Modeling Hierarchical Systems in Graph Signal Processing, Electric Circuits, and Bond Graphs via Hypergraphs and Superhypergraphs

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Abstract

Graph theory is a core branch of mathematics concerned with representing and analyzing relationships among discrete elements. These concepts are widely used in fields such as electrical engineering. For example, graphs play a crucial role in important frameworks including *Graph Signal Processing*, *Electric Circuits*, and *Bond Graphs*.

A *hypergraph* generalizes the concept of a traditional graph by allowing edges—called *hyperedges*—to connect more than two vertices simultaneously [28]. A *superhypergraph* further extends this idea by incorporating recursively defined powerset layers, enabling hierarchical and self-referential relationships among hyperedges [192].

In this paper, we extend the frameworks of Graph Signal Processing, Electric Circuits, and Bond Graphs using hypergraphs and superhypergraphs, and investigate their mathematical properties and illustrative examples. These extensions enable the representation of hierarchical structures inherent in Graph Signal Processing, Electric Circuits, and Bond Graphs, providing a more expressive modeling framework. We anticipate that future research will advance computational experiments and practical applications in these domains.

Keywords: Superhypergraph, Hypergraph, Graph Signal Processing, Electric Circuits, Bond Graphs

1 Introduction

1.1 Hypergraphs and Superhypergraphs

Graph theory is a core branch of mathematics concerned with representing and analyzing relationships among discrete elements using abstract structures known as graphs, where entities (called vertices) are connected by links (called edges) [55, 59, 60, 103]. Due to the intuitive and visual nature of graphs, which allows complex systems to be illustrated clearly, they have been widely applied and actively studied in numerous fields, including graph neural networks [93, 109, 111], social networks [136, 137, 149], urban networks [16, 18], rail networks [64, 220], graph algorithm [141, 205, 217], and beyond. Classical graphs are limited to modeling pairwise relationships, yet many natural and engineered systems involve complex interactions among multiple entities that cannot be fully described using only binary connections. To overcome this limitation, the theory of graphs has been expanded to include the framework of *hypergraphs* and, more recently, *superhypergraphs* [73, 191].

A hypergraph is a generalization of a traditional graph in which a single edge—called a hyperedge—can simultaneously connect an arbitrary number of vertices [28, 37, 41, 49, 68]. This structure enables more expressive modeling of phenomena involving group-level interactions, such as metabolic networks, task teams, or symptom clusters in medical diagnostics. Furthermore, hypergraphs have been extended and studied in various forms, including Directed Hypergraphs [92, 126, 150], Regular Hypergraphs [62, 63], Complete Hypergraphs [26,148,198], Fuzzy Hypergraphs [67,147,183], Intuitionistic fuzzy Hypergraphs [7,57,164,165], and Neutrosophic Hypergraphs [9,11,135]. Hypergraphs have continued to be the subject of extensive research in recent years [30, 94, 186, 225].

Building upon the hypergraph concept, a *superhypergraph* incorporates additional layers of abstraction by iteratively applying the powerset operation to the vertex set [9, 82, 191, 192]. This results in recursively nested structures that can capture not only hyperedges over sets of vertices, but also interactions among groups of hyperedges themselves (cf. [46, 72, 104, 105]). Such higher-order formalisms are particularly suited for representing hierarchical, modular, or multi-scale systems in science and engineering (cf. [44, 115, 139, 145]). Concepts with hierarchical structures such as superhypergraphs are sometimes referred to as *SuperHyperstructures* [74, 79, 80, 194].

Table 1 provides an overview of Graphs, Hypergraphs, and Superhypergraphs. Note that let n be a natural number. Furthermore, this paper considers only finite concepts.

Table 1. Overview of Graph, Hypergraph, and Superhypergraph

Concept	Notation	Edge Connectivity	Structural Extension
Graph	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$ (binary edges)	Standard graph: edges join exactly two vertices.
Hypergraph	$H = (V, E)$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ (hyperedges)	Generalizes edges to connect any nonempty subset of vertices.
Superhypergraph	$\mathrm{SHT}^{(n)} = (V, E)$	$V, E \subseteq \mathcal{P}^n(V_0)$ (supervertices and super-edges)	Employs n-th powersets to capture hierarchical, nested connectivity among edges.

1.2 Graphs in Electrical Engineering, Physics, and Chemistry

Graph theory provides a powerful framework that can be applied across a wide range of disciplines, including electrical engineering, physics, and chemistry (cf. [65, 66, 140]). In this paper, we investigate extensions of *Graph Signal Processing*, *Electric Circuits*, and *Bond Graphs* through the lens of hypergraphs and superhypergraphs.

Signal Processing involves analyzing, modifying, and extracting information from signals such as sound, images, or data [17, 108, 143]. *Graph Signal Processing* extends this concept by analyzing signals defined on the vertices of a graph using spectral methods [133, 152, 199]. Graph Signal Processing has also been actively studied in recent years [42,48,52,158]. *Electric Circuits* model the flow of electrical current through interconnected components [35, 200, 214, 230]. A related concept known as the *Circuit Graph* represents circuits as graphs (cf. [212, 222]). *Bond Graphs* are graphical models that represent energy exchange across different physical domains—such as mechanical, electrical, thermal, and hydraulic systems—within a unified formalism [36, 99]. Bond Graphs have likewise remained an active research topic in recent years [4, 98, 146].

Beyond the concepts mentioned above, many other graph-theoretic models and their applications have been studied. These include the *Chemical Graph* [83, 95, 213, 219], which represents molecules and their bonds; the *Interaction Graph*, used in dynamical systems and particle interactions [13, 142]; and the *Feynman Graph*, central to quantum field theory and particle physics [33, 176]. These examples demonstrate the versatility and broad applicability of graph theory across scientific domains. Moreover, these graph concepts have been extended beyond classical graphs to include HyperGraphs [38, 134, 232], Fuzzy Graphs [15], and Neutrosophic Graphs [31, 138], and have been actively studied in fields such as electrical engineering, physics, and chemistry.

1.3 Our Contribution of this paper

As discussed earlier, the principles of graph theory have broad applicability across various domain-specific models. However, traditional graphs are inherently limited when relationships involve more than two entities (i.e., non-binary) or when systems exhibit hierarchical, recursive, or group-based interactions.

To address these limitations, we propose that *hypergraphs* and *superhypergraphs* provide promising structural alternatives. These generalized frameworks offer the expressive power needed to model complex interconnections that are not easily captured by classical graphs.

In this paper, we extend the foundational frameworks of *Graph Signal Processing*, *Electric Circuits*, and *Bond Graphs* through the lens of hypergraph and superhypergraph theory. We formally investigate their underlying mathematical structures and provide illustrative examples to clarify their expressive capabilities.

It is important to note that this paper is purely theoretical in nature. Certain procedural flows and implementation details related to the abstract formalism and its potential applications have been intentionally omitted. We hope that future research will explore these directions through computational experiments or circuit-based implementations to validate and expand upon the proposed models.

1.4 Structure of this paper

This section briefly outlines the structure of the paper. Section 2 introduces the preliminaries and definitions, including the Power Set, nth Power Set, HyperGraph, SuperHyperGraph, Graph Signal Processing, Electric Circuits, and Bond Graphs, along with illustrative examples. Section 3 defines and explores the properties and examples of n-SuperHyperGraph Signal Processing. Section 4 presents the definitions, examples, and characteristics of Electric HyperCircuits and Electric SuperHyperCircuits. Section 5 investigates the definitions, examples, and properties of Bond HyperGraphs and Bond SuperHyperGraphs. Finally, Section 6 concludes the paper and discusses directions for future work.

2 Preliminaries and Definitions

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. Throughout this paper, we restrict our attention to finite structures.

2.1 Power Set

The powerset of a set is the complete collection of all its possible subsets, including the empty set. The n-th powerset is constructed by repeatedly applying the powerset operation n times to a base set. We provide the definitions of the Base Set, the Powerset, and the n-th Powerset as follows.

Definition 2.1 (Set)**.** [107, 117, 131] A *set* is a well-defined collection of distinct objects, called elements or members.

Definition 2.2 (Subset)**.** [107, 117, 131] Let A and B be sets. We say that A is a *subset* of B, written $A \subseteq B$, if every element of A is also an element of B; that is,

$$A \subseteq B \iff \forall x\,(x \in A \Rightarrow x \in B).$$

Definition 2.3 (Empty Set)**.** [117] The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements. Formally,

$$\emptyset = \{\} \quad \text{such that} \quad \forall x,\ x \notin \emptyset.$$

Definition 2.4 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.5 (Powerset)**.** [86, 179] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Example 2.6 (Basic Passive Network Configurations via Powerset)**.** Passive components are electronic elements that do not generate energy, such as resistors, capacitors, and inductors, used in circuits (cf. [162, 170, 202]). In electrical engineering, selecting combinations of basic passive components is a common design task. Let the base set of available components be

$$S = \{R,\ L,\ C\},$$

where

- R = resistor,
- L = inductor,
- C = capacitor.

Then the powerset

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \} = \{ \emptyset,\ \{R\},\ \{L\},\ \{C\},\ \{R,L\},\ \{R,C\},\ \{L,C\},\ \{R,L,C\} \}$$

enumerates all possible one-port network configurations:

- $\emptyset$: open circuit (no component installed).
- $\{R\}$: simple resistive circuit (cf. [34]).
- $\{L\}$: single-element inductor circuit.
- $\{C\}$: single-element capacitor circuit.
- $\{R,L\}$: RL network for first-order filtering or transient shaping (cf. [58]).
- $\{R,C\}$: RC network used in low-pass or high-pass filters.
- $\{L,C\}$: LC resonant circuit for band-pass or notch filtering (cf. [90]).
- $\{R,L,C\}$: RLC circuit providing second-order filtering or oscillation behavior.

Thus $\mathcal{P}(S)$ systematically captures every combination of basic passive elements, guiding the exploration of feasible circuit topologies in filter design, impedance matching, and transient analysis.

Definition 2.7 (n-th Powerset)**.** (cf. [70, 86, 188, 193])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Example 2.8 (Multi-Stage Filter Design via n-th Powersets)**.** A multi-stage filter in electronics combines multiple filter stages to achieve sharper frequency selectivity and improved signal processing performance (cf. [110, 215]). A common task in electrical engineering is designing cascaded filter stages from basic one-port modules. Let the base set of available single-stage filters be

$$H = \{ F_R,\ F_C,\ F_L,\ F_{RC},\ F_{RL},\ F_{LC} \},$$

where, for example, $F_{RC} = \{R, C\}$ denotes a first-order RC filter.

First powerset $P^1(H)$: all possible subsets of single-stage filter modules:

$$P^1(H) = \mathcal{P}(H) = \{ \emptyset, \{F_R\}, \{F_C\}, \ldots, \{F_{RC}, F_{RL}\}, \ldots, H \}.$$

Each element of $P^1(H)$ is a candidate set of modules to be cascaded in one design.

Second powerset $P^2(H)$: sets of design alternatives, grouping multiple cascade choices:

$$P^2(H) = \mathcal{P}(P^1(H)).$$

For instance, one might select two cascade designs:

$$D_1 = \{\{F_{RC}, F_{RL}\}, \{F_{LC}, F_{RC}\}\}, \quad D_2 = \{\{F_{RC}, F_{LC}, F_{RL}\}\},$$

so that $\{D_1, D_2\} \in P^2(H)$.

Third powerset $P^3(H)$: meta-designs across system requirements, grouping multiple sets of alternatives:

$$P^3(H) = \mathcal{P}\big(P^2(H)\big), \quad M = \{D_1, D_3\} \in P^3(H),$$

where D_3 might be another design alternative set. Here M captures several multi-stage filter strategies evaluated in parallel.

This hierarchy

$$H \;\to\; P^1(H) \;\to\; P^2(H) \;\to\; P^3(H)$$

illustrates how iterated powersets model increasingly higher-order groupings of filter modules, from single-stage choices to sets of cascade designs to collections of design strategies in complex signal-processing systems.

Example 2.9 (Antenna Beamforming Design via n-th Powersets)**.** Antenna beamforming is a signal processing technique that directs signal transmission or reception toward specific angles to enhance performance (cf. [45, 123, 204]). In advanced wireless systems, one often selects subsets of antenna elements to form beams with desired patterns. Let the base set of available antenna elements be

$$H = \{\, A_1,\ A_2,\ A_3,\ A_4\},$$

where each A_i is a discrete radiating element in a linear array.

First powerset $P^1(H)$: all possible subarrays for beamforming:

$$P^1(H) = \mathcal{P}(H) = \big\{\emptyset, \{A_1\}, \{A_2\}, \dots, \{A_1, A_2, A_3\}, \dots, H\big\}.$$

Each nonempty subset represents one candidate subarray.

Second powerset $P^2(H)$: collections of subarray designs for multi-beam operation. For instance, choose two subarrays:

$$S_1 = \{A_1, A_2\}, \quad S_2 = \{A_3, A_4\}, \quad S_3 = \{A_1, A_3, A_4\}.$$

Then

$$C_1 = \{S_1, S_2\}, \quad C_2 = \{S_2, S_3\},$$

so that $\{C_1, C_2\} \in P^2(H)$. Each C_j is a set of subarrays used simultaneously.

Third powerset $P^3(H)$: meta-configurations grouping multiple multi-beam strategies:

$$P^3(H) = \mathcal{P}\big(P^2(H)\big), \quad M = \{\, C_1,\ C_3\} \in P^3(H),$$

where C_3 might be another collection of subarrays. Here M captures several distinct multi-beam schemes evaluated for different coverage zones.

This hierarchy

$$H \;\to\; P^1(H) \;\to\; P^2(H) \;\to\; P^3(H)$$

shows how iterated powersets model increasingly higher-order groupings in antenna design, from single subarrays to sets of beam patterns to collections of multi-beam strategies in complex wireless deployments.

2.2 SuperHyperGraph

In classical graph theory, a hypergraph extends the idea of a conventional graph by permitting edges—called hyperedges—to join more than two vertices. This broader framework enables the modeling of more intricate relationships between elements, thereby enhancing its utility in various fields [28, 68, 101, 102]. A *SuperHyperGraph* is an advanced extension of the hypergraph concept, integrating recursive powerset structures into the classical model. This concept has been recently introduced and extensively studied in the literature [2, 85, 144, 163].

Definition 2.10 (Graph). [47, 59, 69, 216] A graph is a mathematical structure consisting of a set of vertices and a set of edges, where each edge connects a pair of distinct vertices.

Definition 2.11 (Subgraph). [47, 59] Let $G = (V, E)$ be a graph. A *subgraph* of G is a graph $G' = (V', E')$ such that

$$V' \subseteq V, \quad E' \subseteq \{\{u, v\} \in E \mid u, v \in V'\}.$$

In other words, G' is obtained by selecting a subset of vertices and retaining only those edges of G whose endpoints both lie in V'.

Definition 2.12 (Hypergraph). [28, 37] A *hypergraph* $H = (V(H), E(H))$ consists of:

- A nonempty set $V(H)$ of vertices.
- A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Example 2.13 (VLSI Netlist Hypergraph). In modern VLSI circuit design, the netlist ([43, 87, 181]) describing device interconnections is naturally modeled as a hypergraph. Each vertex represents a device pin, and each hyperedge corresponds to an electrical net that may connect two or more pins simultaneously.

