Fermatean Fuzzy Artinian and Noetherian Rings

Abstract

This paper introduces and investigates the concepts of Fermatean fuzzy Artinian and Noetherian rings, extending classical ring-theoretic chain conditions into the framework of Fermatean fuzzy logic. We define Fermatean fuzzy left ideals and characterize the notions of Fermatean fuzzy left Artinian and Noetherian rings via descending and ascending chain conditions, respectively. We establish several structural properties, including the behavior of Fermatean fuzzy level subsets under ring homomorphisms, and the finiteness of the image sets of membership and non-membership degrees in Fermatean fuzzy Artinian and Noetherian rings. We prove that the direct product of Fermatean fuzzy Artinian rings is also Fermatean fuzzy Artinian, and we present a characterization showing that the necessary and sufficient condition for a ring to be Fermatean fuzzy Noetherian is that the value set of every Fermatean fuzzy left ideal is a well-ordered subset of the unit interval. These results contribute to the growing literature on fuzzy algebra and demonstrate the usefulness of Fermatean fuzzy logic in generalizing and enriching classical algebraic concepts.

Keywords: *Fermatean fuzzy sets; Fermatean fuzzy ideals; Artinian rings; Noetherian rings; Fuzzy algebra; Chain conditions; Well-ordered sets; Ring homomorphism; Level subsets*

1 Introduction

Since the introduction of fuzzy sets by Zadeh [17] to model uncertainty and vagueness in human reasoning, fuzzy set theory and its generalizations have developed substantially. In classical fuzzy set theory, each element is assigned a membership degree within the interval [0, 1]. To better capture various forms of uncertainty, several extensions have been proposed, including intuitionistic fuzzy sets (IFS) by Atanassov [5], Pythagorean fuzzy sets (PFS) by Yager [16], and more recently, Fermatean fuzzy sets (FFS) introduced by Senapati and Yager [14]. Fermatean fuzzy sets impose a cubic constraint, requiring that the sum of the cubes of the membership and non-membership degrees does not exceed one. This relaxation better captures hesitation, making FFS particularly suitable for scenarios where the traditional constraints of IFS or PFS prove too restrictive.

Ring theory, a foundational area of abstract algebra, focuses on studying algebraic structures and their ideal-theoretic properties. Two fundamental classes of rings—Artinian and Noetherian—are named after Emil Artin and Emmy Noether, respectively, and are characterized by the termination of descending and ascending chains of ideals. These structures are central to developments in module theory, homological algebra, and algebraic geometry [6, 8, 4].

Fuzzy algebra integrates fuzzy logic into algebraic systems and has emerged as a robust framework for modeling uncertainty within algebraic structures. Early studies by Rosenfeld [12], Liu [10], Mukherjee et al. [11], Rajesh Kumar [9], and Dixit et al. [7] investigated fuzzy subgroups and fuzzy ideals primarily within classical or intuitionistic fuzzy contexts. There are so many works in the field of Fermatean fuzzy sets by various authors [1, 2, 3]. However, introducing Fermatean fuzzy logic offers a promising new dimension, potentially enriching ring-theoretic studies by accommodating higher uncertainty and hesitation.

This paper advances the study of fuzzy algebra by introducing and investigating Fermatean fuzzy Artinian and Noetherian rings. We begin by reviewing the fundamental definitions and properties of Fermatean fuzzy sets and their associated ideals. Building on this foundation, we formulate Fermatean analogues of the classical Artinian and Noetherian conditions and establish several key structural theorems. In particular, we prove that the direct product of Fermatean fuzzy Artinian rings retains the Artinian property and that, under the respective chain conditions, the image set of any Fermatean fuzzy left ideal is necessarily finite. Additionally, we characterize Fermatean fuzzy Noetherian rings through the well-ordered nature of the value sets of their ideals.

2 Preliminaries

This section presents essential preliminaries concerning Fermatean fuzzy sets and Fermatean fuzzy ideals, which are crucial for developing our main results. We review definitions and properties that extend classical algebraic notions into the Fermatean fuzzy setting.

