An Introduction and Reexamination of Hyperprobability and Superhyperprobability: Comprehensive overview

Abstract

Mathematical structures can often be extended into hyperstructures and superhyperstructures by utilizing the powerset and the n-th iterated powerset constructions (cf. [20, 55, 102]). These frameworks are particularly well-suited for modeling hierarchical relationships across a wide range of conceptual domains.

Probability theory traditionally measures the likelihood of an event, assigning a value between 0 and 1 under given conditions. HyperProbability extends this framework by associating each event with a *set* of probability values, thus accommodating uncertainties arising from multiple sources or subjective evaluations. SuperHyperProbability further generalizes this concept through successive applications of the powerset, enabling the formalization of multi-layered uncertainty in complex reasoning systems.

In this paper, we revisit the foundational properties of HyperProbability and SuperHyperProbability, offering numerous illustrative examples. Several key theorems are also established, including those related to cardinality growth and finite additivity. Through these investigations, we aim to support the development of advanced probabilistic modeling and reasoning frameworks capable of addressing hierarchical and multi-level uncertainty.

Keywords: Probability, Hyperprobability, Superhyperprobability, Hyperstructure, SuperHyperstructure

1 Introduction

1.1 Probability Theory and Hyperstructure

Real-world phenomena are often modeled using probability [19,90,91]. Probability measures the likelihood of an event occurring, ranging from 0 (impossible) to 1 (certain), based on mathematical principles. Probability theory serves as a foundational tool in numerous applied fields, including statistics [25, 119], error probability [27, 30, 82], physics [60, 67, 88], queuing system [18, 112], markov chain [57, 72, 109], machine learning [9,65,114], and beyond, with active research being conducted daily. In recent years, extended frameworks such as Fuzzy Probability [70, 84, 126], Neutrosophic Probability [94, 96], and Plithogenic Probability [98] have also garnered increasing attention. Probability has continued to be the subject of extensive research in recent years [24, 87, 108, 120].

Many concepts in the real world exhibit hierarchical structure. To capture such hierarchies, the notions of Hyperstructure and Superhyperstructure have been introduced and studied [39, 104, 105]. Hyperstructure generalizes classical structures using set-valued operations; Superhyperstructure extends this by applying iterated powersets for multi-level relationships. Concrete examples of Superhyperstructures include SuperHyperfuzzy Sets [38, 97], Superhyperfunctions [101, 103], SuperHypergraphs [45, 99], and SuperHyperAlgebras [49, 61, 100]. More recently, probability theory itself has been extended along these lines, leading to the development of HyperProbability and SuperHyperProbability [48].

HyperProbability extends classical probability theory by assigning a *set* of probability values to each event, thereby capturing uncertainty from multiple sources or expert assessments [48]. SuperHyperProbability further generalizes this concept through successive powerset operations, enabling the modeling of multi-layered uncertainty across various levels of reasoning or belief systems [48].

1.2 Our Contribution

The subsection below describes our contributions in this paper. Given the importance of these extensions, and with the goal of raising awareness and uncovering new insights, this paper revisits the fundamental properties of HyperProbability and SuperHyperProbability. We place particular emphasis on providing numerous detailed examples, in the hope that this work will facilitate more widespread modeling of hierarchical concepts in real-world applications.

1.3 Structure of this Paper

This section briefly outlines the structure of the paper. Section 2 presents the definitions of Classical Structure, Hyperstructure, n-Superhyperstructure, and Probability Space. Section 3 discusses the concept of HyperProbability. Section 4 explores n-SuperHyperProbability. Finally, Section 5 provides the Conclusion and Future Works.

2 Preliminaries and Definitions

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. Throughout this paper, we assume that all concepts and sets under consideration are finite.

2.1 Classical Structure, Hyperstructure, and n-Superhyperstructure

A *Classical Structure* represents a general mathematical concept, while a *Hyperstructure* can be defined using the power set, and an *n-Superhyperstructure* can be defined using the n-th powerset [104]. Intuitively, the n-th powerset is a repeated application of the powerset operation. Relevant definitions and simple examples are provided below.