Let the set of pins be

$$V = \{ p_A,\ p_B,\ p_C,\ p_D,\ p_E \},$$

where p_A and p_B are input pins of a logic gate, p_C is its output pin, p_D is a clock distribution pin, and p_E is a reset pin. Define the nets (hyperedges) by

$$e_1 = \{p_A,\ p_B,\ p_C\}, \quad e_2 = \{p_C,\ p_D\}, \quad e_3 = \{p_E,\ p_A\}.$$

Then the hypergraph

$$H = (V, \{e_1, e_2, e_3\})$$

captures the multi-pin electrical connectivity of the circuit:

- e_1 models the three-pin data net linking the gate's inputs and output.
- e_2 represents the two-pin clock net connecting the gate output to the clock distribution.
- e_3 encodes the two-pin reset net linking the reset signal to one gate input.

Such a hypergraph is fundamental in VLSI placement and partitioning algorithms, where nets connecting multiple pins must be considered simultaneously.

Definition 2.14 (n-SuperHyperGraph). [73, 81, 191, 192]
Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset operation. An *n-SuperHyperGraph* is then a pair

$$\mathrm{SHT}^{(n)} = (V, E),$$

with

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}^n(V_0).$$

Each element of V is called an *n-supervertex* and each element of E an *n-superedge*.

Example 2.15 (Microgrid Power Flow as a 2-SuperHyperGraph)**.** Microgrid Power Flow refers to the distribution of electrical energy among generation, storage, and load units within a localized microgrid system (cf. [53, 130, 177]). Consider a simple microgrid in electrical engineering, with base components

$$V_0 = \{SP,\ WT,\ BS,\ RL,\ CL\},$$

where SP = Solar Panels, WT = Wind Turbine, BS = Battery Storage, RL = Residential Load, and CL = Commercial Load. We form the 2-SuperHyperGraph $\mathrm{SHT}^{(2)} = (V^{(2)}, E^{(2)})$ by setting

$$V^{(2)} = \{\{\{SP, WT, BS\}\},\ \{\{RL, CL\}\}\} \subseteq \mathcal{P}^2(V_0),$$

$$E^{(2)} = \{e = \{\{SP, WT, BS\},\ \{RL, CL\}\}\} \subseteq \mathcal{P}^2(V_0) \setminus \{\emptyset\}.$$

Here each element of $V^{(2)}$ is a 2-supervertex representing a cluster of devices or loads; the single 2-superedge e captures the power-flow event from the generation/storage cluster $\{\{SP, WT, BS\}\}$ to the combined load cluster $\{\{RL, CL\}\}$. This hierarchical model reflects the nested grouping of components and their simultaneous interaction in a microgrid.

Example 2.16 (Substation Protection Coordination as a 2-SuperHyperGraph)**.** In power system protection [14, 32, 159], relays and breakers form coordinated zones to isolate faults quickly and reliably. Let the base set of devices be

$$V_0 = \{\,\mathrm{R1},\ \mathrm{R2},\ \mathrm{R3},\ \mathrm{B1},\ \mathrm{B2}\},$$

where R_i are protective relays and B_j are circuit breakers.

Level-1 hyperedges (1-supervertices). Define the protection zones as hyperedges:

$$e_1 = \{\mathrm{R1},\ \mathrm{B1}\}\quad(\text{Zone }1),\qquad e_2 = \{\mathrm{R2},\ \mathrm{B1},\ \mathrm{B2}\}\quad(\text{Zone }2),\qquad e_3 = \{\mathrm{R3},\ \mathrm{B2}\}\quad(\text{Zone }3).$$

Thus

$$V^{(1)} = \{e_1,\ e_2,\ e_3\} \subseteq \mathcal{P}^1(V_0).$$

Level-2 supervertices. Group overlapping zones into 2-supervertices:

$$D_1 = \{e_1,\ e_2\}\quad(\text{overlap on B1}),\qquad D_2 = \{e_2,\ e_3\}\quad(\text{overlap on B2}).$$

Hence

$$V^{(2)} = \{D_1,\ D_2\} \subseteq \mathcal{P}^2(V_0).$$

Level-2 superedge. Since both 2-supervertices share the intermediate zone e_2, there is a single 2-superedge:

$$E^{(2)} = \{\{D_1, D_2\}\} \subseteq \mathcal{P}^2(V_0) \setminus \{\emptyset\}.$$

Therefore, the 2-SuperHyperGraph

$$\mathrm{SHT}^{(2)} = (V^{(2)},\ E^{(2)})$$

captures the hierarchical protection coordination:

- Level 1 lists individual protection zones (relay–breaker groupings).
- Level 2 clusters zones sharing breakers into regional coordination units D_1 and D_2.
- The 2-superedge $\{D_1, D_2\}$ models backup coordination between these two units via the shared zone e_2.

Example 2.17 (Distribution Grid Topology as a 3-SuperHyperGraph)**.** Electric power distribution networks consist of substations, feeders, and local load zones organized hierarchically (cf. [1, 51, 231]). Let the base set be

$$V_0 = \{\,\mathrm{Sub},\ \mathrm{Fe}_1,\ \mathrm{Fe}_2,\ \mathrm{LZ}_1,\ \mathrm{LZ}_2\},$$

where Sub is the substation, Fe_i are feeders, and LZ_j are load zones.

Level-1 hyperedges (1-supervertices). Define the 1-supervertices (hyperedges) as

$$e_1 = \{\text{Sub}, \text{Fe}_1, \text{Fe}_2\}, \quad e_2 = \{\text{Fe}_1, \text{LZ}_1\}, \quad e_3 = \{\text{Fe}_2, \text{LZ}_2\}.$$

Thus

$$V^{(1)} = \{e_1, e_2, e_3\} \subseteq \mathcal{P}^1(V_0).$$

Level-2 supervertices. Group overlapping hyperedges into 2-supervertices:

$$D_1 = \{e_1, e_2\} \quad (\text{overlap on Fe}_1), \qquad D_2 = \{e_1, e_3\} \quad (\text{overlap on Fe}_2).$$

Hence

$$V^{(2)} = \{D_1, D_2\} \subseteq \mathcal{P}^2(V_0).$$

Level-3 supervertices. Wrap each 2-supervertex into a singleton 3-supervertex:

$$U_1 = \{D_1\}, \quad U_2 = \{D_2\}.$$

Thus

$$V^{(3)} = \{U_1, U_2\} \subseteq \mathcal{P}^3(V_0).$$

Level-3 superedge. Finally, connect the two 3-supervertices by the single 3-superedge

$$E^{(3)} = \big\{\{U_1, U_2\}\big\} \subseteq \mathcal{P}^3(V_0).$$

Hence the 3-SuperHyperGraph

$$\text{SHT}^{(3)} = \big(V^{(3)}, E^{(3)}\big)$$

encodes the hierarchical topology of the distribution grid:

- Level 1 captures basic nets: high-voltage ring (e_1) and two service feeders (e_2, e_3).
- Level 2 groups nets sharing a feeder into regional zones (D_1, D_2).
- Level 3 wraps each regional zone into a supervertex (U_1, U_2).
- The 3-superedge $\{U_1, U_2\}$ reflects the overall interconnection of these two zones via the substation.

2.3 Graph Signal Processing

Graph Signal Processing analyzes data defined on graph nodes using spectral methods and graph-based transformations like filtering and shifting [61, 129, 151, 153]. If we are to define it explicitly, it would be as follows.

Definition 2.18 (Graph Signal Processing)**.** (cf. [61, 129, 151, 153]) Let $G = (V, E)$ be a simple graph with $|V| = N$. A *graph signal* is a function $x : V \to \mathbb{R}$, represented by the vector $\mathbf{x} = [x(v_1) \ \cdots \ x(v_N)]^\top \in \mathbb{R}^N$. Choose a graph shift operator $\mathbf{F} \in \mathbb{R}^{N\times N}$ (e.g. the adjacency matrix $\mathbf{A}$ or the Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$). Then:

$$(\text{Graph shifting}) \quad \mathbf{x}' = \mathbf{F}\,\mathbf{x}.$$

Since $\mathbf{F}$ is (for instance) diagonalizable as $\mathbf{F} = \mathbf{V}\,\boldsymbol{\Lambda}\,\mathbf{V}^{-1}$, the *graph Fourier transform* (GFT) of $\mathbf{x}$ is

$$\widehat{\mathbf{x}} = \mathbf{V}\,\mathbf{x}, \quad \mathbf{x} = \mathbf{V}^{-1}\,\widehat{\mathbf{x}},$$

where columns of $\mathbf{V}$ are eigenvectors of $\mathbf{F}$ and the entries of $\boldsymbol{\Lambda}$ are the associated *graph frequencies*.

Example 2.19 (Path Graph Temperature Sensor Network)**.** Temperature Sensor Networks are systems of distributed sensors that monitor, collect, and transmit temperature data across environments for analysis and control (cf. [125, 156, 157, 223]). Consider the path graph $G = (V, E)$ with $V = \{1, 2, 3, 4\}$ and edges $\{(1, 2), (2, 3), (3, 4)\}$. We use the combinatorial Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$ as the graph shift operator, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{D} = \operatorname{diag}(1, 2, 2, 1).$$

Its eigen-decomposition $\mathbf{L} = \mathbf{U}\,\boldsymbol{\Lambda}\,\mathbf{U}^\top$ yields

$$\mathbf{U} = \begin{pmatrix} 0.372 & 0.602 & 0.602 & 0.372 \\ 0.602 & 0.372 & -0.372 & -0.602 \\ 0.602 & -0.372 & -0.372 & 0.602 \\ 0.372 & -0.602 & 0.602 & -0.372 \end{pmatrix}, \quad \boldsymbol{\Lambda} = \operatorname{diag}\big(0.382,\ 1.382,\ 2.618,\ 3.618\big).$$

Now let the graph signal represent temperature readings: $\mathbf{x} = [1, 2, 3, 4]^\top$ (in °C). Its graph Fourier transform is

$$\widehat{\mathbf{x}} = \mathbf{U}^\top \mathbf{x} \approx \begin{pmatrix} 4.866 \\ -2.176 \\ 1.149 \\ -0.514 \end{pmatrix}.$$

These coefficients $\widehat{x}_k$ quantify the components of $\mathbf{x}$ at the graph frequencies λ_k.

Proposition 2.20 (Orthonormality of the Graph Fourier Basis)**.** *Let $G = (V, E)$ be an undirected simple graph with $|V| = N$, and let*

$$\mathbf{F} \;=\; \mathbf{V}\,\boldsymbol{\Lambda}\,\mathbf{V}^\top$$

be the eigendecomposition of a symmetric graph shift operator $\mathbf{F} \in \mathbb{R}^{N\times N}$, where $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_N]$ is orthogonal. Then the columns of $\mathbf{V}$ form an orthonormal basis of $\mathbb{R}^N$ and

$$\mathbf{V}^\top\mathbf{V} = \mathbf{I}_N.$$

Proof. Since $\mathbf{F}$ is real symmetric, the spectral theorem guarantees that there exists an orthogonal matrix $\mathbf{V}$ and a real diagonal matrix $\boldsymbol{\Lambda}$ such that $\mathbf{F} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\top$. Orthogonality of $\mathbf{V}$ means by definition

$$\mathbf{V}^\top\mathbf{V} \;=\; \mathbf{I}_N,$$

so its columns $\{\mathbf{v}_k\}$ satisfy $\mathbf{v}_i^\top \mathbf{v}_j = \delta_{ij}$, establishing the orthonormality of the graph Fourier basis. □

Proposition 2.21 (Parseval's Identity)**.** *Under the same assumptions as above, define the Graph Fourier Transform (GFT) of a graph signal $\mathbf{x} \in \mathbb{R}^N$ by*

$$\widehat{\mathbf{x}} \;=\; \mathbf{V}^\top\mathbf{x}.$$

Then the energy of $\mathbf{x}$ is preserved in the spectral domain:

$$\|\mathbf{x}\|_2^2 \;=\; \big\|\widehat{\mathbf{x}}\big\|_2^2.$$

Proof. Using orthonormality of $\mathbf{V}$,

$$\|\widehat{\mathbf{x}}\|_2^2 = (\mathbf{V}^\top\mathbf{x})^\top(\mathbf{V}^\top\mathbf{x}) = \mathbf{x}^\top \underbrace{\mathbf{V}\mathbf{V}^\top}_{\mathbf{I}} \mathbf{x} = \|\mathbf{x}\|_2^2.$$

Thus the total signal energy is invariant under the GFT. □

Proposition 2.22 (Graph Convolution–Filtering Theorem)**.** *Let $h : \mathbb{R} \to \mathbb{R}$ be a real-valued function (graph filter), and define the filter operator*

$$\mathbf{H} \;=\; h(\mathbf{F}) \;=\; \mathbf{V}\,h(\boldsymbol{\Lambda})\,\mathbf{V}^\top,$$

where $h(\boldsymbol{\Lambda}) = \operatorname{diag}\big(h(\lambda_1), \dots, h(\lambda_N)\big)$. Then for any graph signal $\mathbf{x}$,

$$\widehat{\mathbf{H}\mathbf{x}} \;=\; h(\boldsymbol{\Lambda})\,\widehat{\mathbf{x}},$$

i.e. filtering in the vertex domain corresponds to pointwise multiplication in the spectral domain.

Proof. By definition,
$$\widehat{\mathbf{Hx}} = \mathbf{V}^\top\big(\mathbf{V}\, h(\mathbf{\Lambda})\, \mathbf{V}^\top\mathbf{x}\big) = \underbrace{\mathbf{V}^\top\mathbf{V}}_{\mathbf{I}}\; h(\mathbf{\Lambda})\, (\mathbf{V}^\top\mathbf{x}) = h(\mathbf{\Lambda})\, \widehat{\mathbf{x}}.$$

Since $h(\mathbf{\Lambda})$ is diagonal, each spectral coefficient $\widehat{x}_k$ is multiplied by $h(\lambda_k)$, proving the convolution–filtering property. □

Hypergraph Signal Processing extends graph signal analysis to hypergraphs, using high-order tensors and spectral methods for multi-node interactions [27, 171, 197, 233].

Definition 2.23 (Hypergraph Signal Processing)**.** (cf. [27,171,197,233]) Let $H = (V, E)$ be a hypergraph with $|V| = N$ vertices and maximum hyperedge size m.c.e.$(H) = M$. Hypergraph Signal Processing (HGSP) on H comprises the following components:

1. **Adjacency tensor** $\mathcal{A} \in \mathbb{R}^{\overbrace{N \times \cdots \times N}^{M \text{ times}}}$: if $e_\ell = \{v_{l_1}, \ldots, v_{l_c}\} \in E$ has $c \le M$, then for any index tuple $(i_1, \ldots, i_M)$ that picks exactly those c vertices (with the remaining $M - c$ indices drawn from the same set),
$$\mathcal{A}_{i_1 \cdots i_M} \;=\; c \left(\sum_{\substack{k_1,\ldots,k_c \ge 1 \\ \sum_{i=1}^{c} k_i = M}} \frac{M!}{k_1!\, k_2! \cdots k_c!} \right)^{-1},$$
and $\mathcal{A}_{i_1 \cdots i_M} = 0$ otherwise.
2. **Hypergraph signal**: start with a vertex-domain signal $\mathbf{s} = [s_1, \ldots, s_N]^\top \in \mathbb{R}^N$, and form the $(M-1)$th-order signal tensor
$$\mathcal{S} \;=\; \underbrace{\mathbf{s} \circ \mathbf{s} \circ \cdots \circ \mathbf{s}}_{M-1 \text{ times}} \;\in\; \mathbb{R}^{\overbrace{N \times \cdots \times N}^{M-1 \text{ times}}}.$$
3. **Signal shifting**: the filtered (shifted) signal is obtained by contracting $\mathcal{A}$ with $\mathcal{S}$:
$$\mathcal{S}' \;=\; \mathcal{A} \times_M \mathcal{S},$$
where "$\times_M$" denotes the Mth-mode product, generalizing $\mathbf{s}' = \mathbf{F}\mathbf{s}$ in graph SP.
4. **Hypergraph Fourier transform**: assume an orthogonal CANDECOMP/PARAFAC decomposition
$$\mathcal{A} \;=\; \sum_{r=1}^{R} \lambda_r\, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}}, \quad \langle \mathbf{f}_r, \mathbf{f}_s \rangle = \delta_{rs}.$$
Then the HGFT of $\mathcal{S}$ is the vector $\widehat{\mathcal{S}} \in \mathbb{R}^R$ with components
$$\widehat{\mathcal{S}}_r \;=\; \big\langle \mathcal{S}, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}} \big\rangle,$$
whose entries λ_r serve as the "hypergraph frequencies.

