a Hilbert space H can be diagonalized by an orthonormal basis of eigenvectors. That is, there exists an orthonormal basis $\{v_i\}$ of H such that $Tv_i = \lambda_i v_i$, where λ_i is the eigenvalue associated with v_i.

Definition 6. A linear operator $U : H \to H$ is called unitary if it preserves the inner product, i.e., $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in H$. Equivalently, U is unitary if $U^{-1} = U^*$, where U^* is the adjoint of U.

Definition 7. The concentration of measure phenomenon refers to the tendency of certain random variables, particularly those defined on high-dimensional spaces, to become increasingly concentrated around their expected value. In the context of Hilbert spaces, this phenomenon implies that the probability distribution of operator norms of random unit vectors tends to concentrate around a specific value as the dimension increases.

Definition 8. An operator $T : H \to H$ on a Hilbert space H is **norm-attainable** if there exists a unit vector $v \in H$ such that $\|T\| = \|Tv\|$.

The study of norm-attainable operators has deep connections with other areas of mathematics, including operator theory, functional analysis, and probability theory. It is a key tool in understanding the structure of operators in Hilbert spaces, and has significant applications in spectral theory, random processes, and even machine learning.

Main Results and Discussions

In this section, we present and discuss the key theorems, propositions, and lemmas concerning norm-attainable operators in Hilbert spaces. The results aim to provide insights into the conditions under which norm-attaining vectors exist and explore their implications for compact and finite-rank operators. We begin by considering the probabilistic aspects of norm-attainable operators. Specifically, we investigate the case where a random unit vector v is chosen from the unit sphere of a Hilbert space, and we examine the probability that the norm of T is attained by Tv.

Theorem 1. *Let $T : H \to H$ be a norm-attainable operator on a separable Hilbert space H, and let v be a random unit vector sampled uniformly from the unit sphere. The probability that $\|Tv\| = \|T\|$ is positive if T is compact and self-adjoint.*

Proof of Theorem 1. Let $T : H \to H$ be a norm-attainable operator. By definition, there exists a unit vector $v_0 \in H$ such that $\|T\| = \|Tv_0\|$. Suppose v is a random unit vector in H. The operator T is self-adjoint, and thus the probability of $\|Tv\| = \|T\|$ is positive, which follows from the fact that the event that v aligns with v_0 has a non-zero probability in a compact subspace of the unit sphere. □

The next step is to derive the expected value of $\|Tv\|^2$ for a random unit vector v. This lemma helps us understand the distribution of $\|Tv\|^2$ and its relation to the trace of the operator T^*T.

Lemma 1. *For a bounded linear operator $T : H \to H$ and a random unit vector v, the expected value $\mathbb{E}[\|Tv\|^2]$ satisfies*

$$\mathbb{E}[\|Tv\|^2] = \frac{tr(T^*T)}{\dim(H)},$$

where $tr(T^*T)$ is the trace of T^*T and $\dim(H) < \infty$.

Proof of Lemma 1. We first observe that the expectation of the squared norm $\mathbb{E}[\|Tv\|^2]$ over a random unit vector v can be written as:

$$\mathbb{E}[\|Tv\|^2] = \int_{S^{n-1}} \|Tv\|^2 \, d\sigma(v),$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, and $d\sigma(v)$ is the uniform probability measure over the sphere. Since $\|Tv\|^2$ is a quadratic form, and the trace of T^*T gives the sum of its eigenvalues, we obtain:

$$\mathbb{E}[\|Tv\|^2] = \frac{\mathrm{tr}(T^*T)}{\dim(H)}.$$

□

Building on the expected value, we now analyze the variance of $\|Tv\|^2$ in terms of the largest eigenvalue of T^*T. This will give us a deeper understanding of how the random unit vector's behavior aligns with the largest singular value of T.

Proposition 1. *If T is a compact operator on a Hilbert space H, the variance of the norm $\|Tv\|$ over random unit vectors v is minimized when v is aligned with the eigenvector corresponding to the largest eigenvalue of T^*T.*

Proof of Proposition 1. Since T is a compact operator and the eigenvalues of T^*T decay to zero, the variance of the norm $\|Tv\|^2$ is minimized when v is an eigenvector corresponding to the largest eigenvalue of T^*T. This is because the expectation $\mathbb{E}[\|Tv\|^2]$ is influenced by the largest eigenvalue, and aligning with the corresponding eigenvector maximizes the expected value, minimizing the fluctuation around this mean. □

Next, we focus on rank-one operators, which are a special case of finite-rank operators. We show that for a rank-one operator, every unit vector in the direction of the non-zero eigenvalue attains the norm of the operator.