Definition 1. *[14] A Fermatean fuzzy set $\mathfrak{F}$ in a ring $\mathfrak{R}$ is a set of the form*

$$\mathfrak{F} = \{(\alpha, \mu_{\mathfrak{F}}(\alpha), \eta_{\mathfrak{F}}(\alpha)) | \alpha \in \mathfrak{R}\},$$

where $\mu_{\mathfrak{F}}(\alpha) \in [0, 1]$ and $\eta_{\mathfrak{F}}(\alpha) \in [0, 1]$ are membership and non-membership degrees respectively of $\alpha \in \mathfrak{R}$, satisfying the constraints

$$0 \leq \mu^3_{\mathfrak{F}}(\alpha) + \eta^3_{\mathfrak{F}}(\alpha) \leq 1.$$

Definition 2. *[13] A ring $\mathfrak{R}$ is said to be fuzzy left Artinian if every strictly descending chain of fuzzy left ideals of $\mathfrak{R}$ terminates (is finite) after finitely many steps. That is, for any sequence of fuzzy left ideals*

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq ...,$$

there exists a positive integer n such that $I_n = I_{n+1} = I_{n+2} = ...$
Similarly, a fuzzy subset $\mathfrak{F}$ of $\mathfrak{R}$ is said to be fuzzy left Artinian subset if every strictly ascending chain of fuzzy left ideals contained in $\mathfrak{F}$ becomes stationary after finitely many steps.

Definition 3. *[13] A ring $\mathfrak{R}$ is said to be fuzzy left Noetherian if every strictly ascending chain of fuzzy left ideals of $\mathfrak{R}$ terminates (is finite) after finitely many steps. That is, for any sequence of fuzzy left ideals*

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq ...,$$

there exists a positive integer n such that $I_n = I_{n+1} = I_{n+2} = ...$
Similarly, a fuzzy subset $\mathfrak{F}$ of $\mathfrak{R}$ is said to be fuzzy left Noetherian subset if every strictly ascending chain of fuzzy left ideals contained in $\mathfrak{F}$ becomes stationary after finitely many steps.

Definition 4. *A Fermatean fuzzy subset* $\mathfrak{F}$ *of a ring* $\mathfrak{R}$ *is called a Fermatean fuzzy left ideal of* $\mathfrak{R}$ *if for all* $\alpha, \beta \in \mathfrak{R}$*:*

$$\mu_{\mathfrak{F}}(\alpha - \beta) \geq \min\left(\mu_{\mathfrak{F}}(\alpha), \mu_{\mathfrak{F}}(\beta)\right)\ ,\ \eta_{\mathfrak{F}}(\alpha - \beta) \leq \max\left(\eta_{\mathfrak{F}}(\alpha), \eta_{\mathfrak{F}}(\beta)\right)$$

and

$$\mu_{\mathfrak{F}}(\alpha \cdot \beta) \geq \mu_{\mathfrak{F}}(\beta)\ , \eta_{\mathfrak{F}}(\alpha \cdot \beta) \leq \eta_{\mathfrak{F}}(\beta).$$

Definition 5. *A Fermatean fuzzy subset* $\mathfrak{F}$ *of a ring* $\mathfrak{R}$ *is called a Fermatean fuzzy right ideal of* $\mathfrak{R}$ *if for all* $\alpha, \beta \in \mathfrak{R}$*:*

$$\mu_{\mathfrak{F}}(\alpha - \beta) \geq \min\left(\mu_{\mathfrak{F}}(\alpha), \mu_{\mathfrak{F}}(\beta)\right)\ ,\ \eta_{\mathfrak{F}}(\alpha - \beta) \leq \max\left(\eta_{\mathfrak{F}}(\alpha), \eta_{\mathfrak{F}}(\beta)\right)$$

and

$$\mu_{\mathfrak{F}}(\alpha \cdot \beta) \geq \mu_{\mathfrak{F}}(\alpha)\ ,\ \eta_{\mathfrak{F}}(\alpha \cdot \beta) \leq \eta_{\mathfrak{F}}(\alpha).$$

Definition 6. *Let* $\mathfrak{F}$ *be a Fermatean fuzzy set on a universal set* $\mathfrak{R}$*. A level subset* $\mathfrak{F}_{(l,u)}$ *of* $\mathfrak{R}$ *is defined as:*

$$\mathfrak{F}_{(l,u)} = \{\alpha \in \mathfrak{R} : \mu_{\mathfrak{F}}(\alpha) \geq l\ ,\ \eta_{\mathfrak{F}}(\alpha) \leq u\},$$

where $l, u \in [0, 1]$*.*

Definition 7. *Let* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *be two Fermatean fuzzy sets on rings* $\mathfrak{R}$ *and* $\mathfrak{S}$*, respectively and let* f *be a ring homomorphism. Then:*
the image $f(\mathfrak{F}_1)$ *is defined by*