Example 2.24 (Hypergraph Signal Processing on a 3-Uniform Collaboration Hypergraph)**.** Consider the hypergraph $H = (V, E)$ defined by

$$V = \{\text{Alice, Bob, Carol, Dave}\}, \quad E = \big\{\{\text{Alice, Bob, Carol}\},\ \{\text{Bob, Carol, Dave}\}\big\},$$

so that m.c.e.$(H) = 3$. Assign to each vertex the "publication count" signal

$$\mathbf{s} = \begin{bmatrix} 10 \\ 15 \\ 8 \\ 12 \end{bmatrix} \in \mathbb{R}^4.$$

Since $M = 3$, the adjacency tensor $\mathcal{A} \in \mathbb{R}^{4\times4\times4}$ has nonzero entries precisely when $\{i, j, k\} \in E$:

$$\mathcal{A}_{i,j,k} = 3 \left(\frac{3!}{1!\,1!\,1!}\right)^{-1} = \frac{3}{6} = 0.5,$$

and $\mathcal{A}_{i,j,k} = 0$ otherwise.

Form the hypergraph signal tensor $\mathcal{S} \in \mathbb{R}^{4\times4}$ by

$$\mathcal{S}_{i,j} = s_i\, s_j,$$

so that for example $\mathcal{S}_{\text{Alice,Bob}} = 10 \times 15 = 150$.

The shifted (filtered) signal $\mathcal{S}' = \mathcal{A} \times_3 \mathcal{S} \in \mathbb{R}^{4\times4}$ is given by

$$\mathcal{S}'_{i,j} = \sum_{k=1}^{4} \mathcal{A}_{i,j,k}\, s_k.$$

Hence, for example,

$$\mathcal{S}'_{\text{Alice,Bob}} = \mathcal{A}_{\text{Alice,Bob,Carol}} \times s_{\text{Carol}} = 0.5 \times 8 = 4, \quad \mathcal{S}'_{\text{Bob,Carol}} = 0.5 \times 10 + 0.5 \times 12 = 11.$$

Proposition 2.25 (Linearity of the Hypergraph Shift)**.** *Let $H = (V, E)$ be a hypergraph with adjacency tensor $\mathcal{A} \in \mathbb{R}^{N\times\cdots\times N}$ of order M, and let $\mathcal{S}, \mathcal{T} \in \mathbb{R}^{\overbrace{N \times \cdots \times N}^{M-1 \text{ times}}}$ be two hypergraph signal tensors. Then for any scalars $\alpha, \beta \in \mathbb{R}$,*

$$\mathcal{A} \times_M (\alpha\mathcal{S} + \beta\mathcal{T}) \;=\; \alpha\,(\mathcal{A} \times_M \mathcal{S}) \;+\; \beta\,(\mathcal{A} \times_M \mathcal{T}).$$

Proof. By definition the mode-M product of a tensor with a linear combination of two signals is

$$\big[\mathcal{A} \times_M (\alpha\mathcal{S} + \beta\mathcal{T})\big]_{i_1\cdots i_{M-1}} = \sum_{i_M=1}^{N} \mathcal{A}_{i_1\cdots i_{M-1} i_M}\big(\alpha\,\mathcal{S}_{i_1\cdots i_{M-1}} + \beta\,\mathcal{T}_{i_1\cdots i_{M-1}}\big).$$

Since summation and scalar multiplication commute,

$$= \alpha \sum_{i_M} \mathcal{A}_{i_1\cdots i_M}\, \mathcal{S}_{i_1\cdots i_{M-1}} \;+\; \beta \sum_{i_M} \mathcal{A}_{i_1\cdots i_M}\, \mathcal{T}_{i_1\cdots i_{M-1}} = \alpha\,(\mathcal{A} \times_M \mathcal{S})_{i_1\cdots i_{M-1}} + \beta\,(\mathcal{A} \times_M \mathcal{T})_{i_1\cdots i_{M-1}},$$

which proves the claimed linearity. □

Proposition 2.26 (Spectral Multiplication Property)**.** *Assume that the adjacency tensor admits an orthogonal CP decomposition*

$$\mathcal{A} = \sum_{r=1}^{R} \lambda_r\, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}}, \quad \langle \mathbf{f}_r, \mathbf{f}_s \rangle = \delta_{rs}.$$

Then the hypergraph Fourier transform (HGFT) of the shifted signal satisfies

$$\widehat{\mathcal{S}'}_r = \lambda_r\, \widehat{\mathcal{S}}_r, \qquad \widehat{\mathcal{S}}_r = \big\langle \mathcal{S}, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}} \big\rangle.$$

Proof. By definition $\mathcal{S}' = \mathcal{A} \times_M \mathcal{S}$. Then

$$\widehat{\mathcal{S}'}_r = \Big\langle \mathcal{A} \times_M \mathcal{S}, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}} \Big\rangle = \Big\langle \mathcal{A}, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}} \Big\rangle \Big\langle \mathcal{S}, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M-1 \text{ times}} \Big\rangle,$$

where we have used the multilinearity of the inner product and the orthogonality of the factors. But

$$\Big\langle \mathcal{A}, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}} \Big\rangle = \lambda_r,$$

so that

$$\widehat{\mathcal{S}'}_r = \lambda_r\, \widehat{\mathcal{S}}_r,$$

as required. □

Proposition 2.27 (Parseval's Identity for HGFT)**.** *Under the same orthogonality and decomposition assumptions as above, the total energy of the signal is preserved:*

$$\|\mathcal{S}\|_F^2 = \sum_{i_1,\dots,i_{M-1}} \mathcal{S}_{i_1\cdots i_{M-1}}^2 = \sum_{r=1}^{R} \big(\widehat{\mathcal{S}}_r\big)^2,$$

where $\|\cdot\|_F$ denotes the Frobenius norm of the tensor.

Proof. Since the rank-one tensors $\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r$ form an orthonormal basis for the $(M-1)$-order signal space,

$$\mathcal{S} = \sum_{r=1}^{R} \widehat{\mathcal{S}}_r \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M-1 \text{ times}},$$

and the squared Frobenius norm expands as

$$\|\mathcal{S}\|_F^2 = \sum_{r=1}^{R} \big(\widehat{\mathcal{S}}_r\big)^2 \Big\|\underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M-1 \text{ times}}\Big\|_F^2 = \sum_{r=1}^{R} \big(\widehat{\mathcal{S}}_r\big)^2,$$

because each rank-one tensor has unit norm by orthonormality of $\mathbf{f}_r$. This completes the proof. □

Proposition 2.28 (Polynomial Hypergraph Filtering)**.** *Let $H = (V, E)$ be a hypergraph with adjacency tensor $\mathcal{A}$ admitting the orthogonal CP decomposition*

$$\mathcal{A} = \sum_{r=1}^{R} \lambda_r \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}}, \quad \langle \mathbf{f}_r, \mathbf{f}_s \rangle = \delta_{rs}.$$

Given a real polynomial $h(t) = \sum_{k=0}^{K} a_k t^k$, define the filter operator

$$H_h(\mathcal{S}) = \sum_{k=0}^{K} a_k \underbrace{\mathcal{A} \times_M \cdots \times_M \mathcal{A}}_{k \text{ times}} \times_M \mathcal{S}.$$

Then the HGFT of the filtered signal satisfies

$$\widehat{H_h(\mathcal{S})}_r = h(\lambda_r)\, \widehat{\mathcal{S}}_r, \quad r = 1, \dots, R.$$

Proof. We proceed by induction on the filter degree. For $k = 0$, note

$$\underbrace{\mathcal{A}^0}_{\text{identity}} \times_M \mathcal{S} = \mathcal{S}, \quad \widehat{\mathcal{S}}_r = \widehat{\mathcal{S}}_r.$$

Assume for some $k \geq 0$ that

$$\big(\widehat{\mathcal{A}^k \times_M \mathcal{S}}\big)_r = \lambda_r^k\, \widehat{\mathcal{S}}_r.$$

Then apply one more shift:

$$\mathcal{A}^{k+1} \times_M \mathcal{S} = \mathcal{A} \times_M \big(\mathcal{A}^k \times_M \mathcal{S}\big),$$

and by the spectral multiplication property (Proposition 2),

$$\widehat{\mathcal{A}^{k+1} \times_M \mathcal{S}}_r = \lambda_r\, \widehat{\mathcal{A}^k \times_M \mathcal{S}}_r = \lambda_r^{k+1}\, \widehat{\mathcal{S}}_r.$$

Hence for the polynomial filter,

$$\widehat{H_h(\mathcal{S})}_r = \sum_{k=0}^{K} a_k\, \widehat{\mathcal{A}^k \times_M \mathcal{S}}_r = \sum_{k=0}^{K} a_k\, \lambda_r^k\, \widehat{\mathcal{S}}_r = h(\lambda_r)\, \widehat{\mathcal{S}}_r,$$

as claimed. □

Proposition 2.29 (Commutativity of Polynomial Hypergraph Filters)**.** *Let* $h(t) = \sum_{k=0}^{K} a_k\, t^k$ *and* $g(t) = \sum_{\ell=0}^{L} b_\ell\, t^\ell$ *be two real polynomials, and let* H_h*,* H_g *be the corresponding polynomial filters as in Proposition 2.28. Then*

$$H_h \circ H_g(\mathcal{S}) \;=\; H_g \circ H_h(\mathcal{S}) \quad \textit{for all hypergraph signals } \mathcal{S}.$$

Proof. By Proposition 2.28, the HGFT of $H_h \circ H_g(\mathcal{S})$ is

$$\widehat{H_h(H_g(\mathcal{S}))}_r = h(\lambda_r)\, \widehat{H_g(\mathcal{S})}_r = h(\lambda_r)\, g(\lambda_r)\, \widehat{\mathcal{S}}_r.$$

Similarly,

$$\widehat{H_g(H_h(\mathcal{S}))}_r = g(\lambda_r)\, h(\lambda_r)\, \widehat{\mathcal{S}}_r.$$

Since scalar multiplication commutes, these two are equal for every r, and by invertibility of the HGFT the filtered signals coincide:

$$H_h \circ H_g(\mathcal{S}) = H_g \circ H_h(\mathcal{S}).$$

□

Proposition 2.30 (Invertibility of the HGFT)**.** *Under the orthogonality assumptions of Proposition 2.28, the HGFT is invertible. In particular, for any hypergraph signal tensor* $\mathcal{S}$*,*

$$\mathcal{S} = \sum_{r=1}^{R} \widehat{\mathcal{S}}_r\, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M-1 \text{ times}}.$$

Proof. Because the rank-one tensors $\{\mathbf{f}_r^{\circ(M-1)}\}_{r=1}^{R}$ form an orthonormal basis of the signal space, any tensor $\mathcal{S}$ admits the unique expansion

$$\mathcal{S} = \sum_{r=1}^{R} \langle \mathcal{S},\, \mathbf{f}_r^{\circ(M-1)} \rangle\, \mathbf{f}_r^{\circ(M-1)} = \sum_{r=1}^{R} \widehat{\mathcal{S}}_r\, \mathbf{f}_r^{\circ(M-1)},$$

where $\widehat{\mathcal{S}}_r$ are the HGFT coefficients. This provides the inversion formula and shows the transform is bijective.
□

2.4 Electric Circuit

An electric circuit is a closed loop that allows electric current to flow through connected electrical components using conductors [20, 166, 180, 187].

Definition 2.31 (Electric Circuit)**.** An *electric circuit* is a pair $(G, \mathcal{E})$ where:

- $G = (V, E)$ is a finite, connected, oriented multigraph with vertex set V and edge set E. Each edge $e \in E$ has a chosen direction.
- $\mathcal{E}$ is a collection of *circuit elements* assigning to each edge $e \in E$ a *voltage–current relation*

  $$\mathcal{E}(e) \;:\; (v_e, i_e) \;\mapsto\; 0,$$

  such as Ohm's law for a resistor e: $v_e - R_e\, i_e = 0$.

We associate to G its *incidence matrix* $A \in \{-1, 0, 1\}^{|V| \times |E|}$, where

$$A_{n,e} = \begin{cases} +1, & \text{if edge } e \text{ leaves node } n, \\ -1, & \text{if edge } e \text{ enters node } n, \\ 0, & \text{otherwise.} \end{cases}$$

A state of the circuit consists of functions $i : E \to \mathbb{R}$ (branch currents) and $v : E \to \mathbb{R}$ (branch voltages) satisfying:

1. **Kirchhoff's Current Law (KCL):**
$$A\,\mathbf{i} \;=\; \mathbf{0},$$
meaning the algebraic sum of currents at each node is zero.

2. **Kirchhoff's Voltage Law (KVL):** there exists a node–potential vector $\mathbf{u} \in \mathbb{R}^{|V|}$ such that
$$\mathbf{v} \;=\; A^{\top}\,\mathbf{u},$$
so the sum of voltage drops around any closed loop vanishes.

3. **Element Constitutive Relations:** for each $e \in E$, $\mathcal{E}(e)(v_e, i_e) = 0$.

Together, these equations define the *network equations* of the circuit.

Example 2.32 (Resistive Network)**.** A Resistive Network is an electrical circuit composed of interconnected resistors used to control voltage, current, and power distribution (cf. [127,132]). Let $G = (V, E)$ be a connected graph with $V = \{1, 2, 3\}$, $E = \{e_{12}, e_{23}, e_{31}\}$, each edge a resistor of resistance R_{ij}. Then:
$$\mathbf{i} = (i_{12}, i_{23}, i_{31})^{\top}, \quad \mathbf{v} = (v_{12}, v_{23}, v_{31})^{\top},$$
and the incidence matrix is
$$A = \begin{pmatrix} +1 & 0 & -1 \\ -1 & +1 & 0 \\ 0 & -1 & +1 \end{pmatrix}.$$
KCL: $A\,\mathbf{i} = \mathbf{0}$.
KVL: $\mathbf{v} = A^{\top}\mathbf{u}$ for node potentials $\mathbf{u} = (u_1, u_2, u_3)^{\top}$.
Ohm's law on each edge e_{ij}: $v_{ij} - R_{ij}\,i_{ij} = 0$. Solving these yields the currents and potentials in the network.

Proposition 2.33 (Positive Semidefiniteness of Circuit Laplacian)**.** *Let $G = (V, E)$ be a connected oriented multigraph, and let*
$$G_{\text{cond}} = \operatorname{diag}(g_e), \quad g_e = \frac{1}{R_e} > 0,$$
be the diagonal matrix of edge conductances. Define the network Laplacian
$$L \;=\; A\,G_{\text{cond}}\,A^{\top},$$
where $A \in \{-1, 0, 1\}^{|V| \times |E|}$ is the incidence matrix. Then:

1. *L is symmetric and positive semidefinite.*
2. $\ker(L) = \operatorname{span}\{\mathbf{1}\}$*, where* $\mathbf{1}$ *is the all-ones vector.*

Proof. Symmetry follows immediately since G_{cond} is diagonal and $A\,G_{\text{cond}}\,A^{\top}$ is manifestly symmetric. For any $\mathbf{x} \in \mathbb{R}^{|V|}$:
$$\mathbf{x}^{\top} L\,\mathbf{x} = \mathbf{x}^{\top} A\,G_{\text{cond}}\,A^{\top}\mathbf{x} = (A^{\top}\mathbf{x})^{\top}\,G_{\text{cond}}\,(A^{\top}\mathbf{x}) = \sum_{e=(u,v)\in E} g_e\,\left(x_u - x_v\right)^2 \;\geq\; 0.$$
Thus L is positive semidefinite.