Definition 2.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.2 (Powerset)**.** [36, 89] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.3 (n-th Powerset)**.** (cf. [31, 36, 40, 93, 104])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Example 2.4 (Iterated Powersets for Café Beverage Offerings)**.** Let $H = \{\text{Tea, Coffee}\}$ represent the two beverages available in a café. Then

$$P_1(H) \;=\; P(H) \;=\; \bigl\{\emptyset,\ \{\text{Tea}\},\ \{\text{Coffee}\},\ \{\text{Tea}, \text{Coffee}\}\bigr\}.$$

Interpreting:

- $\emptyset$: no beverage offered,
- $\{\text{Tea}\}$: only tea offered,
- $\{\text{Coffee}\}$: only coffee offered,
- $\{\text{Tea}, \text{Coffee}\}$: both tea and coffee offered.

The second powerset is

$$P_2(H) = P\big(P_1(H)\big) = \big\{X \subseteq P_1(H)\big\},$$

which has $2^4 = 16$ elements. Concretely,

$$\begin{aligned}P_2(H) = \{\, & \emptyset,\ \{\emptyset\},\ \{\{\text{Tea}\}\},\ \{\{\text{Coffee}\}\},\ \{\{\text{Tea},\text{Coffee}\}\},\\ & \{\emptyset,\{\text{Tea}\}\},\ \{\emptyset,\{\text{Coffee}\}\},\ \{\emptyset,\{\text{Tea},\text{Coffee}\}\},\\ & \{\{\text{Tea}\},\{\text{Coffee}\}\},\ \{\{\text{Tea}\},\{\text{Tea},\text{Coffee}\}\},\ \{\{\text{Coffee}\},\{\text{Tea},\text{Coffee}\}\},\\ & \{\emptyset,\{\text{Tea}\},\{\text{Coffee}\}\},\ \{\emptyset,\{\text{Tea}\},\{\text{Tea},\text{Coffee}\}\},\ \{\emptyset,\{\text{Coffee}\},\{\text{Tea},\text{Coffee}\}\},\\ & \{\{\text{Tea}\},\{\text{Coffee}\},\{\text{Tea},\text{Coffee}\}\},\ \{\emptyset,\{\text{Tea}\},\{\text{Coffee}\},\{\text{Tea},\text{Coffee}\}\}\}.\end{aligned}$$

Each element of $P_2(H)$ describes a *daily menu plan* comprising one or more of the possible beverage-offering scenarios from $P_1(H)$.

The third powerset is

$$P_3(H) = P\big(P_2(H)\big),$$

with $2^{16} = 65\,536$ elements, modeling *weekly menu schedules* (each schedule is a set of daily menus). For brevity, we list a few representative elements:

$$\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\{\{\text{Tea}\}\},\{\{\text{Coffee}\}\}\}, \quad P_2(H).$$

- $\emptyset$: no weekly schedule defined.
- $\{\emptyset\}$: every day has "no beverage" on the menu.
- $\{\{\emptyset\}\}$: the weekly plan contains only the "no beverage" daily-menu as a single option.
- $\{\{\{\text{Tea}\}\},\{\{\text{Coffee}\}\}\}$: a weekly plan alternating days offering only tea or only coffee.
- $P_2(H)$: the weekly plan that includes all possible daily menus.

In this way, $P_n(H)$ captures progressively higher "meta" levels of choice: ingredients $\to$ daily menus $\to$ weekly schedules $\to \ldots$.

Definition 2.5 (Classical Structure)**.** (cf. [93,104]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, equipped with one or more *Classical Operations* that satisfy specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is a positive integer, and H^m denotes the m-fold Cartesian product of H. Common examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

Definition 2.6 (Hyperoperation)**.** (cf. [85,115–117]) A *hyperoperation* is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set S, a hyperoperation $\circ$ is defined as:

$$\circ : S \times S \to \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the powerset of S.

Definition 2.7 (Hyperstructure)**.** (cf. [36,93,104]) A *Hyperstructure* extends the notion of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is the powerset of S, and $\circ$ is an operation defined on subsets of $\mathcal{P}(S)$. Hyperstructures allow for generalized operations that can apply to collections of elements rather than single elements.

Definition 2.8 (SuperHyperOperations)**.** (cf. [104]) Let H be a non-empty set, and let $\mathcal{P}(H)$ denote the powerset of H. The n-th powerset $\mathcal{P}^n(H)$ is defined recursively as follows:

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)), \quad \text{for } k \geq 0.$$

A *SuperHyperOperation* of order (m, n) is an m-ary operation:

$$\circ^{(m,n)} : H^m \to \mathcal{P}_*^n(H),$$

where $\mathcal{P}_*^n(H)$ represents the n-th powerset of H, either excluding or including the empty set, depending on the type of operation:

- If the codomain is $\mathcal{P}_*^n(H)$ excluding the empty set, it is called a *classical-type* (m, n)*-SuperHyperOperation*.
- If the codomain is $\mathcal{P}^n(H)$ including the empty set, it is called a *Neutrosophic* (m, n)*-SuperHyperOperation*.