$$f(\mathfrak{F}_1)(\beta) = \left(\sup_{\alpha \in f^{-1}(\beta)} \mu_{\mathfrak{F}}(\alpha), \inf_{\alpha \in f^{-1}(\beta)} \eta_{\mathfrak{F}}(\alpha) \right),\ \textit{if } f^{-1}(\beta) \neq \emptyset,$$

and 0 otherwise.
The preimage $f^{-1}(\mathfrak{F}_2)$ *is defined by*

$$f^{-1}(\mathfrak{F}_2)(\alpha) = \left(\mu_{\mathfrak{F}_2}(f(\alpha)), \eta_{\mathfrak{F}_2}(f(\alpha))\right),\ \textit{if } f^{-1}(\beta) \neq \emptyset,$$

and 0 otherwise. $\forall \alpha \in \mathfrak{R}$*.*

Several researchers have studied decompositions of fuzzy ideals in algebraic structures, such as the work on intuitionistic fuzzy primary ideals in τ-rings by Sharma et al. [15], which forms a conceptual foundation for the current extension into Fermatean fuzzy rings.

3 Fermatean Fuzzy Artinian and Noetherian Rings

This section establishes key results related to Fermatean fuzzy Artinian and Noetherian rings. These results are natural extensions of classical ring theory within the framework of Fermatean fuzzy logic. The following definitions and theorems form the core contributions of this work.

Definition 8. *A ring* $\mathfrak{R}$ *is said to be a Fermatean fuzzy left Artinian ring if every strictly descending chain of Fermatean fuzzy left ideals becomes stationary after finitely many steps. In other words, for any sequence*

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq ...,$$

of Fermatean fuzzy left ideals of $\mathfrak{R}$*, there exists a positive integer* n *such that* $I_n = I_{n+1} = I_{n+2} = ...$

Definition 9. *A ring* $\mathfrak{R}$ *is said to be a Fermatean fuzzy left Noetherian ring if every strictly ascending chain of Fermatean fuzzy left ideals becomes stationary after finitely many steps. In other words, for any sequence*

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq ...,$$

of Fermatean fuzzy left ideals of $\mathfrak{R}$*, there exists a positive integer* n *such that* $I_n = I_{n+1} = I_{n+2} = ...$

Theorem 1. *Let* $f : \mathfrak{R} \to \mathfrak{S}$ *be a ring homomorphism and* $\mathfrak{F}$ *a fermatean fuzzy left ideal of* $\mathfrak{R}$*. Then*

$$f(\mathfrak{F}_{(l,u)}) = (f(\mathfrak{F}))_{(l,u)}.$$

where $\mathfrak{F}_{(l,u)} = \{\mu_{\mathfrak{F}}(\alpha) \geq l\ ,\ \eta(\alpha) \leq u\}$*, and similarly for* $f(\mathfrak{F})_{(l,u)}$*.*

Proof. Let $\beta \in f(\mathfrak{F}_{(l,u)})$. Then there exists an $\alpha \in \mathfrak{F}_{(l,u)} \subseteq \mathfrak{R}$ such that $f(\alpha) = \beta$. Since $\alpha \in \mathfrak{F}_{(l,u)}$, we have

$$\mu_{\mathfrak{F}}(\alpha) \geq l \text{ and } \eta_{\mathfrak{F}}(\alpha) \leq u.$$

By the definition of the image of a Fermatean fuzzy set under a function f, we get

$$\mu_{f(\mathfrak{F})}(\beta) = \sup\{\mu_{\mathfrak{F}}(\alpha)|f(\alpha) = \beta\} \geq \mu_{\mathfrak{F}}(\alpha) \geq l,$$

and

$$\eta_{f(\mathfrak{F})}(\beta) = \inf\{\eta_{\mathfrak{F}}(\alpha)|f(\alpha) = \beta\} \leq \eta_{\mathfrak{F}}(\alpha) \geq u.$$

Therefore, $\beta \in (f(\mathfrak{F}))_{(l,u)}$, which shows that

$$f(\mathfrak{F}_{(l,u)}) \subseteq (f(\mathfrak{F}))_{(l,u)}.$$

Conversely, let $\beta \in (f(\mathfrak{F}))_{(l,u)}$. Then by definition,

$$\mu_{f(\mathfrak{F})}(\beta) \geq l \text{ and } \eta_{f(\mathfrak{F})}(\beta) \leq u.$$