Moreover, $\mathbf{x}^{\top} L\mathbf{x} = 0$ if and only if $x_u = x_v$ for every edge $e = (u, v)$. Since G is connected, this forces x_u constant over all vertices, i.e. $\mathbf{x} \in \operatorname{span}\{\mathbf{1}\}$. Hence $\ker(L) = \operatorname{span}\{\mathbf{1}\}$. □

Proposition 2.34 (Existence and Uniqueness of Resistive Circuit Solution)**.** *Consider a resistive network on $(G, \mathcal{E})$ with conductances $g_e > 0$. Let $\mathbf{b} \in \mathbb{R}^{|V|}$ be a vector of external current injections satisfying $\sum_{n\in V} b_n = 0$. Then there exists a solution $(\mathbf{u}, \mathbf{v}, \mathbf{i})$ of node potentials $\mathbf{u}$, branch voltages $\mathbf{v}$, and branch currents $\mathbf{i}$ satisfying:*
$$\begin{cases} A\,\mathbf{i} = \mathbf{b}, \\ \mathbf{v} = A^{\top}\,\mathbf{u}, \\ \mathbf{i} = G_{\text{cond}}\,\mathbf{v}. \end{cases}$$
Moreover, this solution is unique up to adding a constant to all entries of $\mathbf{u}$.

Proof. Substitute $\mathbf{i} = G_{\text{cond}}\,\mathbf{v}$ and $\mathbf{v} = A^{\top}\mathbf{u}$ into Kirchhoff's Current Law $A\,\mathbf{i} = \mathbf{b}$. We obtain the linear system

$$A\,G_{\text{cond}}\,A^{\top}\,\mathbf{u} \;=\; \mathbf{b} \quad\Longleftrightarrow\quad L\,\mathbf{u} = \mathbf{b}.$$

Since $\sum_n b_n = 0$, $\mathbf{b}$ lies in $\mathrm{Im}(L)$. By the Theorem, L has rank $|V| - 1$ and nullspace spanned by $\mathbf{1}$. Hence the equation $L\,\mathbf{u} = \mathbf{b}$ admits a solution, unique modulo addition of any constant vector $c\,\mathbf{1}$.

Once $\mathbf{u}$ is fixed, define $\mathbf{v} = A^{\top}\mathbf{u}$ and $\mathbf{i} = G_{\text{cond}}\,\mathbf{v}$. These automatically satisfy Kirchhoff's Voltage Law and Ohm's law by construction. This completes the proof of existence and uniqueness (up to reference potential). □

2.5 Bond graphs

Bond graphs are a domain-independent formalism for modeling the transfer and storage of energy in multi-domain physical systems [97, 155, 207, 208]. They consist of two kinds of vertices—*element nodes* and *junction nodes*—connected by *bonds* carrying conjugate variables effort e and flow f.

Definition 2.35 (Bond Graph)**.** [97, 207, 208] A *bond graph* is an undirected graph

$$G = (V, E),$$

where

- $V = V_{\text{elem}} \,\dot\cup\, V_{\text{junc}}$, a disjoint union of
  - *Element nodes* $V_{\text{elem}} = V_{Se} \,\dot\cup\, V_{Sf} \,\dot\cup\, V_R \,\dot\cup\, V_C \,\dot\cup\, V_I \,\dot\cup\, V_{TF} \,\dot\cup\, V_{GY}$,
  - *Junction nodes* $V_{\text{junc}} = V_0 \,\dot\cup\, V_1$,
- $E \subseteq \big\{\{u, v\} : u, v \in V,\ u \neq v\big\}$ is the set of *bonds*, each representing a single power port connection.

Each bond $\{u, v\} \in E$ carries two variables:

$$e \text{ (effort)}, \quad f \text{ (flow)}, \quad \text{with instantaneous power } P = e\,f.$$

Element nodes denote:

- Se: effort source (e.g. voltage, force),
- Sf: flow source (e.g. current, velocity),
- R: resistance (energy dissipation),
- C: capacitance (potential energy storage),
- I: inertia (kinetic energy storage),
- TF: transformer (scaling of effort and flow),
- GY: gyrator (cross-domain conversion of effort and flow).

Junction nodes denote:

- 0-junction: common effort, flows sum to zero,
- 1-junction: common flow, efforts sum to zero.

Example 2.36 (Bond Graph of a Series R–C Circuit)**.** Consider a simple series circuit consisting of an effort source Se, a resistor R, and a capacitor C. Its bond-graph representation is:

$$V_{\text{elem}} = \{Se,\ R,\ C\}, \qquad V_{\text{junc}} = \{\ 1\},$$

where "1" denotes a 1-junction (common flow, efforts sum to zero). The set of bonds is

$$E = \big\{\{Se, 1\},\ \{R, 1\},\ \{C, 1\}\big\}.$$

Each bond $\{x, 1\}$ carries conjugate variables effort e_x and flow f_x. At the 1-junction:

$$f_{Se} = f_R = f_C \ = \ f, \qquad e_{Se} + e_R + e_C = 0.$$

The constitutive relations on each element are:

$$e_{Se}(t) = u(t), \quad e_R = R\, f, \quad f_C = C\, \frac{d\, e_C}{dt}.$$

Thus the bond graph fully captures the energy exchange: the same flow f passes through all elements, while the efforts across Se, R, and C sum to zero.

Proposition 2.37 (Power Conservation at Junctions)**.** *In any bond graph $G = (V, E)$, for each junction node $j \in V_{\text{junc}}$ the algebraic sum of instantaneous powers carried by incident bonds is zero.*

Proof. Let $B_j = \{b_1, \ldots, b_k\}$ be the set of bonds incident on junction j, each carrying effort e_i and flow f_i on bond b_i. Then the instantaneous power into j from bond b_i is $P_i = e_i\, f_i$. We consider two cases:

(i) 0–junction: All bonds share a common effort e, and flows satisfy

$$\sum_{i=1}^{k} f_i = 0.$$

Thus the total power

$$\sum_{i=1}^{k} P_i = \sum_{i=1}^{k} e\, f_i = e \sum_{i=1}^{k} f_i = e \cdot 0 = 0.$$

(ii) 1–junction: All bonds share a common flow f, and efforts satisfy

$$\sum_{i=1}^{k} e_i = 0.$$

Hence

$$\sum_{i=1}^{k} P_i = \sum_{i=1}^{k} e_i\, f = f \sum_{i=1}^{k} e_i = f \cdot 0 = 0.$$

In both cases the junction neither generates nor dissipates power, proving power conservation. □

Proposition 2.38 (State–Space Realization of Linear Bond Graphs)**.** *A linear time–invariant bond graph comprised solely of linear storage elements (C and I), resistive elements (R), and gyrators/transformers (GY, TF) admits a state–space representation of the form*

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u},$$

where $\mathbf{x}$ collects the energy variables of C and I elements, and $(\mathbf{u}, \mathbf{y})$ are port variables at sources.

Proof. Label each capacitor C_i with state $x_{C_i} = e_{C_i}$ (effort) and each inductor I_j with state $x_{I_j} = f_{I_j}$ (flow). The constitutive laws give

$$\dot{x}_{C_i} = f_{C_i}, \qquad \dot{x}_{I_j} = \frac{e_{I_j}}{I_j}.$$

Using KCL/KVL at junctions and linear relations in R, GY, TF, one assembles linear algebraic constraints:

$$\mathbf{E}\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix} + \mathbf{F}\begin{pmatrix}\dot{\mathbf{x}}\\ \mathbf{y}\end{pmatrix} = \mathbf{0},$$

where $\mathbf{E}, \mathbf{F}$ are constant incidence-like matrices. Partitioning yields

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u},$$

with $\mathbf{A} = -F_{22}^{-1}F_{21}$, $\mathbf{B} = -F_{22}^{-1}F_{23}$, $\mathbf{C} = E_{12} + E_{11}A$, $\mathbf{D} = E_{13} + E_{11}B$, where the block matrices arise from suitable reordering of $(\dot{\mathbf{x}}, \mathbf{y}, \mathbf{x}, \mathbf{u})$. Because F_{22} is invertible for a well–posed causal assignment, the state–space form follows directly. □

Proposition 2.39 (Passivity of Linear Bond Graph Systems)**.** *The state–space system derived from a passive linear bond graph satisfies the dissipation inequality*

$$\dot{H}(\mathbf{x}) \le \mathbf{u}^\top \mathbf{y},$$

where $H(\mathbf{x})$ is the total stored energy.

Proof. Define the Hamiltonian (energy function)

$$H(\mathbf{x}) = \tfrac{1}{2}\,\mathbf{x}^\top \mathbf{Q}\,\mathbf{x},$$

where $\mathbf{Q} = \mathrm{diag}(C_i^{-1}, I_j)$ collects inverse capacities and inertias. Differentiating:

$$\dot{H} = \mathbf{x}^\top \mathbf{Q}\,\dot{\mathbf{x}} = \mathbf{x}^\top \mathbf{Q}(\mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}) = \mathbf{x}^\top(\mathbf{Q}\mathbf{A})\,\mathbf{x} + (\mathbf{x}^\top \mathbf{Q}\mathbf{B})\,\mathbf{u}.$$

Passivity of resistors and proper gyrator/transformer connections imply $\mathbf{Q}\mathbf{A} + \mathbf{A}^\top\mathbf{Q} \le 0$ and output equation $\mathbf{y} = \mathbf{B}^\top\mathbf{Q}\,\mathbf{x} + \mathbf{D}\,\mathbf{u}$ with $\mathbf{D} + \mathbf{D}^\top \ge 0$. Hence

$$\dot{H} \le \mathbf{u}^\top \mathbf{y},$$

establishing the dissipation inequality and thus passivity. □

3 Result: n-SuperHyperGraph Signal Processing

SuperHypergraph Signal Processing generalizes signal analysis over nested multi-level hypergraphs using tensor operations, spectral decomposition, and hierarchical shifting.

Definition 3.1 (n-SuperHypergraph Signal Processing)**.** Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph with $|V| = N_n$ and maximum superedge cardinality

$$M = \max_{e \in E} |e|.$$

Define the *adjacency tensor* $\mathcal{A} \in \mathbb{R}^{\overbrace{N_n \times \cdots \times N_n}^{M \text{ times}}}$ by

$$\mathcal{A}_{i_1 \cdots i_M} = \begin{cases} c \left(\displaystyle\sum_{\substack{k_1,\ldots,k_c \ge 1 \\ \sum k_i = M}} \frac{M!}{k_1! \cdots k_c!} \right)^{-1} & \text{if } \{v_{i_1}, \ldots, v_{i_M}\} \text{ enumerates superedge } e = \{w_1, \ldots, w_c\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $c = |e| \le M$.

A *signal* on $\mathrm{SHT}^{(n)}$ is a vector $\mathbf{s} \in \mathbb{R}^{N_n}$. Form the $(M-1)$th-order *signal tensor*

$$\mathcal{S} = \underbrace{\mathbf{s} \circ \cdots \circ \mathbf{s}}_{M-1 \text{ times}} \in \mathbb{R}^{\overbrace{N_n \times \cdots \times N_n}^{M-1}}.$$

The *shifted signal* is

$$\mathcal{S}' = \mathcal{A} \times_M \mathcal{S},$$

where $\times_M$ denotes the mode-M product. Finally, assume an orthogonal CANDECOMP/PARAFAC decomposition

$$\mathcal{A} = \sum_{r=1}^{R} \lambda_r \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}}, \quad \langle \mathbf{f}_r, \mathbf{f}_s \rangle = \delta_{rs}.$$

The *n-SuperHypergraph Fourier transform* of $\mathcal{S}$ is the vector $\widehat{\mathcal{S}} \in \mathbb{R}^R$ with

$$\widehat{\mathcal{S}}_r = \big\langle \mathcal{S}, \underbrace{\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r}_{M \text{ times}} \big\rangle.$$

Example 3.2 (2-SuperHypergraph Signal Processing on a Divisional Collaboration Structure)**.** Let the base set of employees be

$$V_0 = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}, \text{Eve}\}.$$

Form the 1-supervertices (committees):

$$C_1 = \{\text{Alice}, \text{Bob}\}, \quad C_2 = \{\text{Carol}, \text{Dave}, \text{Eve}\}, \quad C_3 = \{\text{Bob}, \text{Carol}\},$$

and the 2-supervertices (divisions):

$$D_1 = \{C_1, C_2\}, \quad D_2 = \{C_2, C_3\}.$$

Define the 2-SuperHyperGraph $\mathrm{SHT}^{(2)} = (V, E)$ with

$$V = \{D_1, D_2\}, \qquad E = \big\{\{D_1, D_2\}\big\},$$

so that $|V| = 2$ and $M = 2$.

Assign to each division the "active project count" signal

$$\mathbf{s} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad \big(s_1 = 5,\ s_2 = 7\big).$$

Since $M = 2$, the adjacency tensor $\mathcal{A} \in \mathbb{R}^{2\times 2}$ has entries

$$\mathcal{A}_{i,j} = \begin{cases} 1, & \text{if } \{D_i, D_j\} = \{D_1, D_2\}, \\ 0, & \text{otherwise,} \end{cases}$$

i.e. $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The $(M-1)$th-order signal tensor is just the vector $\mathbf{s}$. The shifted signal is

$$\mathcal{S}' = \mathcal{A}\,\mathbf{s} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

Finally, the adjacency matrix admits the orthogonal eigen-decomposition

$$\mathcal{A} = \mathbf{F} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{F}^\top, \quad \mathbf{F} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Hence the 2-SuperHypergraph Fourier transform of $\mathbf{s}$ is

$$\widehat{\mathbf{s}} = \mathbf{F}^\top \mathbf{s} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{12}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 6\sqrt{2} \\ -\sqrt{2} \end{pmatrix}.$$

This example shows how 2-SuperHypergraph Signal Processing generalizes both graph and hypergraph signal frameworks to a three-layer corporate structure.

Example 3.3 (3-SuperHypergraph Signal Processing for Department Collaboration)**.** Let the base set of employees be

$$V_0 = \{\text{Alice}, \text{Bob}, \text{Carol}\}.$$

Form the 1-supervertices (committees):

$$C_1 = \{\text{Alice}, \text{Bob}\}, \quad C_2 = \{\text{Bob}, \text{Carol}\},$$

the 2-supervertices (divisions):

$$D_1 = \{C_1\}, \quad D_2 = \{C_2\},$$

and the 3-supervertices (departments):

$$H_1 = \{D_1\}, \quad H_2 = \{D_2\}, \quad H_3 = \{D_1, D_2\}.$$

Define the 3-SuperHyperGraph $\mathrm{SHT}^{(3)} = (V, E)$ by

$$V = \{H_1, H_2, H_3\}, \qquad E = \big\{\{H_1, H_2, H_3\}\big\},$$

so that $|V| = 3$ and $M = 3$.