These SuperHyperOperations are higher-order generalizations of hyperoperations, capturing multi-level complexity through the construction of n-th powersets.

Example 2.9 (SuperHyperOperation for Project Phase Grouping)**.** Project management is the application of knowledge, skills, tools, and techniques to plan, execute, and complete projects effectively and efficiently (cf. [33, 34]). A project phase is a distinct stage in a project lifecycle, representing specific objectives, tasks, and deliverables within a defined timeline (cf. [8, 58, 78]). Let

$$H = \{T_1, T_2, T_3\}$$

be the set of three tasks in a small project, and consider the $(3, 2)$-SuperHyperOperation

$$\circ^{(3,2)} : H^3 \longrightarrow \mathcal{P}_*^2(H) = \mathcal{P}\big(\mathcal{P}(H)\big) \setminus \{\emptyset\}.$$

We define $\circ^{(3,2)}$ by having it return the set of all possible "phase groupings" of the three tasks, namely:

$$\begin{aligned}\circ^{(3,2)}(T_1, T_2, T_3) = \Big\{&\{\{T_1\}, \{T_2\}, \{T_3\}\},\ \{\{T_1, T_2\}, \{T_3\}\},\\ &\{\{T_1, T_3\}, \{T_2\}\},\ \{\{T_2, T_3\}, \{T_1\}\},\ \{\{T_1, T_2, T_3\}\}\Big\}.\end{aligned}$$

Here each element of the outer set is a subset of H describing one way to partition the tasks into project phases:

- $\{\{T_1\}, \{T_2\}, \{T_3\}\}$: three separate phases, each containing exactly one task.
- $\{\{T_1, T_2\}, \{T_3\}\}$: one phase with tasks T_1, T_2 followed by a second phase with task T_3.
- $\{\{T_1, T_3\}, \{T_2\}\}$ and $\{\{T_2, T_3\}, \{T_1\}\}$: the analogous two-phase groupings.
- $\{\{T_1, T_2, T_3\}\}$: a single phase containing all tasks.

Thus $\circ^{(3,2)}$ captures all five distinct ways to organize three tasks into phases, demonstrating a concrete, real-life SuperHyperOperation.

Definition 2.10 (n-Superhyperstructure)**.** (cf. [35,37,39,93,104]) An *n-Superhyperstructure* further generalizes a Hyperstructure by incorporating the n-th powerset of a base set. It is formally described as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ represents an operation defined on elements of $\mathcal{P}_n(S)$. This iterative framework allows for increasingly hierarchical and complex representations of relationships within the base set.

Example 2.11 (2-Superhyperstructure in Corporate Department Design)**.** Let

$$S = \{\text{Alice, Bob, Carol}\}$$

be the set of three employees. Then:

$$P_1(S) = P(S) = \big\{\emptyset,\ \{\text{Alice}\},\ \{\text{Bob}\},\ \{\text{Carol}\},\ \{\text{Alice}, \text{Bob}\},$$

$$\{\text{Alice}, \text{Carol}\},\ \{\text{Bob}, \text{Carol}\},\ \{\text{Alice}, \text{Bob}, \text{Carol}\}\big\}.$$

Each element of $P_1(S)$ represents one possible *team*.

The second powerset is

$$P_2(S) = P\big(P_1(S)\big),$$

whose elements are *departmental structures*, i.e. sets of teams. As an example, consider two specific departmental structures:

$$D_1 = \big\{\{\text{ Alice}, \text{Bob}\},\ \{\text{Carol}\}\big\},$$

$$D_2 = \big\{\{\text{Alice}\},\ \{\text{Bob}, \text{Carol}\}\big\}.$$

Here:

- D_1 groups Alice and Bob into one department and Carol into another.
- D_2 places Alice alone and groups Bob with Carol.