So, there exists some $\alpha \in \mathfrak{R}$ with $f(\alpha) = \beta$ such that

$$\mu_{\mathfrak{F}}(\alpha) \geq l \text{ and } \eta_{\mathfrak{F}}(\alpha) \leq u.$$

That is, $\alpha \in \mathfrak{F}_{(l,u)}$, and hence $\beta = f(\alpha) \in f(\mathfrak{F}_{(l,u)})$,
Therefore,

$$(f(\mathfrak{F}))_{(l,u)} \subseteq f(\mathfrak{F}_{(l,u)}),$$

and combining both inclusions, we conclude that

$$f(\mathfrak{F}_{(l,u)}) = (f(\mathfrak{F}))_{(l,u)}.$$

□

Theorem 2. *Let $\mathfrak{R}$ be a ring with unity. If $\mathfrak{R}$ is a Fermatean fuzzy left Artinian ring, then every Fermatean fuzzy left ideal of $\mathfrak{R}$ has a finite image set. That is, both the membership and non-membership value sets $Im(\mu_{\mathfrak{F}})$ and $Im(\eta_{\mathfrak{F}})$ are finite subsets of the interval* $[0, 1]$.

Proof. Suppose, to the contrary, that the image set of the membership function $\mu_{\mathfrak{F}}$ is infinite. Then there exists an infinite strictly increasing or strictly decreasing sequence

$$a_1, a_2, a_3, ... \in Im(\mu_{\mathfrak{F}}).$$

Case 1: $a_1 < a_2 < a_3 < ...$
Define, for each $t \in \mathbb{N}$, the set

$$\mathfrak{F}_t = \{\alpha \in \mathfrak{R} | \mu_{\mathfrak{F}}(\alpha) \geq a_t\}.$$

Then $\mathfrak{F}_t \subseteq \mathfrak{F}_{t-1}$ for all t, because $a_t > a_{t-1}$. Thus we have a decreasing chain

$$\mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \mathfrak{F}_3 \supset ...,$$

Since $a_{t-1} \in Im(\mu_{\mathfrak{F}})$, there exists $\alpha_{t-1} \in \mathfrak{R}$ such that $\mu_{\mathfrak{F}}(\alpha_{t-1}) = a_{t-1}$. Clearly, $\alpha_{t-1} \in \mathfrak{F}_{t-1}$, but $a_{t-1} \notin \mathfrak{F}_t$, since $\mu_{\mathfrak{F}}(\alpha_{t-1}) < a_t$. Hence each inclusion is strict, and we obtain a strictly descending chain of Fermatean fuzzy left ideals of $\mathfrak{R}$. So this contradicts the assumption of $\mathfrak{R}$ being Fermatean fuzzy left Artinian.
Case 2: $a_1 > a_2 > a_3 > ...$
Define set

$$\mathfrak{F}_t = \{\alpha \in \mathfrak{R} | \mu_{\mathfrak{F}}(\alpha) \geq a_t\}$$

as before. Then

$$\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \mathfrak{F}_3 \subset ...,$$

is a strictly ascending chain of fuzzy left ideals. But, since $\mathfrak{R}$ is Fermatean fuzzy left Artinian (with unity), it must also satisfy the ascending chain condition (Noetherian property), which contradicts this unbounded ascending chain.
Thus, $Im(\mu_{\mathfrak{F}})$ must be finite.
Now suppose $Im(\eta_{\mathfrak{F}})$ is infinite. Then, similarly, we have either a strictly increasing or decreasing sequence

$$b_1, b_2, b_3, ... \in Im(\eta_{\mathfrak{F}}).$$

Subcase 1: $b_1 < b_2 < b_3 < ...$
Define $\mathfrak{F}_k = \{\beta \in \mathfrak{R} | \eta_{\mathfrak{F}}(\beta) \leq b_k\}$. Then

$$\mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \mathfrak{F}_3 \supset ...,$$

is a strictly descending chain, contradicting the Artinian property.
Subcase 2: $b_1 > b_2 > b_3 > ...$
Then

$$\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \mathfrak{F}_3 \subset ...,$$

is a strictly ascending chain, again contradicting the Noetherian nature implied by Artinian rings with unity. Hence, $Im(\eta^3_{\mathfrak{F}})$ must also be finite.
Therefore, both the membership and non-membership image sets of any Fermatean fuzzy left ideal of a Fermatean fuzzy left Artinian ring with unity are finite. □

Theorem 3. *If $\mathfrak{R}$ is a Fermatean fuzzy left Noetherian ring with unity, then every Fermatean fuzzy left ideal $\mathfrak{F}$ of $\mathfrak{R}$ has a finite image set; i.e., both $Im(\mu_{\mathfrak{F}})$ and $Im(\eta_{\mathfrak{F}})$ are finite subsets of* $[0, 1]$.