Assign to each department the "active project count" signal

$$\mathbf{s} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

The adjacency tensor $\mathcal{A} \in \mathbb{R}^{3\times3\times3}$ has entries

$$\mathcal{A}_{i,j,k} = \begin{cases} \dfrac{3}{\sum_{k_1+k_2+k_3=3} \frac{3!}{k_1!\,k_2!\,k_3!}} = \dfrac{3}{6} = 0.5, & \{i, j, k\} = \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

Form the order-2 signal tensor $\mathcal{S} \in \mathbb{R}^{3\times3}$ by

$$\mathcal{S}_{i,j} = s_i\, s_j,$$

so that for instance $\mathcal{S}_{1,2} = 3 \times 4 = 12$. The shifted signal $\mathcal{S}' = \mathcal{A} \times_3 \mathcal{S} \in \mathbb{R}^{3\times3}$ has entries

$$\mathcal{S}'_{i,j} = \sum_{k=1}^{3} \mathcal{A}_{i,j,k}\, s_k,$$

giving

$$\mathcal{S}' = \begin{pmatrix} 0 & 0.5\times5 & 0.5\times4 \\ 0.5\times5 & 0 & 0.5\times3 \\ 0.5\times4 & 0.5\times3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2.5 & 2.0 \\ 2.5 & 0 & 1.5 \\ 2.0 & 1.5 & 0 \end{pmatrix}.$$

Theorem 3.4 (GSP and HGSP as Special Cases)**.** *Let* $\mathrm{NSP}^{(n)}$ *denote the n-SuperHypergraph Signal Processing above, with parameters n and M. Then:*

1. *If* $n = 0$, $\mathrm{NSP}^{(0)}$ *coincides with Hypergraph Signal Processing on the hypergraph* $H = (V_0, E)$.
2. *If moreover* $M = 2$, $\mathrm{NSP}^{(0)}$ *further reduces to Graph Signal Processing on the simple graph* (V_0, E).

Proof. For $n = 0$, one has $V \subseteq \mathcal{P}^0(V_0) = V_0$ and $E \subseteq \mathcal{P}^0(V_0) = V_0$, so $\mathrm{SHT}^{(0)}$ is exactly a hypergraph on V_0. By construction, the adjacency tensor and all subsequent operations in $\mathrm{NSP}^{(0)}$ agree with those of Hypergraph Signal Processing.

If additionally $M = 2$, then all superedges have size at most 2, so $\mathcal{A}$ is a matrix (order-2 tensor). The tensor definitions collapse to vector–matrix operations: $\mathcal{S} = \mathbf{s}$, $\mathcal{S}' = \mathbf{A}\,\mathbf{s}$, and the CANDECOMP/PARAFAC decomposition reduces to the eigen-decomposition of $\mathbf{A}$. These are precisely the definitions of Graph Signal Processing. □

Theorem 3.5 (Underlying n-SuperHyperGraph Structure)**.** *The n-SuperHypergraph Signal Processing* $\mathrm{NSP}^{(n)}$ *is built intrinsically on the combinatorial structure of the n-SuperHyperGraph* $\mathrm{SHT}^{(n)}$.

Proof. By definition, every element of the domain V of signals is an n-supervertex in $\mathcal{P}^n(V_0)$, and every nonzero entry of the adjacency tensor $\mathcal{A}$ corresponds exactly to an n-superedge in $E \subseteq \mathcal{P}^n(V_0)$. All signal operations (outer-product, mode products, tensor decompositions) are indexed by these supervertices and superedges. Hence the entire signal-processing pipeline is a direct translation of the combinatorial data of $\mathrm{SHT}^{(n)}$ into multilinear algebra, proving that $\mathrm{NSP}^{(n)}$ inherits and requires the full n-SuperHyperGraph structure. □

Theorem 3.6 (Spectral Diagonalization of the Shift Operator)**.** *Let* $\mathcal{A} = \sum_{r=1}^{R} \lambda_r\,(\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r)$ *be the orthogonal CANDECOMP/PARAFAC decomposition of the adjacency tensor and let* $\mathcal{S}' = \mathcal{A} \times_M \mathcal{S}$. *Then for each spectral component r,*

$$\widehat{\mathcal{S}'}_r \;=\; \lambda_r\,\widehat{\mathcal{S}}_r,$$

where $\widehat{\mathcal{S}}_r = \langle \mathcal{S},\, \mathbf{f}_r \circ \cdots \circ \mathbf{f}_r \rangle$.

Proof. By definition,

$$\widehat{\mathcal{S}'}_r = \big\langle \mathcal{S}',\, \mathbf{f}_r \circ \cdots \circ \mathbf{f}_r \big\rangle = \big\langle \mathcal{A} \times_M \mathcal{S},\, \mathbf{f}_r^{\otimes M} \big\rangle.$$

Using the multilinear contraction property,

$$\big\langle \mathcal{A} \times_M \mathcal{S},\, \mathbf{f}_r^{\otimes M} \big\rangle = \big\langle \mathcal{A},\, \mathbf{f}_r^{\otimes (M-1)} \circ\ \mathcal{S} \times_M \mathbf{f}_r \big\rangle = \big\langle \mathcal{A},\, \mathbf{f}_r^{\otimes M} \big\rangle\,\widehat{\mathcal{S}}_r.$$

But from the CP-decomposition,

$$\big\langle \mathcal{A},\, \mathbf{f}_r^{\otimes M} \big\rangle = \lambda_r\,\big\langle \mathbf{f}_r^{\otimes M},\, \mathbf{f}_r^{\otimes M} \big\rangle = \lambda_r,$$

by orthonormality. Hence $\widehat{\mathcal{S}'}_r = \lambda_r\,\widehat{\mathcal{S}}_r$. □

Theorem 3.7 (Inversion Formula)**.** *The collection* $\{\mathbf{f}_r^{\otimes M}\}_{r=1}^{R}$ *forms an orthonormal basis for the signal-tensor space. Consequently, any signal tensor* $\mathcal{S}$ *admits the expansion*

$$\mathcal{S} = \sum_{r=1}^{R} \widehat{\mathcal{S}}_r\,(\mathbf{f}_r \circ \cdots \circ \mathbf{f}_r), \qquad \widehat{\mathcal{S}}_r = \big\langle \mathcal{S},\, \mathbf{f}_r^{\otimes M} \big\rangle.$$

Proof. Orthonormality of the rank-one factors implies $\langle \mathbf{f}_r^{\otimes M}, \mathbf{f}_s^{\otimes M} \rangle = \delta_{rs}$. Any tensor in $\mathbb{R}^{N_n^{\times (M-1)}}$ can be uniquely decomposed in this basis. The coefficients are given by the inner products $\widehat{\mathcal{S}}_r$. Summing over r yields the reconstruction formula. □

Theorem 3.8 (Parseval's Identity)**.** *For any signal tensor* $\mathcal{S}$*,*

$$\|\mathcal{S}\|^2 = \sum_{i_1,\ldots,i_{M-1}} \mathcal{S}_{i_1 \cdots i_{M-1}}^2 = \sum_{r=1}^{R} (\widehat{\mathcal{S}}_r)^2.$$

Proof. From the inversion formula, $\mathcal{S} = \sum_r \widehat{\mathcal{S}}_r\,\mathbf{f}_r^{\otimes M}$, so

$$\|\mathcal{S}\|^2 = \Big\langle \sum_r \widehat{\mathcal{S}}_r\,\mathbf{f}_r^{\otimes M}, \sum_s \widehat{\mathcal{S}}_s\,\mathbf{f}_s^{\otimes M} \Big\rangle = \sum_{r,s} \widehat{\mathcal{S}}_r\,\widehat{\mathcal{S}}_s\,\langle \mathbf{f}_r^{\otimes M}, \mathbf{f}_s^{\otimes M} \rangle = \sum_r (\widehat{\mathcal{S}}_r)^2.$$

□

Theorem 3.9 (Filter Diagonalization). *Let $\mathcal{A} = \sum_{r=1}^{R} \lambda_r \mathbf{f}_r^{\otimes M}$ be the orthogonal CANDECOMP/PARAFAC decomposition of the adjacency tensor. For any real polynomial $g(t) = \sum_{k=0}^{K} a_k t^k$, define the filter operator*

$$\mathcal{H} = \sum_{k=0}^{K} a_k \underbrace{\mathcal{A} \times_M \mathcal{A} \times_M \cdots \times_M \mathcal{A}}_{k \text{ times}} \in \mathbb{R}^{\overbrace{N_n \times \cdots \times N_n}^{M \text{ times}}}.$$

Then for any signal tensor $\mathcal{S}$,

$$\widehat{\mathcal{H} \times_M \mathcal{S}}_r = g(\lambda_r)\,\widehat{\mathcal{S}}_r, \quad r = 1, \dots, R,$$

i.e. the filter acts as pointwise multiplication by $g(\lambda_r)$ in the spectral domain.

Proof. Since $\mathcal{A}^{\times k} = \sum_{r=1}^{R} \lambda_r^k \mathbf{f}_r^{\otimes M}$ by repeated application of the CP decomposition, it follows that

$$\mathcal{H} = \sum_{k=0}^{K} a_k \mathcal{A}^{\times k} = \sum_{r=1}^{R} \Big(\sum_{k=0}^{K} a_k \lambda_r^k \Big) \mathbf{f}_r^{\otimes M} = \sum_{r=1}^{R} g(\lambda_r)\,\mathbf{f}_r^{\otimes M}.$$

Hence for any $\mathcal{S}$,

$$\widehat{\mathcal{H} \times_M \mathcal{S}}_r = \langle \mathcal{H} \times_M \mathcal{S}, \mathbf{f}_r^{\otimes M} \rangle = g(\lambda_r)\,\langle \mathcal{S}, \mathbf{f}_r^{\otimes M} \rangle = g(\lambda_r)\,\widehat{\mathcal{S}}_r.$$

□

Theorem 3.10 (Shift-Invariant Operator Characterization). *A multilinear operator $\mathcal{H}\colon \mathbb{R}^{N_n^{\times(M-1)}} \to \mathbb{R}^{N_n^{\times(M-1)}}$ commutes with the shift $\mathcal{A} \times_M (\cdot)$ if and only if it is simultaneously diagonalizable, i.e.,*

$$\mathcal{H} = \sum_{r=1}^{R} h_r \mathbf{f}_r^{\otimes M},$$

for some scalars h_r. In this case, $\mathcal{H} \times_M \mathcal{A} = \mathcal{A} \times_M \mathcal{H}$.

Proof. ($\Rightarrow$) If $\mathcal{H} \circ (\mathcal{A}\times_M) = (\mathcal{A}\times_M) \circ \mathcal{H}$, then $\mathcal{H}$ preserves each one-dimensional eigenspace spanned by $\mathbf{f}_r^{\otimes M}$. By orthonormality, $\mathcal{H}(\mathbf{f}_r^{\otimes M}) = h_r \mathbf{f}_r^{\otimes M}$ for some h_r.

($\Leftarrow$) Conversely, if $\mathcal{H} = \sum h_r \mathbf{f}_r^{\otimes M}$, then

$$\mathcal{H} \times_M \mathcal{A} = \sum_{r=1}^{R} h_r \lambda_r \mathbf{f}_r^{\otimes M} = \mathcal{A} \times_M \mathcal{H}.$$

□

Theorem 3.11 (Operator Norm and Spectral Radius). *Let $T\colon \mathcal{S} \mapsto \mathcal{A} \times_M \mathcal{S}$ be the shift operator. Then its induced spectral norm equals the maximum absolute hypergraph frequency:*

$$\|T\|_2 = \max_{1 \le r \le R} |\lambda_r|.$$

Proof. Since T is diagonalizable in the orthonormal basis $\{\mathbf{f}_r^{\otimes M}\}$, its operator norm is the largest magnitude of its eigen-values, which are exactly $\{\lambda_r\}_{r=1}^{R}$. □

4 Result: Electric HyperCircuit and Electric SuperHyperCircuit

We define the concepts of the Electric HyperCircuit and the Electric SuperHyperCircuit, and provide concrete examples and mathematical theorems to illustrate their structures and properties.

Definition 4.1 (Electric HyperCircuit). An *electric hypercircuit* is a pair $(H, \mathcal{E})$ where:

- $H = (V, E, I, \pi, \sigma)$ is a finite oriented hypergraph:

- V is the set of *nodes*.
- E is the set of *hyperedges* (multi-terminal elements).
- I is a finite set of *incidences*, with surjections $\pi : I \to V$ (attaching each incidence to a node) and $e : I \to E$ (attaching each incidence to a hyperedge).
- $\sigma : I \to \{+1, -1\}$ is an *orientation* on incidences.

• $\mathcal{E} = \{\mathcal{E}_e\}_{e \in E}$ assigns to each hyperedge e a *constitutive relation*

$$\mathcal{E}_e\big(v_e, i_e\big) = 0, \quad v_e = (v_k)_{k \in I_e}, \ i_e = (i_k)_{k \in I_e},$$

where $I_e = \{k \in I : e(k) = e\}$ is the set of incidences of e.

A *state* of the hypercircuit consists of functions $i : I \to \mathbb{R}$ (port currents) and $v : I \to \mathbb{R}$ (port voltages) satisfying:

1. **Kirchhoff's Current Law (KCL):**

$$\sum_{k \in I:\, \pi(k) = n} i(k) = 0, \quad \forall\, n \in V.$$

2. **Kirchhoff's Voltage Law (KVL):** there exists a node-potential function $u : V \to \mathbb{R}$ such that

$$v(k) = \sigma(k)\, u\big(\pi(k)\big), \quad \forall\, k \in I.$$

3. **Element Constitutive Relations:** for each $e \in E$,

$$\mathcal{E}_e\big(v_e, i_e\big) = 0.$$

Example 4.2 (Common–Emitter BJT Amplifier as an Electric HyperCircuit)**.** A BJT (Bipolar Junction Transistor) is a semiconductor device that amplifies or switches signals using current-controlled junctions (cf. [21, 54, 185]). A Common–Emitter BJT Amplifier is a transistor circuit configuration that amplifies voltage signals with significant gain and phase inversion (cf. [112, 113]). Consider the hypercircuit $(H, \mathcal{E})$ defined as follows:

Hypergraph structure $H = (V, E, I, \pi, e, \sigma)$**:**

$$V = \{V_{CC},\ B,\ C,\ E\}, \qquad E = \{R_B,\ R_C,\ T\},$$

where

$$I = \{\, k_{V_{CC},R_B},\ k_{B,R_B},\ k_{V_{CC},R_C},\ k_{C,R_C},\ k_{B,T},\ k_{C,T},\ k_{E,T}\,\}.$$

The attachment maps are

$$\begin{aligned}
\pi(k_{V_{CC},R_B}) &= V_{CC}, & e(k_{V_{CC},R_B}) &= R_B,\\
\pi(k_{B,R_B}) &= B, & e(k_{B,R_B}) &= R_B,\\
\pi(k_{V_{CC},R_C}) &= V_{CC}, & e(k_{V_{CC},R_C}) &= R_C,\\
\pi(k_{C,R_C}) &= C, & e(k_{C,R_C}) &= R_C,\\
\pi(k_{B,T}) &= B, & e(k_{B,T}) &= T,\\
\pi(k_{C,T}) &= C, & e(k_{C,T}) &= T,\\
\pi(k_{E,T}) &= E, & e(k_{E,T}) &= T.
\end{aligned}$$

Orient all incidences from the first-listed node to the second, so $\sigma(k_{X,e}) = +1$ if X is listed first, and -1 otherwise.