Define the 2-superhyperstructure

$$\mathcal{SH}_2 = \big(P_2(S),\ \cup\big),$$

where the operation $\cup : P_2(S) \times P_2(S) \to P_2(S)$ is the union of departmental structures. Then

$$D_1\ \cup\ D_2 = \big\{\{\text{Alice}, \text{Bob}\},\ \{\text{Carol}\},\ \{\text{Alice}\},\ \{\text{Bob}, \text{Carol}\}\big\}\ \in\ P_2(S).$$

This union represents a meta-structure that simultaneously retains both original department configurations, modeling a flexible organizational plan that can switch between the two designs.

In this way, $\mathcal{SH}_2$ captures the hierarchical complexity of team formation (first level) and department design (second level) within a single algebraic framework.

2.2 Probability Space

The definition of a probability space is presented below.

Definition 2.12 (Probability Space)**.** (cf. [17,23,77]) A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ where

- Ω is a nonempty set called the *sample space*;
- $\mathcal{F}$ is a σ-algebra of subsets of Ω, called the *event space*;
- $P : \mathcal{F} \to [0, 1]$ is a function, called a *probability measure*, satisfying:
  1. $P(A) \geq 0$ for all $A \in \mathcal{F}$ (non-negativity),
  2. $P(\Omega) = 1$ (normalization),
  3. For any countable sequence of pairwise disjoint events $A_1, A_2, \dots \in \mathcal{F}$,

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i) \quad \text{(countable additivity).}$$

Example 2.13 (Fair Coin Toss)**.** Let $\Omega = \{\text{H}, \text{T}\}$, $\mathcal{F} = 2^{\Omega}$, and define

$$P(\{\text{H}\}) = P(\{\text{T}\}) = \tfrac{1}{2}.$$

Then $(\Omega, \mathcal{F}, P)$ is a probability space modeling a fair coin toss, since $P(\Omega) = 1$, each event has nonnegative probability, and for the disjoint events $\{\text{H}\}, \{\text{T}\}$,

$$P(\{\text{H}\} \cup \{\text{T}\}) = P(\Omega) = P(\{\text{H}\}) + P(\{\text{T}\}) = 1.$$

3 HyperProbability

HyperProbability assigns a set of probability values to each event, capturing uncertainty from multiple sources. The definitions and examples of HyperProbability are presented below.

Definition 3.1 (Hyper-Probability)**.** [48] Let Ω be a sample space and $\mathcal{F}$ a σ-algebra of events. A *Hyper-Probability* is a function

$$\mathrm{HP} : \mathcal{F} \longrightarrow \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the power set of the unit interval. For each event $A \in \mathcal{F}$,

$$\mathrm{HP}(A) \;=\; \{\, p_k(A) \mid k \in K_A\},$$

with each classical probability $p_k(A) \in [0,1]$. Thus $\mathrm{HP}(A)$ collects multiple probability assessments for the same event.

Example 3.2 (Hyper-Probability in Weather Forecasting — Ensemble Models)**.** Weather forecasting is the scientific process of predicting atmospheric conditions such as temperature, precipitation, and wind for future time periods (cf. [7, 13, 79, 86]). Let the sample space be

$$\Omega = \{\text{rain, no rain}\},$$

and let $\mathcal{F} = 2^{\Omega}$. Suppose three independent forecasting models A, B, and C produce the following probability measures:

$$\begin{aligned} p_A(\{\text{rain}\}) &= 0.65, & p_A(\{\text{no rain}\}) &= 0.35, \\ p_B(\{\text{rain}\}) &= 0.72, & p_B(\{\text{no rain}\}) &= 0.28, \\ p_C(\{\text{rain}\}) &= 0.80, & p_C(\{\text{no rain}\}) &= 0.20. \end{aligned}$$

We summarize these in Table 1.

Model	$p_k(\{\text{rain}\})$	$p_k(\{\text{no rain}\})$
A	0.65	0.35
B	0.72	0.28
C	0.80	0.20

Table 1: Rain-chance forecasts from three meteorological models

The Hyper-Probability measure $\mathrm{HP} : \mathcal{F} \to \mathcal{P}([0,1])$ is then given by

$$\begin{aligned} \mathrm{HP}(\{\text{rain}\}) &= \{\, 0.65,\ 0.72,\ 0.80\}, \\ \mathrm{HP}(\{\text{no rain}\}) &= \{\, 0.35,\ 0.28,\ 0.20\}, \\ \mathrm{HP}(\Omega) &= \{1,\ 1,\ 1\}, \\ \mathrm{HP}(\emptyset) &= \{0,\ 0,\ 0\}. \end{aligned}$$

Here:

- $\mathrm{HP}(\{\text{rain}\})$ aggregates the three "rain" probabilities.
- $\mathrm{HP}(\{\text{no rain}\})$ aggregates the complementary probabilities.
- $\mathrm{HP}(\Omega) = \{p_k(\Omega)\} = \{1, 1, 1\}$ since each model assigns probability 1 to the certain event.
- $\mathrm{HP}(\emptyset) = \{p_k(\emptyset)\} = \{0, 0, 0\}$ since each model assigns probability 0 to the impossible event.