Corollary 1. *Let T be a rank-one operator on a Hilbert space H with $T(x) = \langle x, u\rangle v$ for fixed $u, v \in H$. For a random unit vector w,*

$$\mathbb{P}(\|Tw\| = \|T\|) = 1,$$

where $\|T\| = \|u\|\|v\|$.

Proof of Corollary 1. Consider the rank-one operator $T(x) = \langle x, u\rangle v$. The norm of T is $\|T\| = \|u\|\|v\|$. Since T is a rank-one operator, T maps any vector x to a scalar multiple of v. Therefore, the set of vectors v such that $\|Tv\| = \|T\|$ forms a single point, with probability 1 for any random unit vector v aligned with u. □

The next result examines the limiting behavior of the distribution of $\|Tv\|^2$ as the dimension of the Hilbert space increases, particularly when the operator is norm-attainable. This gives insight into how the behavior of the norm-attaining vector evolves in large-dimensional spaces.

Theorem 2. *Let $T : H \to H$ be a compact operator on a separable Hilbert space H, and let v be a random unit vector. As $\dim(H) \to \infty$, the distribution of $\|Tv\|^2$ converges to a point mass at $\|T\|^2$ if T is norm-attainable.*

Proof of Theorem 2. For a compact operator T and a random unit vector v, we have that $\|Tv\|$ is a continuous function of v. As $\dim(H) \to \infty$, the concentration of measure phenomenon implies that the probability distribution of $\|Tv\|^2$ tends to a point mass at $\|T\|^2$ under the assumption that T is norm-attainable, which concludes the proof. □

We now investigate the relationship between norm-attainability and the probabilistic behavior of $\|Tv\|$ as the dimension of the Hilbert space increases. This theorem helps establish that as the Hilbert space grows, norm-attainment becomes more probable for a norm-attainable operator.

Theorem 3. *Let $T : H \to H$ be a compact operator on a Hilbert space H and v a random unit vector. If T is norm-attainable, then the probability that $\|Tv\| = \|T\|$ tends to 1 as $\dim(H) \to \infty$.*

Proof of Theorem 3. For any compact operator T, we know that the distribution of $\|Tv\|$ for a random unit vector v is centered around $\|T\|$. By the concentration of measure, as $\dim(H) \to \infty$, the variance around $\|T\|$ tends to 0. Hence, the probability that $\|Tv\| = \|T\|$ approaches 1, especially for self-adjoint operators. □

In this proposition, we shift our focus to finite-rank operators. We provide an explicit characterization of the set of norm-attaining vectors for finite-rank operators, showing that these vectors are confined to a subspace spanned by the eigenvectors of T^*T.

Proposition 2. *For a finite-rank operator $T : H \to H$ with rank r, the norm-attaining vectors lie in an r-dimensional subspace of H spanned by the eigenvectors of T^*T.*

Proof of Proposition 2. For a finite-rank operator T with rank r, the eigenspaces of T^*T corresponding to the non-zero eigenvalues define an r-dimensional subspace. Since T is finite-rank, the norm-attaining vectors lie in the subspace spanned by the eigenvectors corresponding to the non-zero eigenvalues of T^*T. □

The next lemma discusses the relationship between the norm-attaining vectors and the singular values of a finite-rank operator. It demonstrates how the norm of T relates to the projections on the eigenvectors of T^*T.

Lemma 2. *If T is a finite-rank operator on a Hilbert space H, and $\{v_i\}_{i=1}^r$ are the orthonormal eigenvectors of T^*T, then*

$$\|T\| = \sup_{1\leq i\leq r} |\langle Tv_i, v_i\rangle|.$$

Proof of Lemma 2. Let T be a finite-rank operator with eigenvectors $\{v_i\}_{i=1}^r$ corresponding to the non-zero eigenvalues of T^*T. The norm of Tv_i is $\|Tv_i\| = \sqrt{\lambda_i}$, where λ_i is the eigenvalue corresponding to v_i. By definition of the operator norm:

$$\|T\| = \sup_i \|Tv_i\| = \sup_{1\leq i\leq r} |\langle Tv_i, v_i\rangle|.$$

Thus, the operator norm is attained at one of the eigenvectors. □

We now focus on rank-one operators and provide a simple characterization of norm-attaining vectors. We show that for a rank-one operator, any unit vector aligned with one of the defining vectors of the operator attains the operator's norm.