Constitutive relations $\mathcal{E}$:

$$\begin{aligned}
\mathcal{E}_{R_B}: \quad & v_{R_B} - R_B\, i_{R_B} = 0, \quad v_{R_B} = v(k_{V_{CC},R_B}) - v(k_{B,R_B}),\ i_{R_B} = i(k_{V_{CC},R_B});\\
\mathcal{E}_{R_C}: \quad & v_{R_C} - R_C\, i_{R_C} = 0, \quad v_{R_C} = v(k_{V_{CC},R_C}) - v(k_{C,R_C}),\ i_{R_C} = i(k_{V_{CC},R_C});\\
\mathcal{E}_{T}: \quad & v(k_{B,T}) - v(k_{E,T}) - V_{BE} = 0,\\
& i(k_{E,T}) - i(k_{B,T}) - i(k_{C,T}) = 0,\\
& i(k_{C,T}) - \alpha\, i(k_{E,T}) = 0,
\end{aligned}$$

where V_{BE} is the base–emitter threshold and α the common-base gain.

KCL and KVL: A state consists of $v : I \longrightarrow \mathbb{R}$, $i : I \longrightarrow \mathbb{R}$, and node potentials $u : V \longrightarrow \mathbb{R}$, satisfying

$$\sum_{k:\,\pi(k)=n} i(k) = 0 \quad (\text{KCL at each } n \in V), \qquad v(k) = \sigma(k)\, u\big(\pi(k)\big) \quad (\text{KVL for each } k \in I).$$

This hypercircuit model captures the two resistors and the three-terminal transistor in one unified oriented hypergraph framework.

Theorem 4.3 (Generalization of Electric Circuit)**.** *If each hyperedge $e \in E$ has exactly two incidences $I_e = \{k_1, k_2\}$ and $\mathcal{E}_e\big(v_e, i_e\big)$ depends only on the voltage difference and a single current (as in Ohm's law), then the electric hypercircuit reduces to the classical electric circuit on the graph $G = (V, E)$.*

Proof. When $|I_e| = 2$, index the two incidences by k_1, k_2 with $\pi(k_1) = n_1,\ \pi(k_2) = n_2$. KVL gives

$$v(k_1) = u(n_1), \quad v(k_2) = -u(n_2) \implies v_{n_1 n_2} = u(n_1) - u(n_2),$$

recovering the usual branch voltage. KCL at each node $\sum_{k:\,\pi(k)=n} i(k) = 0$ becomes the sum of incident branch currents. Finally, if $\mathcal{E}_e(v_e, i_e) \equiv v_{n_1 n_2} - R_e\, i_e = 0$, we obtain Ohm's law. Thus the hypercircuit equations coincide with the network equations of an electric circuit on the graph G. □

Theorem 4.4 (Underlying Hypergraph Structure)**.** *The electric hypercircuit $(H, \mathcal{E})$ is intrinsically built on the combinatorial data of the oriented hypergraph H.*

Proof. By definition, the set of nodes V, hyperedges E, incidences I, and orientation σ completely determine the incidence relations π and e. All circuit equations—KCL, KVL, and constitutive relations—are indexed by these hypergraph components (V, I, E). Therefore the signal-processing and network-analysis formalisms operate directly on the hypergraph structure, proving that the electric hypercircuit inherits and requires the full hypergraph. □

Theorem 4.5 (Existence and Uniqueness of Linear Hypercircuit Solution)**.** *Let $(H, \mathcal{E})$ be an electric hypercircuit in which each hyperedge $e \in E$ has a linear, time-invariant constitutive relation*

$$i_e = G_e\, v_e, \quad G_e \in \mathbb{R}^{|I_e| \times |I_e|},\ G_e = G_e^\top > 0.$$

Define the oriented incidence matrix $B \in \mathbb{R}^{|V| \times |I|}$ by

$$B_{n,k} = \begin{cases} \sigma(k), & \pi(k) = n,\\ 0, & \text{otherwise}, \end{cases}$$

and let $\Sigma = \operatorname{diag}(\sigma(1), \ldots, \sigma(|I|))$. Then for any vector of external current injections $b \in \mathbb{R}^{|V|}$ satisfying $\sum_{n \in V} b_n = 0$, there exists a node-potential vector $u \in \mathbb{R}^{|V|}$ and port currents $i \in \mathbb{R}^{|I|}$ satisfying Kirchhoff's laws and constitutive relations, unique up to an additive constant in u.

Proof. Kirchhoff's Voltage Law gives

$$v = \Sigma\, B^\top u,$$

and each hyperedge's relation yields

$$i \;=\; G\,v \;=\; G\,\Sigma\,B^{\top}u, \quad G = \bigoplus_{e\in E} G_e.$$

Kirchhoff's Current Law reads

$$B\,i \;=\; b \;\Longrightarrow\; B\,G\,\Sigma\,B^{\top}u \;=\; b.$$

Since each G_e is positive definite, $Y \equiv B\,G\,\Sigma\,B^{\top}$ is symmetric positive semidefinite with $\operatorname{rank}(Y) = |V| - 1$. The condition $\sum b_n = 0$ ensures $b \in \operatorname{Im}(Y)$, so $Y\,u = b$ admits a solution modulo $\ker(Y) = \operatorname{span}\{\mathbf{1}\}$. Once u is fixed, one recovers v and i uniquely. □

Theorem 4.6 (Superposition Principle)**.** *Under the same linearity assumptions, if two sets of external injections $b^{(1)}$ and $b^{(2)}$ produce solutions $(u^{(1)}, i^{(1)})$ and $(u^{(2)}, i^{(2)})$, then the combined injection $b = b^{(1)} + b^{(2)}$ yields the solution*

$$u = u^{(1)} + u^{(2)}, \quad i = i^{(1)} + i^{(2)}.$$

Proof. The hypercircuit equations are linear:

$$Y\,u^{(j)} = b^{(j)}, \quad i^{(j)} = G\,\Sigma\,B^{\top}u^{(j)}, \quad j = 1, 2.$$

By linearity of matrix equations,

$$Y\,(u^{(1)} + u^{(2)}) = Y\,u^{(1)} + Y\,u^{(2)} = b^{(1)} + b^{(2)},$$

and similarly for i. Hence $(u^{(1)} + u^{(2)},\, i^{(1)} + i^{(2)})$ satisfies KCL, KVL, and constitutive relations for the combined excitation b. □

Theorem 4.7 (Reciprocity of Passive Hypercircuits)**.** *In a passive linear hypercircuit (all G_e symmetric), the nodal admittance matrix $Y = B\,G\,\Sigma\,B^{\top}$ is symmetric. Consequently, the transfer impedance between any two nodes is reciprocal.*

Proof. Since each G_e is symmetric and Σ is diagonal,

$$Y^{\top} = (B\,G\,\Sigma\,B^{\top})^{\top} = B\,\Sigma^{\top}\,G^{\top}\,B^{\top} = B\,G\,\Sigma\,B^{\top} = Y.$$

Symmetry of Y implies that for any two distinct nodes $n, m \in V$, the entry $Y_{nm} = Y_{mn}$, which governs the small-signal transfer between n and m, yielding reciprocity. □

Definition 4.8 (Electric n-SuperHyperCircuit)**.** Let V_0 be a finite base set of fundamental nodes and let $\mathrm{SHT}^{(n)} = (V, E)$ be an oriented n-SuperHyperGraph with

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0),$$

and incidence structure (I, π, e, σ) where

$$I = \{(v, e) : v \in e,\ e \in E\},\ \pi(v, e) = v,\ e(v, e) = e,\ \sigma(v, e) \in \{+1, -1\}.$$

An *electric n-superhypercircuit* is the pair $(\mathrm{SHT}^{(n)}, \mathcal{E})$ where $\mathcal{E} = \{\mathcal{E}_e\}_{e\in E}$ assigns to each superedge e a constitutive relation

$$\mathcal{E}_e\big(v_e, i_e\big) = 0, \quad v_e = \big(v(k)\big)_{k\in I_e},\ i_e = \big(i(k)\big)_{k\in I_e},$$

with $I_e = \{k \in I : e(k) = e\}$. A *state* consists of port-voltage and port-current functions

$$v : I \to \mathbb{R}, \quad i : I \to \mathbb{R},$$

and a supervertex potential $u : V \to \mathbb{R}$, satisfying:

1. $\displaystyle\sum_{k\in I:\,\pi(k)=v} i(k) = 0$ for all supervertices $v \in V$ (generalized KCL).
2. $v(k) = \sigma(k)\,u\big(\pi(k)\big)$ for all ports $k \in I$ (generalized KVL).

3. $\mathcal{E}_e\big(v_e, i_e\big) = 0$ for each superedge $e \in E$ (constitutive laws).

Example 4.9 (Electric 2-SuperHyperCircuit for a BJT Amplifier Subnetwork)**.** A BJT amplifier uses a bipolar junction transistor to amplify input signals, commonly employed in analog circuits for voltage or current gain (cf. [29, 112, 114]). Let the base set of fundamental nodes be

$$V_0 = \{V_{CC},\ B,\ C,\ E\},$$

and consider the electric hypercircuit with three hyperedges:

$$e_1 = \{V_{CC}, B\} \quad (R_B), \quad e_2 = \{V_{CC}, C\} \quad (R_C), \quad e_3 = \{B, C, E\} \quad (T).$$

We form the 2-supervertices by grouping overlapping hyperedges:

$$D_1 = \{\, e_1,\ e_2\}, \qquad D_2 = \{\, e_2,\ e_3\}.$$

Thus the set of 2-supervertices is

$$V = \{D_1, D_2\},$$

and there is a single 2-superedge connecting them:

$$E = \big\{\{D_1, D_2\}\big\}.$$

The incidence set is

$$I = \{\, k_1 = (D_1, E),\ k_2 = (D_2, E)\},$$

with $\pi(k_i) = D_i$, $e(k_i) = E$, and choose $\sigma(k_i) = +1$.

We assign to each 2-superedge E the constitutive relations of an ideal connection:

$$\mathcal{E}_E : \quad v(k_1) - v(k_2) = 0, \qquad i(k_1) + i(k_2) = 0,$$

where $v(k_i)$ and $i(k_i)$ are the port-voltage and port-current at incidence k_i.

A state consists of port-functions $v : I \to \mathbb{R},\ i : I \to \mathbb{R}$ and supervertex potentials $u : V \to \mathbb{R}$ satisfying:

$$\sum_{k:\,\pi(k)=D_i} i(k) = 0, \quad v(k) = \sigma(k)\, u\big(\pi(k)\big), \quad \mathcal{E}_E\big(v_E, i_E\big) = 0.$$

Concretely,

$$i(k_1) + i(k_2) = 0, \quad v(k_1) = v(k_2),$$

ensuring that the two subnetworks $\{R_B, R_C\}$ and $\{R_C, T\}$ are perfectly connected in this 2-superhypercircuit.

Example 4.10 (Electric 3-SuperHyperCircuit for a BJT Amplifier Meta-Connection)**.** Let the base set of fundamental nodes be

$$V_0 = \{V_{CC},\ B,\ C,\ E\},$$

and consider the three 1-superedges (ordinary hyperedges)

$$e_1 = \{V_{CC}, B\}, \quad e_2 = \{V_{CC}, C\}, \quad e_3 = \{B, C, E\}.$$

Form the 2-supervertices (elements of $\mathcal{P}^2(V_0)$) by grouping overlapping 1-superedges:

$$D_1 = \{e_1, e_2\}, \quad D_2 = \{e_2, e_3\}, \quad D_3 = \{e_3, e_1\}.$$

Thus

$$V^{(2)} = \{D_1, D_2, D_3\}.$$

Next form the 3-supervertices (elements of $\mathcal{P}^3(V_0)$) by grouping overlapping 2-supervertices:

$$A_1 = \{D_1, D_2\}, \quad A_2 = \{D_2, D_3\}, \quad A_3 = \{D_3, D_1\},$$

so

$$V^{(3)} = \{A_1, A_2, A_3\}.$$

Finally, the single 3-superedge

$$S = \{A_1, A_2, A_3\}$$

yields the oriented 3-SuperHyperGraph $\mathrm{SHT}^{(3)} = (V^{(3)}, \{S\})$.

The incidence set is

$$I = \{\, k_i = (A_i, S) \mid i = 1, 2, 3\},$$

with attachments $\pi(k_i) = A_i$, $e(k_i) = S$, and orientation $\sigma(k_i) = +1$.

Assign to the 3-superedge S the constitutive (ideal coupling) relations

$$\mathcal{E}_S : \qquad v(k_1) - v(k_2) = 0, \quad v(k_2) - v(k_3) = 0, \quad i(k_1) + i(k_2) + i(k_3) = 0.$$

A *state* consists of port-voltage and port-current functions $v : I \to \mathbb{R},\ i : I \to \mathbb{R}$ and a supervertex potential $u : V^{(3)} \to \mathbb{R}$, satisfying:

$$\sum_{k:\,\pi(k)=A_i} i(k) = 0, \qquad v(k) = \sigma(k)\, u\big(\pi(k)\big), \qquad \mathcal{E}_S\big(v_S, i_S\big) = 0.$$

Concretely:

$$i(k_1) + i(k_2) + i(k_3) = 0, \quad v(k_1) = v(k_2) = v(k_3),$$

ensuring an ideal three-port connection that unifies the two subnetworks $\{e_1, e_2\}$, $\{e_2, e_3\}$, and $\{e_3, e_1\}$ into a single 3-superhyperconnection.

Theorem 4.11 (Reduction to Hypercircuit and Circuit)**.** *The electric n-superhypercircuit* $(\mathrm{SHT}^{(n)}, \mathcal{E})$*:*

1. *For $n = 0$, $V \subseteq V_0$ and $E \subseteq V_0$, so $\mathrm{SHT}^{(0)}$ is an oriented hypergraph on V_0. The above equations recover exactly those of an electric hypercircuit.*
2. *If furthermore each superedge e has $|I_e| = 2$ and $\mathcal{E}_e(v_e, i_e)$ depends only on the voltage difference and the single branch current, then $\mathrm{SHT}^{(0)}$ is a graph and the hypercircuit reduces to a classical electric circuit on $G = (V_0, E)$.*

Proof. When $n = 0$, each supervertex is a base node and each superedge is a subset of nodes in V_0. The incidence set I and orientation σ coincide with those of an oriented hypergraph. Hence KCL and KVL match the hypercircuit laws, and $\mathcal{E}_e$ are the same constitutive relations.

If in addition $|I_e| = 2$, label the two incidences k_1, k_2 with $\pi(k_1) = n_1,\ \pi(k_2) = n_2$. Then

$$v(k_1) = u(n_1),\ v(k_2) = -u(n_2) \implies v_{n_1 n_2} = u(n_1) - u(n_2),$$

and KCL becomes the node-current sum law. If $\mathcal{E}_e(v_e, i_e) : v_{n_1 n_2} - R_e\, i_e = 0$, one recovers Ohm's law. Thus the model reduces to the classical electric circuit network equations. □

Theorem 4.12 (Intrinsic n-SuperHyperGraph Structure)**.** *The electric n-superhypercircuit* $(\mathrm{SHT}^{(n)}, \mathcal{E})$ *is built intrinsically on the oriented n-SuperHyperGraph* $\mathrm{SHT}^{(n)}$.

Proof. All components—supervertices V, superedges E, incidences I, attachment maps π, e, and orientations σ—are data of $\mathrm{SHT}^{(n)}$. The network laws (generalized KCL, KVL) and constitutive equations are formulated directly in terms of these hypergraph elements. No additional structure or external indexing is required. Therefore the circuit model inherently carries and exploits the full combinatorial structure of the n-SuperHyperGraph. □

Theorem 4.13 (Positive Semidefiniteness of Hypercircuit Admittance)**.** *Let* $(\mathrm{SHT}^{(n)}, \mathcal{E})$ *be a linear electric n-superhypercircuit in which each superedge e has a symmetric positive-definite constitutive conductance matrix G_e. Define the global oriented incidence matrix*

$$B \in \mathbb{R}^{|V|\times|I|}, \quad B_{v,k} = \sigma(k) \text{ if } \pi(k) = v, \text{ else } 0,$$

and the block-diagonal conductance $G = \bigoplus_{e\in E} G_e$. Then the nodal admittance matrix

$$Y \;=\; B\, G\, B^{\top}$$

is symmetric positive semidefinite with $\ker(Y) = \mathrm{span}\{\mathbf{1}\}$.

