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Incomplete metrics on ℝ with usual topology

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Abstract

The set of all real numbers ℝ with the usual metric is complete, and it generates the usual topology on ℝ. Motivated from the fact that incomplete metric on ℝ that produces usual topology on ℝ provides some insight about the relation between topological and metric structure of ℝ, present paper discusses the existence of an incomplete metric on ℝ that generates the usual topology on it. The paper demonstrates that completeness is a metric property, not a topological one. We proved some general results that lead to a method to identify infinitely many incomplete metrics on ℝ. Moreover, the existence of such incomplete metrics on ℝ highlights the presence of metrics on ℝ which are not norm induced.

Keywords: Metric Space; Complete Metric; Incomplete Metric; Topology

2010 Mathematics Subject Classification: 54E35, 54E50, 54A05

1 Introduction

A metric space is complete if every Cauchy sequence in that space converges to a point in the space itself. So ℝ with the usual metric (ie, absolute difference between two real numbers) is complete. Also, the usual metric generates the usual topology on ℝ.

The present paper discusses the existence of an incomplete metric on the set of real numbers ℝ that still induces the usual topology on the space. The paper establishes some general results that pave the way for identifying such incomplete metrics that produce the usual topology on ℝ. Similar investigations have been conducted by other researchers, exploring the interplay between metric completeness and topological properties. (see (Kanovei,V et.al., 2020), (Peng Chen. , 2023), (Borkowski, M. et. al., 2017), (Ishiki, Y. , 2023), (R. LASHKARIPOUR et. al., 2020), (Jelena Vujaković et. al. , 2021), (Kodekalmath et. al., 2022)).

The fundamental definitions and results employed in the paper are derived from the sources (Joshi, 2006), (Satish et.al., 2009), (Qamrul, 2010), (Bartle et.al., 2007), (Limaye, 2008), and (K.C. Rao , 2006).

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2 Preliminaries

The concept of distance allows you to quantify how far apart or dissimilar two elements in a set are. A metric is a way to measure the distance between two elements of a non-empty set.

Definition 2.1. Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a function. Then d is said to be a metric on X if d satisfies the following four conditions.

(i) $d(x, y) \geq 0, \ \forall x, y \in X;$

(ii) $d(x, y) = 0$ if and only if $x = y;$

(iii) $d(x, y) = d(y, x), \ \forall x, y \in X;$

(iv) $d(x, y) \leq d(x, z) + d(z, y), \ \forall x, y, z \in X.$

Then the pair (X, d) is said to be a metric space.

Definition 2.2. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for given $\epsilon > 0$ there exist $M \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq M$.

Definition 2.3. A metric space is said to be complete if every Cauchy sequence in the space converges to a point in the space itself.

Definition 2.4. An open ball with center x and radius r in a metric space (X, d) is defined as $B_r(x) = \{y \in X \ / \ d(x, y) < r\}$.

Definition 2.5. A subset S of a metric space is said to be open if for all $x \in S$, $\exists \ r > 0$ such that $B_r(x) \subset S$.

Definition 2.6. Two metrics d_1 and d_2 on X are equivalent if the open sets in (X, d_1) are the open sets in (X, d_2) and vice versa.

Definition 2.7. A metric d on a set X is said to be bounded if there exists a real number M such that $d(x, y) \leq M$ for all $x, y \in X$.

Remark 2.1. If (X, d) is a metric space and $\rho(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$, then

(i) $\rho(x, y)$ is a bounded metric on X;

(ii) d and ρ are equivalent metrics on X.

Definition 2.8 (Homeomorpism on Metric Space)**.** Let (X, d_1) and (Y, d_2) be metric spaces. A bijection $f : X \to Y$ is said to be a homeomorpism if the functions f and f^{-1} are continuous on (X, d_1) and (Y, d_2), respectively.

Theorem 2.1. *Let d_1 and d_2 be two metrics on X. Then the following statements are equivalent.*

(i) The metrics d_1 and d_2 are equivalent on X;

(ii) The identity map $I : (X, d_1) \to (X, d_2)$ is a homeomorphism;

(iii) A sequence $\{x_n\}$ converges to $x \in X$ with respect to d_1 if and only if $\{x_n\}$ converges to $x \in X$ with respect to d_2.

Definition 2.9. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is said to be an isometry if $d_1(x, y) = d_2(f(x), f(y))$ for all $x, y \in X$.

Let (X, d_1) and (Y, d_2) be metric spaces. If $f : (X, d_1) \to (Y, d_2)$ is an isometry, then f is one-to-one and uniformly continuous.

Remark 2.2. If an isometry is also onto, then the spaces X and Y are said to be isometric.

Let (X, d_1) and (Y, d_2) be metric spaces. If $f : (X, d_1) \to (Y, d_2)$ is an isometry and onto, then $f : (X, d_1) \to (Y, d_2)$ is a homeomorphism.