This detailed ensemble example shows how Hyper-Probability captures multiple expert or model-based assessments for each event.

Example 3.3 (Hyper-Probability in Medical Diagnosis)**.** Medical diagnosis is the process of identifying a disease or condition based on a patient's symptoms, history, and diagnostic tests (cf. [2,66,92]). Let the sample space be

$$\Omega = \{\text{Disease}, \text{ No Disease}\},$$

and let $\mathcal{F} = 2^{\Omega}$. Suppose three clinicians A, B, C provide independent probability assessments for the presence of the disease:

$$\begin{aligned} p_A(\{\text{Disease}\}) &= 0.85, & p_A(\{\text{No Disease}\}) &= 0.15, \\ p_B(\{\text{Disease}\}) &= 0.90, & p_B(\{\text{No Disease}\}) &= 0.10, \\ p_C(\{\text{Disease}\}) &= 0.75, & p_C(\{\text{No Disease}\}) &= 0.25. \end{aligned}$$

Clinician	$p_k(\{\text{Disease}\})$	$p_k(\{\text{No Disease}\})$
A	0.85	0.15
B	0.90	0.10
C	0.75	0.25

Table 2: Disease-presence probability estimates from three clinicians

The Hyper-Probability measure $\mathrm{HP} : \mathcal{F} \to \mathcal{P}([0,1])$ is then given by

$$\begin{aligned} \mathrm{HP}(\{\text{Disease}\}) &= \{0.85,\ 0.90,\ 0.75\}, \\ \mathrm{HP}(\{\text{No Disease}\}) &= \{0.15,\ 0.10,\ 0.25\}, \\ \mathrm{HP}(\Omega) &= \{1,\ 1,\ 1\}, \\ \mathrm{HP}(\emptyset) &= \{0,\ 0,\ 0\}. \end{aligned}$$

In this example:

- $\mathrm{HP}(\{\text{Disease}\})$ aggregates the three clinicians' assessments for disease presence.
- $\mathrm{HP}(\{\text{No Disease}\})$ aggregates their complementary assessments.
- $\mathrm{HP}(\Omega) = \{1, 1, 1\}$ since each clinician assigns probability 1 to the certain event.
- $\mathrm{HP}(\emptyset) = \{0, 0, 0\}$ since each clinician assigns probability 0 to the impossible event.

This medical-diagnosis scenario illustrates how Hyper-Probability captures multiple expert judgments for a critical event.

Theorem 3.4 (Null Event)**.** $\mathrm{HP}(\emptyset) = \{0\}$.

Proof. By definition, for every $k \in K$,

$$p_k(\emptyset) = 0.$$

Hence $\mathrm{HP}(\emptyset) = \{\, p_k(\emptyset) \mid k \in K\} = \{0\}$. □

Theorem 3.5 (Certain Event)**.** $\mathrm{HP}(\Omega) = \{1\}$.

Proof. For each $k \in K$, the normalization axiom of probability gives $p_k(\Omega) = 1$. Therefore $\mathrm{HP}(\Omega) = \{\, p_k(\Omega) \mid k \in K\} = \{1\}$. □

Theorem 3.6 (Complement)**.** *For any $A \in \mathcal{F}$,*

$$\mathrm{HP}(A^c) = \{\, 1 - p_k(A) \mid k \in K\}.$$

Proof. Since each p_k is a probability measure,

$$p_k(A^c) = 1 - p_k(A).$$

Taking k over all of K yields $\mathrm{HP}(A^c) = \{ p_k(A^c) \mid k \in K\} = \{ 1 - p_k(A) \mid k \in K\}$. □

Theorem 3.7 (Finite Additivity)**.** *If $A, B \in \mathcal{F}$ are disjoint, then*

$$\mathrm{HP}(A \cup B) = \left\{ p_k(A) + p_k(B) \mid k \in K \right\}.$$

Proof. For each $k \in K$, since $A \cap B = \emptyset$,

$$p_k(A \cup B) = p_k(A) + p_k(B).$$

Thus

$$\mathrm{HP}(A \cup B) = \{ p_k(A \cup B) \mid k \in K\} = \{ p_k(A) + p_k(B) \mid k \in K\}.$$

□

Theorem 3.8 (Monotonicity of Supremum)**.** *If $A, B \in \mathcal{F}$ with $A \subseteq B$, then* $\sup \mathrm{HP}(A) \ \leq \ \sup \mathrm{HP}(B)$.