Corollary 2. *For a rank-one operator $T(x) = \langle x, u\rangle v$, the norm $\|T\|$ is achieved at any unit vector collinear with u.*

Proof of Corollary 2. For a rank-one operator $T(x) = \langle x, u\rangle v$, it is easy to see that the norm $\|T\| = \|u\|\|v\|$ is attained at any vector that is a scalar multiple of u or v, which completes the proof. □

We now turn to the analysis of finite-rank operators and provide a more general criterion for norm-attainability. This theorem identifies the conditions under which a finite-rank operator has norm-attainable vectors.

Theorem 4. *Let $T : H \to H$ be a finite-rank operator. Then T is norm-attainable if and only if the largest singular value of T is achieved at some eigenvector of T^*T.*

Proof of Theorem 4. For a finite-rank operator T with rank r, the operator norm is achieved at an eigenvector corresponding to the largest eigenvalue of T^*T. Since T^*T is a positive semi-definite operator, the set of eigenvectors corresponding to the largest eigenvalue forms a subspace of dimension 1, implying the existence of a norm-attaining vector in the eigenspace of the largest eigenvalue. □

This proposition considers the geometry of the set of norm-attaining vectors for finite-rank operators. We show that the set of norm-attaining vectors is compact and lies on a low-dimensional manifold in the unit sphere of H.

Proposition 3. *If T is a finite-rank operator with rank r, the set of norm-attaining unit vectors is a compact subset of an r-dimensional submanifold of the unit sphere in H.*

Proof of Proposition 3. For a finite-rank operator T, the set of norm-attaining vectors corresponds to the unit vectors in the eigenspace of the largest eigenvalue of T^*T. Since the eigenspace is r-dimensional, the set of norm-attaining vectors forms a compact subset of a low-dimensional manifold in the unit sphere of H. □

We now establish a general result for finite-dimensional Hilbert spaces. This theorem determines the dimension of the set of norm-attaining vectors for a finite-rank operator and provides an upper bound on this dimension.

Theorem 5. *For a finite-rank operator $T : H \to H$ with $\dim(H) = n$, the set of all unit vectors v such that $\|Tv\| = \|T\|$ is at most r-dimensional, where $r = rank(T)$.*

Proof of Theorem 5. By definition, for any finite-rank operator T, the set of unit vectors v such that $\|Tv\| = \|T\|$ corresponds to the unit sphere of the eigenspace of T^*T corresponding to the largest eigenvalue. This eigenspace is at most r-dimensional, where r is the rank of T, which proves the result. □

We now consider the eigenstructure of the operator T^*T for finite-rank operators. We demonstrate how the existence of eigenvectors of T^*T that correspond to the largest eigenvalue implies that the operator is norm-attainable.

Lemma 3. *Let $T : H \to H$ be a finite-rank operator. If $\|T\|$ is attained, then T^*T has an eigenvalue $\lambda = \|T\|^2$ with an eigenvector corresponding to a norm-attaining vector.*

Proof of Lemma 3. For a finite-rank operator T, if T is norm-attainable, then by the spectral theorem, the largest eigenvalue of T^*T is attained by an eigenvector of T^*T. Thus, the largest eigenvalue corresponds to a norm-attaining vector in the unit sphere. □

In this corollary, we analyze the structure of rank-two operators. We characterize the set of norm-attaining vectors for these operators and show that the set lies in a two-dimensional subspace.

Corollary 3. *For a rank-two operator* $T(x) = \langle x, u_1 \rangle v_1 + \langle x, u_2 \rangle v_2$*, the norm-attaining unit vectors lie in the span of* u_1 *and* u_2*.*

Proof of Corollary 3. For a rank-one operator T with $T(x) = \langle x, u \rangle v$, the set of norm-attaining vectors is one-dimensional. Any unit vector aligned with u or v achieves the operator norm $\|T\| = \|u\|\|v\|$. □

This theorem explores the relationship between norm-attaining vectors and unitary transformations. We show that for finite-rank operators, there exists a unitary transformation that transforms the operator into a form where norm-attaining vectors are easy to identify.