Definition 2.10. Let X be a set and $\mathcal{T}$ be a set of subsets of X. The pair $(X, \mathcal{T})$ is said to be a topological space if $\mathcal{T}$ satisfying following three conditions:

(i) ϕ, $X \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under arbitrary unions;

(iii) $\mathcal{T}$ is closed under finite intersections.

The set $\mathcal{T}$ is called a topology on X and elements of $\mathcal{T}$ are called open subsets of X(with respect to $\mathcal{T}$).

Definition 2.11. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a subset of X. $\mathcal{B}$ is said to be a base for $\mathcal{T}$ if every member of $\mathcal{T}$ can be written as union of elements of $\mathcal{B}$.

Definition 2.12. Let X be a set. $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X. $\mathcal{T}_2$ is said to be stronger than $\mathcal{T}_1$ if $\mathcal{T}_1$ is a subset of $\mathcal{T}_2$.

Theorem 2.2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Let $\mathcal{B}_2$ be a basis for $\mathcal{T}_2$. $\mathcal{T}_2$ is stronger than $\mathcal{T}_1$ if and only if every element of $\mathcal{T}_1$ contains at least one element of $\mathcal{B}_2$.*

Theorem 2.3. *If (X, d) is a metric space, then the set of all open subsets of X is a topology on X, denoted by τ_d. So every metric space is also a topological space.*

Remark 2.3. Equivalent metrics on a space X induce the same topology on the space.

Definition 2.13 (Homeomorphism on Topological Space)**.** Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two topological spaces. A bijection $f : X \to Y$ is said to be a homeomorphism if the functions f and f^{-1} are continuous on $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$, respectively.

Let (X, d_1) and (Y, d_2) be metric spaces. If $f : (X, d_1) \to (Y, d_2)$ is an isometry and onto, then $f : (X, \tau_{d_1}) \to (Y, \tau_{d_2})$ is a homeomorphism.

3 Main Results

This section establishes the existence of an incomplete metric on $\mathbb{R}$ which induces the usual topology on it. A specific method for constructing infinitely many such incomplete metrics is developed as an outcome of the following discussion.

Theorem 3.1. *If (Y, d) is a metric space and $f : X \to Y$ is a one-to-one function, then $d_f(x, y) = d(f(x), f(y))$ is a metric on X.*

Proof. To prove that d_f is a metric on the set X, we need to verify the four axioms of a metric.

(i) Since d is a metric, $d(x, y) \geq 0$ $\forall x, y \in X$. It follows that,

$$d_f(x, y) = d(f(x), f(y)) \geq 0, \ \forall x, y \in X$$

(ii)

$$\begin{aligned} d_f(x, y) = 0 &\Leftrightarrow d(f(x), f(y)) = 0 \\ &\Leftrightarrow f(x) = f(y) \\ &\Leftrightarrow x = y \end{aligned}$$

(iii)

$$\begin{aligned} d_f(x,y) &= d(f(x),f(y)) \\ &= d(f(y),f(x)) \\ &= d_f(y,x),\ \forall x,y \in X. \end{aligned}$$

(iv)

$$\begin{aligned} d_f(x,y) &= d(f(x),f(y)) \\ &\leq d(f(x),f(z)) + d(f(z),f(y)) \\ &= d_f(x,z) + d_f(z,y) \end{aligned}$$

That is $d_f(x,y) \leq d_f(x,z) + d_f(z,y)$, $\forall x,y,z \in X$. Hence d_f is a metric on X.

□

Theorem 3.2. *Let $(X,\mathcal{T})$ be a topological space and (Y,d) be a metric space. If $f:(X,\mathcal{T}) \to (Y,\tau_d)$ is a homeomorphism, then $(X,\mathcal{T})$ is homeomorphic to (X,τ_{d_f}), where d_f is defined as in Theorem 3.1.*

Proof. Since f is a homeomorphism from $(X,\mathcal{T})$ to (Y,τ_d), f is a bijection from X to Y. Therefore, $f:(X,d_f) \to (Y,d)$ is an isometry by the definition of d_f. From Lemma 2, f is a homeomorphism between (X,τ_{d_f}) and (Y,τ_d). Given that $(X,\mathcal{T})$ is homeomorphic to (Y,τ_d). Hence it follows that $(X,\mathcal{T})$ is homeomorphic to (X,τ_{d_f}). □

Remark 3.1. The homeomorphism between $(X,\mathcal{T}_1)$ and $(X,\mathcal{T}_2)$ may not guarantee that the underlying topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are the same. For example, consider the set of integers $\mathbb{Z}$. Let $\mathcal{T}_1$ be the topology consisting of the empty set, $\mathbb{Z}$ itself, and subsets of $\mathbb{Z}$ of the form $\{k \in \mathbb{Z} : k \leq n,\ n \in \mathbb{Z}\}$. Similarly, let $\mathcal{T}_2$ be another topology consisting of the empty set, $\mathbb{Z}$ itself, and subsets of $\mathbb{Z}$ of the form $\{k \in \mathbb{Z} : k \geq n,\ n \in \mathbb{Z}\}$. Then the map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(x) = -x$ is a homeomorphism from $(\mathbb{Z},\mathcal{T}_1)$ to $(\mathbb{Z},\mathcal{T}_2)$. But they have no non-trivial open sets in common.