Proof. Symmetry: $Y^\top = (B\,G\,B^\top)^\top = B\,G^\top\,B^\top = B\,G\,B^\top = Y$. Positive semidefiniteness: for any $u \in \mathbb{R}^{|V|}$,

$$u^\top Y\,u = u^\top B\,G\,B^\top u = (B^\top u)^\top G\,(B^\top u) \geq 0,$$

since G is block-diagonal with each $G_e \succ 0$. Finally, $u^\top Y u = 0$ if and only if $B^\top u = 0$, i.e. all port-voltage differences vanish, forcing u constant on the connected superhypergraph. Hence $\ker(Y) = \mathrm{span}\{\mathbf{1}\}$. □

Theorem 4.14 (Reciprocity of Passive n-SuperHypercircuits)**.** *Let* $(\mathrm{SHT}^{(n)}, \mathcal{E})$ *be a passive linear electric n-superhypercircuit with oriented incidence matrix* $B \in \{-1, 0, 1\}^{|V|\times|I|}$ *and block-diagonal conductance* $G \succ 0$. *Define the nodal admittance matrix*

$$Y \;=\; B\,G\,B^\top.$$

Then:

1. *Y is symmetric:*
$$Y^\top = (B\,G\,B^\top)^\top = B\,G^\top\,B^\top = B\,G\,B^\top = Y.$$
2. *Consequently, for any two supervertices* $v, w \in V$*, the transfer admittance satisfies*
$$Y_{vw} \;=\; Y_{wv}.$$

Proof. Immediate from the symmetry of Y. Physically, injecting a current at v and measuring the resulting voltage at w yields the same relation when the roles of v and w are exchanged. □

Theorem 4.15 (Superposition Principle)**.** *If* $(\mathrm{SHT}^{(n)}, \mathcal{E})$ *is linear as above, and two external current injection patterns* $b^{(1)}, b^{(2)} \in \mathbb{R}^{|V|}$ *(with zero total sum) produce nodal potentials* $u^{(1)}, u^{(2)}$*, then the combined injection* $b = b^{(1)} + b^{(2)}$ *produces*

$$u \;=\; u^{(1)} + u^{(2)}, \quad i \;=\; i^{(1)} + i^{(2)},$$

where $i^{(j)} = G\,B^\top u^{(j)}$.

Proof. Linearity of the global equation $Y\,u = b$ implies

$$Y(u^{(1)} + u^{(2)}) = Y\,u^{(1)} + Y\,u^{(2)} = b^{(1)} + b^{(2)} = b.$$

Similarly, $i = G\,B^\top u$ is linear in u. Uniqueness up to a constant follows as in previous theorem. □

Theorem 4.16 (Energy Conservation)**.** *In any state* (v, i, u) *of a passive linear electric n-superhypercircuit, the total instantaneous power supplied by the ports equals the rate of change of stored energy:*

$$P_{\text{in}} = \sum_{k\in I} e_k\,f_k = \frac{d}{dt}\Big(\tfrac{1}{2}\,u^\top\,Q\,u\Big),$$

where $e_k = \sigma(k)\,u(\pi(k))$, $f = G\,v$*, and* Q *is the block-diagonal matrix of storage coefficients from capacitive/inertial superedges.*

Proof. Define the stored energy $E(u) = \frac{1}{2}\,u^\top Q\,u$. Differentiating,

$$\dot{E} = u^\top Q\,\dot{u}.$$

From KCL and constitutive laws, $B\,i = 0$ (no net injection) and $i = G\,B^\top u$. Multiply $u^\top B\,i = 0$ by u to get

$$u^\top B\,G\,B^\top u = 0.$$

On the other hand, the total port power $P_{\text{in}} = v^\top i = (B^\top u)^\top G\,(B^\top u)$. Hence

$$P_{\text{in}} = (B^\top u)^\top G\,(B^\top u) = u^\top B\,G\,B^\top u = 0 = \dot{E},$$

demonstrating conservation of energy (no dissipation in ideal storage elements). □

Theorem 4.17 (Existence and Uniqueness of Solution)**.** *For a passive linear electric n-superhypercircuit with nodal admittance* Y*, given any external injection* b *with* $\sum b_v = 0$*, there exists a unique (modulo a constant) potential vector* u *and unique port currents* i *satisfying KCL, KVL, and constitutive laws.*

Proof. Combine Theorems: positive semidefiniteness and superposition guarantee existence for each homogeneous and particular component, and symmetry of Y ensures uniqueness up to an additive constant in u. □

5 Result: Bond HyperGraph and Bond SuperHyperGraph

A Bond HyperGraph is a hypergraph from a bond graph, mapping each junction to a hyperedge connecting incident element nodes. A Bond SuperHyperGraph is an n-level superhypergraph extending a bond graph, where nested hyperedges represent hierarchical junction groupings across supervertices. We define the concepts of the Bond HyperGraph and the Bond SuperHyperGraph as follows.

Definition 5.1 (Bond HyperGraph)**.** Let V_{elem} be the set of bond-graph element nodes and V_{junc} the set of junction nodes, and let

$$G = \big(V_{\text{elem}} \dot{\cup} V_{\text{junc}},\ E\big)$$

be the classical bond graph. The *Bond HyperGraph* is the hypergraph

$$H = \big(V_{\text{elem}},\ \mathcal{E}\big),$$

where

$$\mathcal{E} = \big\{\, e_j \subseteq V_{\text{elem}} : j \in V_{\text{junc}},\ e_j = \{\, u \in V_{\text{elem}} : \{u, j\} \in E\}\big\}.$$

Each hyperedge e_j collects exactly those element nodes incident on junction j.

Example 5.2 (Bond HyperGraph of a Series R–C Circuit Driven by a Voltage Source)**.** An RC circuit is an electrical circuit composed of a resistor and capacitor, used for filtering, timing, and signal processing applications (cf. [88, 89, 91]). Consider the bond graph with element nodes and junctions as follows:

$$V_{\text{elem}} = \{Se,\ R,\ C\}, \qquad V_{\text{junc}} = \{j_1,\ j_2\}.$$

The bond connections are

$$E = \big\{\{Se, j_1\},\ \{R, j_1\},\ \{R, j_2\},\ \{C, j_2\}\big\},$$

where Se is an effort source, R a resistor, C a capacitor, and j_1, j_2 are 1-junctions.

Forming the Bond HyperGraph $H = (V_{\text{elem}}, \mathcal{E})$, each junction j_k induces a hyperedge

$$e_{j_1} = \{\, Se,\ R\}, \qquad e_{j_2} = \{\, R,\ C\},$$

so that

$$\mathcal{E} = \{\, e_{j_1},\ e_{j_2}\}.$$

Thus H is the hypergraph with vertex set $\{Se, R, C\}$ and hyperedge set $\{\{Se, R\}, \{R, C\}\}$, exactly capturing which element nodes meet at each junction.

Theorem 5.3 (Generalization of Bond Graph)**.** *Every bond graph G arises from a unique Bond HyperGraph H via the construction above, and conversely any Bond HyperGraph H defines a bond graph G in which each hyperedge e_j becomes a junction node j connected by bonds to every element $u \in e_j$.*

Proof. Starting from G, we form $H = (V_{\text{elem}}, \mathcal{E})$ by setting each hyperedge e_j to be the neighborhood of junction j. Conversely, given H, define

$$V_{\text{junc}} = \mathcal{E}, \qquad E = \big\{\{u, e_j\} : u \in e_j,\ e_j \in \mathcal{E}\big\}.$$

Then $G' = (V_{\text{elem}} \dot{\cup} V_{\text{junc}}, E)$ is a bond graph whose junction-neighborhoods recover exactly the hyperedges of H. These two operations are inverse to one another, proving the bijective correspondence. □

Theorem 5.4 (Underlying Hypergraph Structure)**.** *The Bond HyperGraph $H = (V_{\text{elem}}, \mathcal{E})$ carries by definition the full structure of a finite hypergraph: its vertex set is V_{elem} and its hyperedge set is $\mathcal{E} \subseteq \mathcal{P}(V_{\text{elem}})$.*

Proof. By construction, $\mathcal{E}$ is a collection of subsets of V_{elem}, and there are no additional constraints: H satisfies exactly the axioms of a finite hypergraph. All bond-graph junction connectivity is encoded solely in these hyperedges. □

Theorem 5.5 (Degree Correspondence)**.** *Let $G = (V_{\text{elem}} \dot{\cup} V_{\text{junc}}, E)$ be a bond graph and $H = (V_{\text{elem}}, \mathcal{E})$ its Bond HyperGraph. For each element node $u \in V_{\text{elem}}$, the degree of u in G (number of bonds incident on u) equals the number of hyperedges in $\mathcal{E}$ that contain u:*

$$\deg_G(u) \;=\; \big|\{\, e_j \in \mathcal{E} : u \in e_j\}\big|.$$

Proof. In G, each bond connecting u to a junction j contributes one to $\deg_G(u)$. By definition of H, each such junction $j \in V_{\text{junc}}$ yields a hyperedge $e_j = \{ v \in V_{\text{elem}} : \{v, j\} \in E\}$. Thus u appears in e_j exactly when $\{u, j\} \in E$. Counting all bonds incident on u is therefore identical to counting all hyperedges e_j with $u \in e_j$, proving the claimed equality. □

Theorem 5.6 (Connectivity Equivalence)**.** *The bond graph G is (vertex-)connected if and only if its Bond HyperGraph H is connected in the sense that its incidence bipartite graph (with parts V_{elem} and $\mathcal{E}$) is connected.*

Proof. By construction, the incidence bipartite graph of H has an edge between $u \in V_{\text{elem}}$ and hyperedge $e_j \in \mathcal{E}$ precisely when $\{u, j\} \in E$ in G. But G itself is exactly that same bipartite graph between elements and junctions (with junction-names identified with hyperedges). Hence connectivity of one is equivalent to connectivity of the other. □

Theorem 5.7 (Primal Graph Reconstruction)**.** *Let $H = (V_{\text{elem}}, \mathcal{E})$ be a Bond HyperGraph. Its 2-section (primal graph)*

$$G_H = \big(V_{\text{elem}}, E_H\big), \quad E_H = \big\{\{u, v\} : \exists\, e \in \mathcal{E},\ \{u, v\} \subseteq e\big\}$$

is exactly the element-adjacency projection of the original bond graph G, where two elements are adjacent whenever they share a common junction.

Proof. By definition of the primal graph of a hypergraph, u and v are connected by an edge in G_H if there exists a hyperedge e_j containing both. But e_j collects precisely those element nodes incident on junction j. Therefore u, v share e_j if and only if both are connected to the same junction j in G, which is exactly the adjacency rule in the element-projection of G. □

Theorem 5.8 (Dual Primal Graph and Junction Adjacency)**.** *Form the dual hypergraph $H^* = (\mathcal{E}, V_{\text{elem}})$ of H, where each element $u \in V_{\text{elem}}$ defines a hyperedge $\{e_j \in \mathcal{E} : u \in e_j\}$ in H^*. Then the primal graph of H^* on vertex-set $\mathcal{E}$ is isomorphic to the junction-projection graph of G, in which two junctions are adjacent whenever they share an element node.*

Proof. In H^*, two dual-vertices $e_{j_1}, e_{j_2} \in \mathcal{E}$ are adjacent if they both belong to some dual-hyperedge, i.e. there exists $u \in V_{\text{elem}}$ with $u \in e_{j_1} \cap e_{j_2}$. But this condition is exactly that junctions j_1, j_2 in G each connect to the same element node u, establishing adjacency in the junction-projection of G. Hence the two graphs coincide. □

Theorem 5.9 (Rank of Incidence Matrix)**.** *Let $B \in \{0, 1\}^{|V_{\text{elem}}| \times |\mathcal{E}|}$ be the incidence matrix of H, with $B_{u,j} = 1$ iff $u \in e_j$. Then*

$$\operatorname{rank}(B) = |V_{\text{elem}}| - c,$$

where c is the number of connected components of G (equivalently of H's incidence graph).

Proof. Since G is connected on each component and B is the biadjacency matrix between elements and junctions, standard results on the rank of the incidence matrix of a connected bipartite graph apply: its rank equals the number of vertices minus the number of connected components. Here the "vertices" on one side are V_{elem}, and the result follows by restriction to that side. □

Definition 5.10 (Bond n-SuperHyperGraph)**.** Let V_0 be the finite set of *element nodes* in a bond-graph domain. For each integer $k \geq 0$ define

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

A *Bond n-SuperHyperGraph* is a pair

$$\text{BnSHT}^{(n)} = (V, E),$$

where

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{(the } n\text{-supervertices)}, \qquad E \subseteq \mathcal{P}^n(V_0) \quad \text{(the } n\text{-superedges)},$$

together with the canonical incidence relation that each n-superedge $e \in E$ attaches to its member n-supervertices in V.

Example 5.11 (Bond 2-SuperHyperGraph of a Series R–L–C Circuit)**.** An RLC circuit is an electrical circuit consisting of a resistor (R), inductor (L), and capacitor (C) connected in series or parallel (cf. [118, 154, 182]). Let the base set of element nodes be

$$V_0 = \{Se,\ R,\ L,\ C\},$$

and consider the bond graph with three 1-junctions j_1, j_2, j_3 defined by the bonds

$$E = \big\{\{Se, j_1\},\ \{R, j_1\},\ \{R, j_2\},\ \{L, j_2\},\ \{L, j_3\},\ \{C, j_3\}\big\}.$$

The corresponding Bond HyperGraph $H = (V_{\text{elem}}, \mathcal{E})$ has

$$V_{\text{elem}} = \{Se, R, L, C\}, \qquad \mathcal{E} = \big\{\, e_{j_1},\ e_{j_2},\ e_{j_3}\big\},$$

where

$$e_{j_1} = \{Se, R\}, \quad e_{j_2} = \{R, L\}, \quad e_{j_3} = \{L, C\}.$$

Now form the 2-supervertices (elements of $\mathcal{P}^2(V_0)$) by grouping overlapping hyperedges:

$$D_1 = \{\, e_{j_1},\ e_{j_2}\}, \qquad D_2 = \{\, e_{j_2},\ e_{j_3}\}.$$

Thus the set of 2-supervertices is

$$V_2 = \{D_1,\ D_2\}.$$

A natural 2-superedge arises by connecting those two 2-supervertices that share the common hyperedge e_{j_2}:

$$E_2 = \big\{\{D_1,\ D_2\}\big\}.$$

Therefore, the Bond 2-SuperHyperGraph is

$$\text{BnSHT}^{(2)} \;=\; \big(V_2,\ E_2\big) \;=\; \big(\{D_1, D_2\},\ \{\{D_1, D_2\}\}\big).$$

This 2-SuperHyperGraph encodes a higher-level "meta-junction" that links the two 1-junction subnetworks $\{Se, R\}$–$\{R, L\}$ and $\{R, L\}$–$\{L, C\}$, thus generalizing both the bond graph and its hypergraph representation.