Proof. Suppose, for contradiction, that there exists a Fermatean fuzzy left ideal $\mathfrak{F}$ of $\mathfrak{R}$ such that $Im(\mu_{\mathfrak{F}})$ is infinite. Then $Im(\mu_{\mathfrak{F}})$ contains a sequence that is either strictly increasing or strictly decreasing.
Case 1: $\mu_{\mathfrak{F}}$ has a strictly decreasing sequence $a_1 > a_2 > a_3 > ...$
Define a sequence of level sets

$$V_n = \{\alpha \in \mathfrak{R} | \mu_{\mathfrak{F}}(\alpha) \geq a_n\}, \ n \in \mathbb{N}.$$

Then $V_1 \subset V_2 \subset V_3 \subset ...$ forms a strictly ascending chain of Fermatean fuzzy left ideals. This contradicts the assumption that $\mathfrak{R}$ is Fermatean fuzzy left Noetherian, which requires all such ascending chains to terminate.
Case 2: $\mu_{\mathfrak{F}}$ has a strictly increasing sequence $a_1 < a_2 < a_3 < ...$
Again, define $V_n = \{\alpha \in \mathfrak{R} | \mu_{\mathfrak{F}}(\alpha) \geq a_n\}$. This time,

$$V_1 \supset V_2 \supset V_3 \supset ...$$

is a strictly descending chain of fuzzy left ideals. While this chain does not directly violate the Noetherian condition, it implies the existence of infinitely many distinct levels of $\mu_{\mathfrak{F}}$, which cannot happen if the chain is to stabilize, a requirement of Noetherian structure in Fermatean fuzzy context with unity. Thus, such infinite variation is not possible.
Now suppose that $\eta_{\mathfrak{F}}$ is infinite. Then there exists either a strictly increasing or strictly decreasing sequence $b_1, b_2, b_3, ...Im(\eta_{\mathfrak{F}})$.
Subcase 1: $b_1 < b_2 < b_3 < ...$
Define

$$W_n = \{\alpha \in \mathfrak{R} | \eta_{\mathfrak{F}}(\alpha) \leq b_n\}.$$

Then $W_1 \supset W_2 \supset W_3 \supset ...$ forms a strictly descending chain of fuzzy left ideals. As in Case 2 above, this would imply an infinite variation in the non-membership degrees, contradicting the stabilization condition implied by the Noetherian property.
Subcase 2: $b_1 > b_2 > b_3 > ...$
Then

$$W_1 \supset W_2 \supset W_3 \supset ...$$

forms a strictly descending chain of fuzzy left ideals, contradicting the assumption that $\mathfrak{R}$ is Fermatean fuzzy left Noetherian.
Hence, both $Im(\mu_{\mathfrak{F}})$ and $Im(\eta_{\mathfrak{F}})$ must be finite. □

Theorem 4. *Let us consider $\mathfrak{R}$ and $\mathfrak{S}$ to be Fermatean fuzzy left Artinian rings with unity. Then the direct product $\mathfrak{R} \times \mathfrak{S}$ is also a Fermatean fuzzy left Artinian ring.*

Proof. Consider a Fermatean fuzzy left ideal $\mathfrak{F}$ of the ring $\mathfrak{R} \times \mathfrak{S}$. We define associated Fermatean fuzzy subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively by:

$$\mu_{\mathfrak{F}_1}(\alpha) = \mu_{\mathfrak{F}}(\alpha, 0), \quad \eta_{\mathfrak{F}_1}(\alpha) = \eta_{\mathfrak{F}}(\alpha, 0), \quad \forall \alpha \in \mathfrak{R},$$

$$\mu_{\mathfrak{F}_2}(\beta) = \mu_{\mathfrak{F}}(0, \beta), \quad \eta_{\mathfrak{F}_2}(\beta) = \eta_{\mathfrak{F}}(0, \beta), \quad \forall \beta \in \mathfrak{S}.$$