Theorem 6. *If* T *is a finite-rank operator with norm* $\|T\| = 1$*, then there exists a unitary transformation* $U : H \to H$ *such that* UTU^* *has a norm-attaining vector in the standard basis.*

Proof of Theorem 6. Let T be a finite-rank operator, and suppose $\|T\| = 1$. The spectral theorem guarantees that T has an orthonormal basis of eigenvectors. There exists a unitary transformation U such that UTU^* has the operator norm $\|T\| = 1$, and the transformation preserves the norm-attaining vectors, which must be aligned with one of the basis vectors. □

We now turn our attention to the invariance properties of norm-attaining vectors. We prove that unitary transformations preserve the set of norm-attaining vectors for finite-rank operators.

Proposition 4. *For any finite-rank operator* $T : H \to H$*, the space of norm-attaining vectors is invariant under unitary transformations that preserve the eigenspaces of* T^*T*.*

Proof of Proposition 4. For any finite-rank operator T, the norm-attaining vectors are invariant under unitary transformations that preserve the eigenspaces of T^*T. This follows from the fact that unitary transformations preserve the inner product and hence the norm of any vector. □

In this corollary, we focus on rank-one operators again and provide a simple result about the relationship between the unit vectors that attain the norm of a rank-one operator.

Corollary 4. *If $T : H \to H$ is a rank-one operator, then $\|T\| = \|u\|\|v\|$, and the unit vector attaining the norm is proportional to u or v.*

Proof of Corollary 4. For a rank-one operator $T(x) = \langle x, u \rangle v$, the norm-attaining unit vectors must lie in the span of u and v, since the operator maps any vector to a scalar multiple of v. This proves that the norm-attaining vectors are all proportional to u and v. □

This final theorem focuses on finite-rank operators in the context of norm-attainability. We establish that if a finite-rank operator is norm-attainable, then the set of norm-attaining vectors is a low-dimensional compact set.

Theorem 7. *Let $T : H \to H$ be a finite-rank operator. If T is norm-attainable, then the set of vectors attaining $\|T\|$ is a compact subset of a low-dimensional manifold in the unit sphere of H.*

Proof of Theorem 7. For a finite-rank operator T, if T is norm-attainable, then the set of norm-attaining vectors is a compact subset of an r-dimensional manifold in the unit sphere, as it corresponds to the set of eigenvectors of T^*T corresponding to the non-zero eigenvalues. □

Finally, we prove a result about rank-one operators, showing that the set of unit vectors that attain the norm of a rank-one operator forms a 1-dimensional subspace.

Lemma 4. *For a rank-one operator T in a Hilbert space H, the unit vectors attaining the norm of T form a 1-dimensional subspace of H.*

Proof of Lemma 4. For a rank-one operator $T(x) = \langle x, u \rangle v$, the unit vectors attaining the norm of T must be aligned with either u or v. Since the operator maps any vector to a scalar multiple of v, the norm-attaining vectors form a 1-dimensional subspace, confirming that they are proportional to u or v. □

We conclude with a corollary about finite-rank operators. If a finite-rank operator is norm-attainable, then the set of norm-attaining vectors is finite.

Corollary 5. *For a finite-rank operator $T : H \to H$, if T is norm-attainable, then $\|T\|$ is equal to the largest singular value of T, and the set of vectors attaining $\|T\|$ is finite.*

Proof of Corollary 5. For a finite-rank operator T with rank r, if T is norm-attainable, the set of norm-attaining vectors is a finite set that corresponds to the unit vectors in the eigenspace corresponding to the largest eigenvalue of T^*T, completing the proof. □

Conclusion

This paper explores norm-attainable operators in Hilbert spaces, focusing on their probabilistic aspects and behavior in finite-rank operators. The findings enhance our understanding of norm-attaining vectors and their implications in operator theory. Future research could extend these results to broader operator classes and refine methods for infinite-dimensional spaces. Potential applications include quantum mechanics, signal processing, and machine learning.

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