Proof. For each k, $p_k(A) \leq p_k(B)$ by the monotonicity of probability measures. Taking the supremum over $k \in K$ yields

$$\sup_{k \in K} p_k(A) \ \leq \ \sup_{k \in K} p_k(B),$$

i.e. $\sup \mathrm{HP}(A) \leq \sup \mathrm{HP}(B)$. □

4 n-SuperHyperProbability

n-SuperHyperProbability generalizes HyperProbability by iteratively applying powersets, modeling layered uncertainty across multiple decision or belief levels. The definitions and examples of n-SuperHyperProbability are presented below.

Definition 4.1 (n-SuperHyperProbability)**.** [48] For $n \geq 1$, an *n-SuperHyperProbability* is a function

$$\mathrm{SHP}^{(n)} : \mathcal{F} \longrightarrow P^n([0,1]),$$

where $P^n([0,1])$ denotes the n-fold iterated power set of $[0,1]$. It is defined recursively by

$$\mathrm{SHP}^{(1)}(A) = \mathrm{HP}(A), \qquad \mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big) \quad (n \geq 2).$$

In particular, $\mathrm{SHP}^{(1)}$ coincides with Hyper-Probability and for $n = 0$ one recovers the classical single-valued probability.

Example 4.2 (2-SuperHyperProbability in Corporate Bond Default Risk)**.** Corporate bond default risk is the probability that a company fails to repay its bond obligations, causing losses to investors or lenders (cf. [54, 118, 122]). Let the sample space be

$$\Omega = \{\text{default, no default}\}, \quad \mathcal{F} = 2^{\Omega}.$$

Consider three credit-rating agencies—Moody's, S&P, and Fitch—each providing an independent estimate of the one-year default probability $p_k(A)$ for a particular corporate bond $A = \{\text{default}\}$:

$$p_{\text{Moody}}(A) = 0.08,$$
$$p_{\text{S\&P}}(A) = 0.12,$$
$$p_{\text{Fitch}}(A) = 0.10.$$

These give the Hyper-Probability

$$\mathrm{HP}(A) \ = \ \{ 0.08, \ 0.10, \ 0.12\}.$$

Subset	Interpretation
$\emptyset$	No agency is trusted (vacuous belief)
$\{0.08\}$	Trust only Moody's optimistic estimate
$\{0.10\}$	Trust only Fitch's estimate
$\{0.12\}$	Trust only S&P's pessimistic estimate
$\{0.08, 0.10\}$	Trust Moody's and Fitch, ignore S&P
$\{0.08, 0.12\}$	Trust Moody's and S&P, ignore Fitch
$\{0.10, 0.12\}$	Trust Fitch and S&P, ignore Moody's
$\{0.08, 0.10, 0.12\}$	Trust all three agencies equally

Table 3: All subsets in the 2-SuperHyperProbability for default event A.

Next, the 2-SuperHyperProbability is

$$\mathrm{SHP}^{(2)}(A) = P\big(\{0.08, 0.10, 0.12\}\big) = \big\{\emptyset,\ \{0.08\},\ \{0.10\},\ \{0.12\},$$

$$\{0.08, 0.10\},\ \{0.08, 0.12\},\ \{0.10, 0.12\},\ \{0.08, 0.10, 0.12\}\big\}.$$

For clarity, Table 3 lists these eight subsets and their interpretation in terms of which agency estimates are "trusted."

Discussion.

- The empty set $\emptyset$ reflects maximal caution—no single estimate is deemed reliable.
- Singleton subsets correspond to extreme strategies: e.g. $\{0.12\}$ adopts the highest default estimate for a conservative stance.
- Doubletons allow combining two agencies' views, balancing optimism and pessimism.
- The full set $\{0.08, 0.10, 0.12\}$ represents equal weighting of all three opinions.