We now define the membership and non-membership degrees in the product ring by:

$$\mu_{\mathfrak{F}}((\alpha, \beta)) = \min\{\mu_{\mathfrak{F}_1}(\alpha), \mu_{\mathfrak{F}_2}(\beta)\},$$

$$\eta_{\mathfrak{F}}((\alpha, \beta)) = \max\{\eta_{\mathfrak{F}_1}(\alpha), \eta_{\mathfrak{F}_2}(\beta)\}.$$

Since both $\mathfrak{R}$ and $\mathfrak{S}$ are Fermatean fuzzy left Artinian, their respective image sets $Im(\mu^3_{F_1})$, $Im(\eta^3_{F_1})$, $Im(\mu^3_{F_2})$, and $Im(\eta^3_{F_2})$ are finite subsets of $[0, 1]$.
Now, since the minimum and maximum of two finite sets are also finite, the image sets

$$Im(\mu_{\mathfrak{F}}) = \min\{(a, b) | a \in Im(\mu_{\mathfrak{F}_1}), b \in Im(\mu_{\mathfrak{F}_2})\},$$

$$Im(\eta_{\mathfrak{F}}) = \min\{(a, b) | a \in Im(\eta_{\mathfrak{F}_1}), b \in Im(\eta_{\mathfrak{F}_2})\},$$

are also finite.
Thus, every Fermatean fuzzy left ideal in $\mathfrak{R} \times \mathfrak{S}$ has a finite number of membership and non-membership values. Hence, $\mathfrak{R} \times \mathfrak{S}$ satisfies the descending chain condition on Fermatean fuzzy left ideals.
Therefore, $\mathfrak{R} \times \mathfrak{S}$ is Fermatean fuzzy left Artinian. □

Theorem 5. *A ring $\mathfrak{R}$ is Fermatean fuzzy left Noetherian if and only if the value set of every Fermatean fuzzy left ideal of $\mathfrak{R}$ is a well-ordered subset of the interval $[0, 1]$.*

Proof. If $\mathfrak{R}$ is a Fermatean fuzzy left Noetherian ring, then every Fermatean fuzzy left ideal has a well-ordered value set:
Let us Consider $\mathfrak{F}$ to be a Fermatean fuzzy left ideal of $\mathfrak{R}$. Assume, for contradiction, that $Im(\mu_{\mathfrak{F}})$ or $Im(\eta_{\mathfrak{F}})$ is not well-ordered.
Suppose $Im(\mu_{\mathfrak{F}})$ is not well-ordered. Then there exists a strictly decreasing sequence $a_1 > a_2 > a_3 > ... \in Im(\mu_{\mathfrak{F}})$. Define:

$$V_n = \{\alpha \in \mathfrak{R} | \mu_{\mathfrak{F}}(\alpha) \geq a_n\}.$$

Then:

$$V_1 \subset V_2 \subset V_3 \subset ...$$

is a strictly ascending chain of Fermatean fuzzy left ideals, contradicting the Noetherian property of $\mathfrak{R}$.
A similar contradiction arises if $Im(\eta_{\mathfrak{F}})$ is not well-ordered (using descending chain arguments).
Hence, both $Im(\mu_{\mathfrak{F}})$ and $Im(\eta_{\mathfrak{F}})$ must be well-ordered subsets of $[0, 1]$.
Conversely,
If every Fermatean fuzzy left ideal has a well-ordered value set, then $\mathfrak{R}$ is Fermatean fuzzy left Noetherian.
Suppose, for contradiction, that $\mathfrak{R}$ is not Fermatean fuzzy left Noetherian. Then there exists a strictly ascending chain of left ideals:

$$V_1 \subset V_2 \subset V_3 \subset V_3 ... \quad \text{(non-terminating)}.$$

Define a Fermatean fuzzy set $\mathfrak{F} = \langle \mu_{\mathfrak{F}}, \eta_{\mathfrak{F}} \rangle$ on $\mathfrak{R}$ as follows:

- For each $\alpha \in \cup_{n\in N} V_n$, define:

$$\mu_{\mathfrak{F}}(\alpha) = \frac{1}{m}, \quad \text{where} \quad m = \min\{n \in \mathbb{N} | \alpha \in V_n\},$$

$$\eta_{\mathfrak{F}}(\alpha) = \frac{1}{m+1}, \quad \text{where} \quad m = \max\{n \in \mathbb{N} | \alpha \in V_n\},$$