In particular, Theorem 3.2 does not guarantee that the topologies $\mathcal{T}$ and τ_{d_f} are equal. But the next theorem shows that they are the same.

Theorem 3.3. *Let $(X,\mathcal{T})$ be a topological space and (Y,d) be a metric space. If $f:(X,\mathcal{T}) \to (Y,\tau_d)$ is a homeomorphism, then the metric d_f on X induces the topology $\mathcal{T}$.*

Proof. Let V be an open set in X and let $x \in V$. Since $f:(X,\mathcal{T}) \to (Y,\tau_d)$ is a homeomorphisum, $f(V)$ is open in Y and $f(x) \in f(V)$. So $f(V)$ contains a base element $B_\epsilon^d(f(x))$. Since f is one-one, $f^{-1}[B_\epsilon^d(f(x))] \subseteq V$.

let $z \in B_\epsilon^{d_f}(x) \Rightarrow d_f(z,x) < \epsilon \Rightarrow d(f(z),f(x)) < \epsilon \Rightarrow f(z) \in B_\epsilon^d(f(x)) \Rightarrow f[B_\epsilon^{d_f}(x)] \subseteq B_\epsilon^d(f(x))$. Conversely, let $y \in B_\epsilon^d(f(x))$. Since f is onto, there exist a $z \in X$ such that $y = f(z)$ and $d(f(z),f(x)) < \epsilon \Rightarrow d_f(z,x) < \epsilon \Rightarrow z \in B_\epsilon^{d_f}(x)$. That is every member of $B_\epsilon^d(f(x))$ is of the form $f(z)$ for some $z \in B_\epsilon^{d_f}(x)$. That means, $B_\epsilon^d(f(x)) \subseteq f[B_\epsilon^{d_f}(x)]$. Therefore $B_\epsilon^d(f(x)) = f[B_\epsilon^{d_f}(x)]$. Considering f^{-1} on both sides, we get, $f^{-1}[B_\epsilon^d(f(x))] = f^{-1}[f[B_\epsilon^{d_f}(x)]]$. But $f^{-1}[f[B_\epsilon^{d_f}(x)]] = B_\epsilon^{d_f}(x)$, since f is one-one. That is $f^{-1}[B_\epsilon^d(f(x))] = B_\epsilon^{d_f}(x)$.

Therefore $B_\epsilon^{d_f}(x) \subseteq V$. By Theorem 2.2 topology induced by d_f is stronger than $\mathcal{T}$. That is $\mathcal{T} \subseteq \tau_{d_f}$. Conversely, consider a base element of τ_{d_f}. Let it be $B_\epsilon^{d_f}(x)$ for some $x \in X$. The open ball $B_\epsilon^d(f(x))$ is open in Y with respect to τ_d. Since f is continuous with respect to $\mathcal{T}$ and τ_d, it follows that $f^{-1}[B_\epsilon^d(f(x))] \in \mathcal{T}$. Already we have shown that $B_\epsilon^{d_f}(x) = f^{-1}[B_\epsilon^d(f(x))]$. Therefore $B_\epsilon^{d_f}(x) \in \mathcal{T}$. This implies that $\mathcal{T}$ is stronger than the topology τ_{d_f}. Therefore $\tau_{d_f} \subseteq \mathcal{T}$. Hence it follows that $\mathcal{T} = \tau_{d_f}$. □

Theorem 3.4. *Let X be a non-empty set. If (Y, d) is a complete metric space and $f : X \to Y$ is a bijection, then (X, d_f) is a complete metric space, where d_f is defined as in Theorem 3.1.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Then for a give $\epsilon > 0$ there exist $M \in \mathbb{N}$ such that $d_f(x_k, x_m) < \epsilon$ for all $k, m \geq M$. By the definition of d_f, $d(f(x_k), f(x_m)) < \epsilon$ for all $k, m \geq M$. Therefore the sequence $\{f(x_n)\}$ is a Cauchy sequence in Y. Since Y is complete, $\{f(x_n)\}$ converges to y for some $y \in Y$. There exist $x \in X$ such that $y = f(x)$, since f is onto. Continuity of f^{-1} implies that the sequence $\{f^{-1}[f(x_n)]\}$ converges to $f^{-1}[f(x)]$. But $f^{-1}[f(x_n)] = x_n$ and $f^{-1}[f(x)] = x$, since f is one-one. Therefore the sequence $\{x_n\}$ converges to $x \in X$. Hence (X, d_f) is complete. $\square$