Example 5.12 (Bond 3-SuperHyperGraph of a Series *R*–*L*–*C* Circuit)**.** Let the base set of element nodes be

$$V_0 = \{Se,\ R,\ L,\ C\},$$

and consider the bond graph with three 1-junctions j_1, j_2, j_3 defined by the bonds

$$E = \big\{\{Se, j_1\}, \{R, j_1\},\ \{R, j_2\}, \{L, j_2\},\ \{L, j_3\}, \{C, j_3\}\big\}.$$

The corresponding Bond HyperGraph $H = (V_{\text{elem}}, \mathcal{E})$ has

$$V_{\text{elem}} = \{Se, R, L, C\}, \qquad \mathcal{E} = \{\, e_1, e_2, e_3\},$$

where

$$e_1 = \{Se, R\}, \quad e_2 = \{R, L\}, \quad e_3 = \{L, C\}.$$

Form the 2-supervertices (elements of $\mathcal{P}^2(V_0)$) by grouping overlapping hyperedges:

$$D_1 = \{e_1, e_2\}, \quad D_2 = \{e_2, e_3\}, \quad D_3 = \{e_3, e_1\}.$$

Thus the set of 2-supervertices is

$$V_2 = \{D_1, D_2, D_3\},$$

and there is a natural 2-superedge for each pair of adjacent 2-supervertices:

$$E_2 = \big\{\{D_1, D_2\}, \{D_2, D_3\}, \{D_3, D_1\}\big\}.$$

Now form the 3-supervertices (elements of $\mathcal{P}^3(V_0)$) by grouping adjacent 2-supervertices:

$$A_1 = \{D_1, D_2\}, \quad A_2 = \{D_2, D_3\}, \quad A_3 = \{D_3, D_1\}.$$

Hence

$$V_3 = \{A_1, A_2, A_3\}.$$

Finally, the single 3-superedge connects all three 3-supervertices:

$$E_3 = \big\{\{A_1, A_2, A_3\}\big\}.$$

Therefore, the Bond 3-SuperHyperGraph is

$$\mathrm{BnSHT}^{(3)} = (V_3,\ E_3) = \Big(\ \{A_1, A_2, A_3\},\ \{\{A_1, A_2, A_3\}\}\Big).$$

This structure captures a three-level meta-junction that links the overlapping sub-circuits $\{Se, R\}$–$\{R, L\}$, $\{R, L\}$–$\{L, C\}$, and $\{L, C\}$–$\{Se, R\}$ in one unified 3-superhypergraph.

Theorem 5.13 (Reduction to Bond HyperGraph and Bond Graph)**.** *Let* $\mathrm{BnSHT}^{(n)} = (V, E)$ *be a Bond n-SuperHyperGraph on base set* V_0*. Then:*

1. *If* $n = 1$*, and we take*
$$V = \big\{\{v\} : v \in V_0\big\}, \quad E = \big\{\, e_j : j \in V_{\mathrm{junc}}\big\},$$
where each $e_j \subseteq V_0$ *is the set of element-nodes incident on junction* j*, then* $\mathrm{BnSHT}^{(1)}$ *coincides with the* Bond HyperGraph*.*
2. *If moreover each hyperedge* $e_j \in E$ *has* $|e_j| = 2$*, then this Bond HyperGraph is exactly the classical* Bond Graph*.*

Proof. **(1)** For $n = 1, \mathcal{P}^1(V_0) = \mathcal{P}(V_0)$. Choosing $V = \{\{v\} : v \in V_0\}$ identifies each singleton with the original element node. Setting $E = \{e_j : j \in V_{\mathrm{junc}}\}$ reproduces exactly the hyperedges of the Bond HyperGraph, since each e_j collects the element-nodes attached to junction j.

(2) If each e_j has size two, then every hyperedge is a pair of singletons $\{\,\{u\}, \{v\}\,\}$. Collapsing the singletons back to their underlying nodes yields an undirected graph with vertex set V_0 and edge set $\{\{u, v\} : e_j = \{u, v\}\}$. This is precisely the Bond Graph. □

Theorem 5.14 (Intrinsic n-SuperHyperGraph Structure)**.** *Any Bond n-SuperHyperGraph* $\mathrm{BnSHT}^{(n)} = (V, E)$ *is by definition an n-SuperHyperGraph: its supervertex set V and superedge set E satisfy*

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0),$$

and the incidence relation is the natural membership relation of superedges on supervertices.

Proof. The construction of $\mathrm{BnSHT}^{(n)}$ uses exactly the data of an n-SuperHyperGraph on base set V_0. By hypothesis V and E are subsets of $\mathcal{P}^n(V_0)$, and each superedge $e \in E$ attaches precisely to the supervertices it contains. No additional structure is needed, hence $\mathrm{BnSHT}^{(n)}$ inherits the full combinatorial and incidence structure of an n-SuperHyperGraph. □

Theorem 5.15 (Skeleton Consistency)**.** *Let* $\mathrm{BnSHT}^{(n)} = (V^{(n)}, E^{(n)})$ *be a Bond n-SuperHyperGraph over base set* V_0*. For each* $k = n-1, n-2, \ldots, 1$*, define recursively*

$$V^{(k)} = \bigcup_{S \in V^{(k+1)}} S, \qquad E^{(k)} = \big\{\ F \subseteq V^{(k)} :\ F \subseteq e \text{ for some } e \in E^{(k+1)}\big\}.$$

Then for every $1 \le k \le n$*,* $\big(V^{(k)}, E^{(k)}\big)$ *is a Bond k-SuperHyperGraph. In particular:*

- $\big(V^{(1)}, E^{(1)}\big)$ *coincides with the Bond HyperGraph.*
- $\big(V^{(0)}, E^{(0)}\big)$ *is the classical Bond Graph.*

Proof. We prove by downward induction on k. For $k = n$, the result is given. Suppose $\left(V^{(k+1)}, E^{(k+1)}\right)$ is a Bond $(k+1)$-SuperHyperGraph with $V^{(k+1)} \subseteq \mathcal{P}^{k+1}(V_0)$, $E^{(k+1)} \subseteq \mathcal{P}^{k+1}(V_0)$. By definition,

$$V^{(k)} = \bigcup_{S \in V^{(k+1)}} S \subseteq \bigcup_{S \in \mathcal{P}^{k+1}(V_0)} S = \mathcal{P}^k(V_0),$$

and each $F \in E^{(k)}$ is a subset of some $e \in E^{(k+1)} \subseteq \mathcal{P}^{k+1}(V_0)$, so $F \subseteq \bigcup e \subseteq \mathcal{P}^k(V_0)$. The canonical incidence (membership) relation restricts correctly. Hence $\left(V^{(k)}, E^{(k)}\right)$ satisfies the definition of a Bond k-SuperHyperGraph. Taking $k = 1$ and then $k = 0$ yields the Bond HyperGraph and Bond Graph, respectively. □

Theorem 5.16 (Connectivity Inheritance)**.** *If the underlying Bond Graph (the 0-skeleton $\left(V^{(0)}, E^{(0)}\right)$) is connected, then for every $1 \leq k \leq n$, the Bond k-SuperHyperGraph $\left(V^{(k)}, E^{(k)}\right)$ is connected in the sense that its 2-section graph is connected.*

Proof. Recall that the 2-section of a hypergraph (V, E) is the graph on V where two vertices are adjacent if they belong to a common hyperedge. We show by induction on k that the 2-section of $\left(V^{(k)}, E^{(k)}\right)$ is connected.

Base ($k = 0$). The 2-section of the Bond Graph is itself, which is connected by hypothesis.

Inductive Step. Assume the 2-section of $\left(V^{(k)}, E^{(k)}\right)$ is connected. Consider $\left(V^{(k+1)}, E^{(k+1)}\right)$. By skeleton consistency, every superedge $e \in E^{(k+1)}$ is a subset of $V^{(k)}$. Thus in the 2-section of $\left(V^{(k+1)}, E^{(k+1)}\right)$, any two k-supervertices $S_1, S_2 \in V^{(k+1)}$ that share an underlying $k-1$-vertex become adjacent if $S_1 \cap S_2 \neq \emptyset$. Since the 2-section at level k is connected, one can traverse from any k-supervertex to any other by stepping through overlapping sets. Therefore the 2-section at level $k+1$ is also connected. □

Theorem 5.17 (Superedge-Induced Subgraph Connectivity)**.** *In a Bond n-SuperHyperGraph* $\mathrm{BnSHT}^{(n)} = (V^{(n)}, E^{(n)})$*, for each superedge $e \in E^{(n)}$, the induced subgraph of the underlying Bond Graph on the union of all base-nodes in e is connected.*

Proof. Let $e \in E^{(n)}$ be an n-superedge. By recursive definition of skeletons, each element of e is a $(n-1)$-supervertex, whose member set is connected at the $(n-2)$-level, and so on down to base-level. Since hyperedges at each level correspond to junction connectivity in the lower level, the union of all base-nodes in e forms a connected set in the Bond Graph. More formally, for any two base-nodes u, v in $\bigcup e$, there exists a chain of overlapping supervertices linking them, which projects to a path in the 2-section of the 0-skeleton. Hence the induced subgraph is connected. □

Theorem 5.18 (Clique Characterization of $(k+1)$-Superedges)**.** *Let* $\mathrm{BnSHT}^{(n)} = (V^{(n)}, E^{(n)})$ *be a Bond n-SuperHyperGraph, and let $1 \leq k < n$. Consider the primal (2-section) graph $G^{(k)}$ of $(V^{(k)}, E^{(k)})$, whose vertices are the k-supervertices and whose edges join any two that lie together in some k-superedge. Then each $(k+1)$-superedge $e \in E^{(k+1)}$ induces a clique in $G^{(k)}$, and conversely any maximal clique of $G^{(k)}$ arises from a unique $(k+1)$-superedge.*

Proof. By definition, a $(k+1)$-superedge $e \subseteq V^{(k)}$ consists of those k-supervertices grouped together because they share a common $(k-1)$-level face. In the 2-section $G^{(k)}$, two k-supervertices are adjacent exactly if they belong to some common $(k-1)$-superedge; but membership in the same $(k+1)$-superedge implies pairwise sharing of lower–level faces, hence adjacency. Thus e is a clique. Maximality follows because if a clique could be extended, that would contradict the maximal grouping in $E^{(k+1)}$. Conversely, any maximal clique in $G^{(k)}$ collects all k-supervertices pairwise overlapping in a common $(k-1)$-face, so by construction it defines a unique $(k+1)$-superedge. □

Theorem 5.19 (Chain–Height Bound)**.** *In a Bond n-SuperHyperGraph* $\mathrm{BnSHT}^{(n)}$*, the longest chain of strict inclusions*

$$v_0 \subsetneq v_1 \subsetneq \cdots \subsetneq v_m$$

among supervertices has length at most n, i.e. $m \leq n$.

Proof. By definition each $v_k \in V^{(k)}$ is a subset of $\mathcal{P}^k(V_0)$, and every element of v_k is itself an element of $\mathcal{P}^{k-1}(V_0)$. Thus a strict inclusion $v_{k-1} \subsetneq v_k$ necessarily increases the "powerset depth" by one. Starting from $v_0 \subseteq V_0$ (depth 0), one can strictly ascend at most to depth n, proving $m \le n$. □

Theorem 5.20 (Degree Propagation Across Levels)**.** *Let* $\mathrm{BnSHT}^{(n)}$ *be connected, and let* $\deg_k(v)$ *denote the degree of a k-supervertex $v \in V^{(k)}$ in the primal graph $G^{(k)}$. Then for each $1 \le k < n$,*

$$\deg_{k+1}(e) \;=\; |\{\, v \in V^{(k)} : v \in e \}| \quad\Longrightarrow\quad \deg_k(v) \;=\; |\{\, e \in E^{(k+1)} : v \in e \}|.$$

In particular, the number of $(k+1)$-superedges incident on a given k-supervertex equals its degree in $G^{(k)}$.

Proof. By construction, in the primal graph $G^{(k)}$ two k-supervertices are adjacent exactly when they share membership in some $(k+1)$-superedge. Thus the number of neighbors of v in $G^{(k)}$, i.e. $\deg_k(v)$, counts exactly those $(k+1)$-superedges to which v belongs, establishing the equality. □

Theorem 5.21 (Duality of Incidence Matrices)**.** *For each level k, let $B^{(k)} \in \{0,1\}^{|V^{(k)}|\times|E^{(k)}|}$ be the incidence matrix of the Bond k-SuperHyperGraph. Then its transpose $\bigl(B^{(k)}\bigr)^{\top}$ is the incidence matrix of the dual hypergraph, and*

$$\operatorname{rank}\bigl(B^{(k)}\bigr) = \operatorname{rank}\bigl(B^{(k)\top}\bigr).$$

Proof. By definition $B^{(k)}_{v,e} = 1$ iff $v \in e$. Transposition thus swaps the roles of supervertices and superedges, yielding the dual. Over any field, a matrix and its transpose have equal rank, giving the asserted equality. □

Theorem 5.22 (Skeleton Associativity)**.** *Forming the k-skeleton of an n-SuperHyperGraph and then the ℓ-skeleton ($\ell < k$) yields the same result as directly forming the ℓ-skeleton. Concretely, for $0 \le \ell < k \le n$,*

$$\mathrm{Skeleton}_\ell\bigl(\mathrm{Skeleton}_k(\mathrm{BnSHT}^{(n)})\bigr) \;=\; \mathrm{Skeleton}_\ell\bigl(\mathrm{BnSHT}^{(n)}\bigr).$$

Proof. Both construction procedures extract supervertices of depth ℓ by successive membership unwinding. Whether one unwinds from n down to k and then to ℓ, or directly from n to ℓ, the resulting collection of ℓ-supervertices (and their induced ℓ-superedges) is identical. This follows from the transitive nature of set membership in iterated powersets. □

6 Conclusion and Future Works

In this paper, we extended the frameworks of *Graph Signal Processing*, *Electric Circuits*, and *Bond Graphs* by incorporating the mathematical structures of *hypergraphs* and *superhypergraphs*. We examined their formal properties and provided illustrative examples to demonstrate their applicability and expressiveness.

In future work, we aim to conduct computational experiments related to these frameworks in order to explore their practical applications in real-world scenarios more concretely. In addition, we plan to investigate theoretical extensions and applications to foundational concepts such as *Ohm's Law* [206, 221], *Kirchhoff's Laws* [172, 178], *AC/DC Analysis* [3, 19], *Transfer Functions* [121, 203], and *Integrated Circuits* [209, 218].

And as a direction for future work, we plan to integrate advanced uncertainty-handling frameworks into the proposed models by incorporating various set-theoretic generalizations, including Fuzzy Sets [227–229], Intuitionistic Fuzzy Sets [22–25], Vague Sets [8, 12, 40, 96], Rough Sets [39, 167–169], HyperRough Sets [71, 77, 78], Bipolar Fuzzy Sets [5, 234, 235], Tripolar Fuzzy Sets [173–175], HyperFuzzy Sets [120, 196], Picture Fuzzy Sets [50, 106], Hesitant Fuzzy Sets [6, 210, 211, 224], spherical fuzzy sets [10, 128], Neutrosophic Sets [116, 189, 195], Quadripartitioned Neutrosophic Sets [122, 226], HyperPlithogenic Sets [75, 76], and Plithogenic Sets [73, 84, 190]. These advanced frameworks are expected to significantly enhance the expressive power and practical applicability of hypergraph-based models, particularly in capturing complex, multi-level, and hierarchical uncertainty across a variety of domains. We also hope to explore possible extensions using structures such as directed graphs [119, 184, 201], bidirected graphs [56, 100, 124], and multidirected graphs [160, 161].

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability

This research is purely theoretical, involving no data collection or analysis. I encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Author's Contribution

The author solely conceived the idea, developed the theoretical framework, performed the mathematical analysis, and wrote the manuscript.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

Competing interests

Author has declared that no competing interests exist.

Consent to Publish declaration

The author approved to Publish declarations.

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