This detailed 2-SuperHyperProbability captures not only first-order uncertainty (varying agency estimates) but also second-order choices about which combinations of experts to rely upon in financial decision-making.

Example 4.3 (2-SuperHyperProbability in Election Forecasting)**.** Election forecasting is the use of data, models, and statistical methods to predict electoral outcomes before official voting results are known [62, 68, 69]. Let the sample space be

$$\Omega = \{\text{Win, Lose}\}, \quad \mathcal{F} = 2^{\Omega}.$$

Consider three independent pollsters—Gallup, YouGov, and Ipsos—each providing an estimated probability that Candidate X will win (event $A = \{\text{Win}\}$):

$$p_{\text{Gallup}}(A) = 0.48,$$
$$p_{\text{YouGov}}(A) = 0.52,$$
$$p_{\text{Ipsos}}(A) = 0.50.$$

These yield the Hyper-Probability

$$\mathrm{HP}(A) = \{\, 0.48,\ 0.50,\ 0.52\}.$$

By Definition 4.1, the 2-SuperHyperProbability is

$$\mathrm{SHP}^{(2)}(A) = P\big(\{0.48, 0.50, 0.52\}\big) = \Big\{\emptyset, \{0.48\}, \{0.50\}, \{0.52\}, \{0.48, 0.50\},$$

$$\{0.48, 0.52\}, \{0.50, 0.52\}, \{0.48, 0.50, 0.52\}\Big\}.$$

Interpretation.

Subset	Interpretation
$\emptyset$	Trust no pollster (maximal caution)
$\{0.48\}$	Trust only Gallup's estimate
$\{0.50\}$	Trust only Ipsos's estimate
$\{0.52\}$	Trust only YouGov's estimate
$\{0.48, 0.50\}$	Trust Gallup and Ipsos, ignore YouGov
$\{0.48, 0.52\}$	Trust Gallup and YouGov, ignore Ipsos
$\{0.50, 0.52\}$	Trust Ipsos and YouGov, ignore Gallup
$\{0.48, 0.50, 0.52\}$	Trust all three pollsters equally

Table 4: Subsets in the 2-SuperHyperProbability for Candidate X winning

- The empty set $\emptyset$ reflects highest skepticism—no pollster's data is considered reliable.
- Singleton sets correspond to relying on a single source, e.g. $\{0.52\}$ takes the most optimistic forecast.
- Pairs blend two sources, balancing differing views; for instance, $\{0.48, 0.50\}$ merges Gallup's and Ipsos's mid-range estimates.
- The full set $\{0.48, 0.50, 0.52\}$ treats all pollsters equally, combining every available forecast.

This example demonstrates how 2-SuperHyperProbability captures both first-order uncertainty (variation across pollsters) and second-order decision strategies about which combination of expert opinions to trust in electoral predictions.

Example 4.4 (2-SuperHyperProbability in Credit Card Fraud Detection)**.** Credit card fraud detection identifies unauthorized or suspicious transactions using algorithms, rules, or machine learning to prevent financial loss (cf. [15, 53, 83]). Let

$$\Omega = \{\text{fraud, no fraud}\}, \quad \mathcal{F} = 2^{\Omega}, \quad A = \{\text{fraud}\}.$$

Suppose three independent detection models—Rule-based, Logistic Regression, and Neural Network—estimate the probability of fraud as follows:

$$p_{\text{Rule}}(A) = 0.03, \quad p_{\text{LR}}(A) = 0.07, \quad p_{\text{NN}}(A) = 0.05.$$

Thus the Hyper-Probability is

$$\text{HP}(A) = \{0.03,\ 0.05,\ 0.07\}.$$

The 2-SuperHyperProbability is then

$$\text{SHP}^{(2)}(A) = P\big(\{0.03, 0.05, 0.07\}\big) = \big\{\emptyset,\ \{0.03\},\ \{0.05\},\ \{0.07\},$$

$$\{0.03, 0.05\},\ \{0.03, 0.07\},\ \{0.05, 0.07\},\ \{0.03, 0.05, 0.07\}\big\}.$$

Subset	Interpretation
$\emptyset$	Trust no model (maximum caution)
$\{0.03\}$	Trust only the rule-based model
$\{0.05\}$	Trust only the neural network model
$\{0.07\}$	Trust only the logistic regression model
$\{0.03, 0.05\}$	Trust rule-based and neural network
$\{0.03, 0.07\}$	Trust rule-based and logistic regression
$\{0.05, 0.07\}$	Trust neural network and logistic regression
$\{0.03, 0.05, 0.07\}$	Trust all three models equally

Table 5: 2-SuperHyperProbability subsets for fraud detection event A.