- For $\alpha \notin \cup_{n\in N}$, define:

$$\mu_{\mathfrak{F}}(\alpha) = 0, \quad \eta_{\mathfrak{F}}(\alpha) = 1.$$

Now, we shall verify that $\mathfrak{F}$ is a Fermatean fuzzy left ideal.
Let $\alpha, \beta \in \mathfrak{R}$, and assume that,

- $\alpha \in V_k \backslash V_{k+1} \Rightarrow \mu_{\mathfrak{F}}(\alpha) = \frac{1}{k}$
- $\beta \in V_m \backslash V_{m+1} \Rightarrow \mu_{\mathfrak{F}}(\beta) = \frac{1}{m}$,
- $\alpha - \beta \in V_{n+1} \Rightarrow \mu_{\mathfrak{F}}(\alpha - \beta) = \frac{1}{n}$.

Assume without loss of generality that $m \leq k$, then:

$$\mu_{\mathfrak{F}}(\alpha) \wedge \mu_{\mathfrak{F}}(\beta) = \min\left(\frac{1}{k}, \frac{1}{m}\right) = \frac{1}{m}.$$

Since $\alpha, \beta \in V_m \Rightarrow \alpha - \beta \in V_n$ for some $n \leq m$, we get:

$$\mu_{\mathfrak{F}}(\alpha - \beta) = \frac{1}{n} \geq \frac{1}{m} = \mu_{\mathfrak{F}}(\alpha) \wedge \mu_{\mathfrak{F}}(\beta).$$

Similarly, for the non-membership part:

- $\eta_{\mathfrak{F}}(\alpha) = \frac{1}{k+1}, \eta_{\mathfrak{F}}(\beta) = \frac{1}{m+1}$,
- $\eta_{\mathfrak{F}}(\alpha - \beta) = \frac{1}{n+1} \leq \max\left(\frac{1}{k+1}, \frac{1}{m+1}\right) = \eta_{\mathfrak{F}}(\alpha) \vee \eta_{\mathfrak{F}}(\beta)$.

Also, for multiplication $\alpha \cdot \beta \in V_n \backslash V_{n+1}$, where $n \leq \min(k, m)$, we get:

$$\mu_{\mathfrak{F}}(\alpha \cdot \beta) = \frac{1}{n} \geq \frac{1}{m+1} = \mu_{\mathfrak{F}}(\beta),$$

$$\eta_{\mathfrak{F}}(\alpha \cdot \beta) = \frac{1}{n+1} \leq \eta_{\mathfrak{F}}(\alpha).$$

hence, all conditions for a Fermatean fuzzy left ideal are satisfied.
Now consider the values of $\mu_{\mathfrak{F}}(\alpha)$ for $\alpha \in V_1, V_2, V_3, ...,$ We have:

$$\mu_{\mathfrak{F}}(\alpha) = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$$

which is a strictly decreasing sequence in $Im(\mu_{\mathfrak{F}})$ and similarly:

$$\eta_{\mathfrak{F}}(\alpha) = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$$

is also strictly decreasing.
This contradicts our assumption that the value sets $Im(\mu_{\mathfrak{F}})$, $Im(\eta_{\mathfrak{F}})$ are well-ordered subsets of $[0, 1]$, which cannot contain infinite strictly decreasing sequences.
Thus, our assumption is false, and the ring $\mathfrak{R}$ must be Fermatean fuzzy left Noetherian. □

4 Conclusion

In this paper, we extended the concepts of Fermatean fuzzy left Artinian and Noetherian rings by employing strictly descending and ascending chains, respectively. It was established that in a Fermatean fuzzy left Artinian or Noetherian ring, every Fermatean fuzzy left ideal possesses a finite image set. Additionally, we showed that ring homomorphisms preserve the level structure of Fermatean fuzzy level subsets. We also established that the direct product of Fermatean fuzzy Artinian rings is itself Fermatean fuzzy Artinian. Finally, we demonstrated that a ring is Fermatean fuzzy Noetherian if and only if all its Fermatean fuzzy left ideals have well-ordered value sets. Future research could explore these notions in the context of modules over Fermatean fuzzy rings or investigate topological structures arising from Fermatean fuzzy subsets.

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Conflicts of Interest

The authors declare that there are no competing interests.

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