Example 3.5. *Let $X = (-1, 1)$ and $Y = \mathbb{R}$. Consider $(X, \mathcal{T})$ and (Y, d), where $\mathcal{T}$ is the usual topology and d is the usual metric, on X and Y respectively. Define $f : X \to Y$ as $f(x) = \tan\left(\frac{\pi}{2}x\right)$. Then f is a homeomorphism. By Theorem 3.1,*

$$d_f(x, y) = d\left(\tan\left(\frac{\pi}{2}x\right), \tan\left(\frac{\pi}{2}y\right)\right) = \left|\tan\left(\frac{\pi}{2}x\right) - \tan\left(\frac{\pi}{2}y\right)\right|$$

is a metric on X. By Theorem 3.3, τ_{d_f} is the same as the usual topology. Since (Y, d) is complete, it follows from Theorem 3.4 that (X, d_f) is complete. Hence $(-1, 1)$ is complete with respect to metric $d_f(x, y) = \left|\tan\left(\frac{\pi}{2}x\right) - \tan\left(\frac{\pi}{2}y\right)\right|$ and d_f induces the usual topology on X.

Remark 3.2. As defined in the previous example, there are infinitely many such homeomorphisms from $X = (-1, 1)$ to $Y = \mathbb{R}$; for instance, $kf(x) = k\tan\left(\frac{\pi}{2}x\right)$ is also a homeomorphism from X to Y, where $k > 0$. Then Theorem 3.4 guarantees that there are infinitely many complete metrics on $(-1, 1)$ which generate the usual topology on it.

Theorem 3.6. *Let X be a non-empty set and (Y, d) be a metric space. If $f : X \to Y$ is a bijection and if (X, d_f) is a complete metric space, then (Y, d) is complete.*

Proof. Let $\{y_n\}$ be a Cauchy sequence in Y. Then for a given $\epsilon > 0$ there exist $M \in \mathbb{R}$ such that $d(y_k, y_m) < \epsilon$ for all $k, m \geq M$. Since f is onto, there exist a sequence $\{x_n\}$ in X such that $f(x_n) = y_n$. Therefore $d(f(x_k), f(x_m)) < \epsilon$ for all $k, m \geq M$. It follows from the definition of d_f that $d_f(x_k, x_m) < \epsilon$ for all $k, m \geq M$. This means that the sequence $\{x_n\}$ is a Cauchy sequence in X. Therefore $\{x_n\}$ converges to some $x \in X$. Since f is continuous, the sequence $\{f(x_n)\}$ converges to $f(x) \in Y$. Hence (Y, d) is complete. $\square$

Example 3.7. *Let $X = \mathbb{R}$ and $Y = (-1, 1)$. Consider $(X, \mathcal{T})$ and (Y, d), where $\mathcal{T}$ is the usual topology and d is the usual metric, on X and Y respectively. Define $f : X \to Y$ as $f(x) = \frac{2}{\pi}\tan^{-1} x$. Then f is a homeomorphism. By Theorem 3.1,*

$$d_f(x, y) = d\left(\frac{2}{\pi}\tan^{-1} x - \frac{2}{\pi}\tan^{-1} y\right) = \frac{2}{\pi}|\tan^{-1} x - \tan^{-1} y|$$

is a metric on X. From Theorem 3.6, it follows that, X is incomplete with respect to d_f. Hence $\mathbb{R}$ is incomplete with respect to the metric $d_f = \frac{2}{\pi}|\tan^{-1} x - \tan^{-1} y|$ and it induces the usual topology on $\mathbb{R}$.

Remark 3.3. As defined in the previous example, there are infinitely many such homeomorphisms from $X = \mathbb{R}$ to $Y = (-1, 1)$; for instance, $f(x) = \frac{2}{\pi}\tan^{-1}(kx)$ is also homeomorphism from X to Y, where $k \neq 0$. Then Theorem 3.6 guarantees that there are infinitely many incomplete metrics on $\mathbb{R}$ which generate the usual topology on it.

4 Conclusion

Corresponding to every metric, there exists a bounded metric such that both induce the same topology. This paper guarantees the existence of infinitely many bounded incomplete metrics on $\mathbb{R}$ that induce the usual topology on $\mathbb{R}$. Every norm induced metric ($d(x, y) = ||x - y||$) on $\mathbb{R}$ is complete. The existence of infinitely many incomplete metrics on $\mathbb{R}$ ensures the presence of infinitely many metrics on $\mathbb{R}$ that are not norm induced.

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Competing Interests

Author has declared that no competing interests exist.

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