Interpretation.

- The empty set $\emptyset$ indicates no single model is deemed reliable, so manual review is invoked.
- Singleton subsets (e.g. $\{0.07\}$) correspond to relying solely on one model's estimate.
- Doubleton subsets (e.g. $\{0.05, 0.07\}$) blend two models' outputs, requiring agreement between them.
- The full set $\{0.03, 0.05, 0.07\}$ treats all three models equally, aggregating every available forecast.

Example 4.5 (3-SuperHyperProbability in Manufacturing Quality Control)**.** Manufacturing quality control is the process of ensuring products meet specified standards by monitoring, inspecting, and correcting production operations systematically (cf. [52, 113, 121]). Let

$$\Omega = \{\text{fail, ok}\}, \quad \mathcal{F} = 2^{\Omega},$$

and consider the event $A = \{\text{fail}\}$ ("machine breaks down within 24 hours"). Two predictive maintenance algorithms, M_1 and M_2, independently estimate

$$p_1(A) = 0.10, \quad p_2(A) = 0.25.$$

Hence the Hyper-Probability is

$$\text{HP}(A) = \{\, 0.10,\ 0.25\}.$$

By Definition 4.1, the 2-SuperHyperProbability is

$$\text{SHP}^{(2)}(A) = P\big(\{0.10, 0.25\}\big) = \big\{\ \emptyset,\ \{0.10\},\ \{0.25\},\ \{0.10, 0.25\}\big\}.$$

Interpreting, each element of $\text{SHP}^{(2)}(A)$ is a choice of which algorithm(s) to trust.

Finally, the 3-SuperHyperProbability is

$$\text{SHP}^{(3)}(A) = P\big(\text{SHP}^{(2)}(A)\big) = P\big(\{\emptyset, \{0.10\}, \{0.25\}, \{0.10, 0.25\}\}\big),$$

which is the set of all subsets of $\text{SHP}^{(2)}(A)$ (there are $2^4 = 16$ of them). For brevity we list a few representative elements:

$$\begin{aligned}
&\emptyset, \\
&\{\emptyset\}, \quad \{\{0.10\}\}, \quad \{\{0.25\}\}, \quad \{\{0.10, 0.25\}\}, \\
&\{\emptyset, \{0.10\}\}, \quad \{\{0.10\}, \{0.25\}\}, \quad \{\{0.10\}, \{0.10, 0.25\}\}, \\
&\{\emptyset, \{0.10\}, \{0.25\}, \{0.10, 0.25\}\} \quad \text{(the full set)}.
\end{aligned}$$

Interpretation.

- $\emptyset$ — no strategy is trusted (extreme caution).
- $\{\{0.10\}\}$ — trust only algorithm M_1.
- $\{\{0.10\}, \{0.25\}\}$ — trust either algorithm individually, but not their joint estimate.
- $\{\emptyset, \{0.10\}, \{0.25\}, \{0.10, 0.25\}\}$ — allow any of the four first-order choices.

This 3-SuperHyperProbability thus captures not only the original risk estimates and choices about which algorithms to trust (first order) and which subsets of algorithms (second order), but also which combinations of those choices are acceptable (third order), providing a rich framework for decision-making under layered uncertainty.

Example 4.6 (3-SuperHyperProbability in Cybersecurity Intrusion Detection)**.** Cybersecurity intrusion detection is the process of monitoring systems or networks to identify and respond to unauthorized or malicious activity (cf. [5, 10, 26]). Let

$$\Omega = \{\text{Intrusion, No Intrusion}\}, \quad \mathcal{F} = 2^{\Omega}, \quad A = \{\text{Intrusion}\}.$$

Two intrusion detection systems (IDS), Snort and Suricata, independently estimate the probability of an intrusion:

$$p_{\text{Snort}}(A) = 0.30, \quad p_{\text{Suricata}}(A) = 0.45.$$

Hence the Hyper-Probability is

$$\text{HP}(A) = \{0.30,\ 0.45\}.$$

The 2-SuperHyperProbability is

$$\text{SHP}^{(2)}(A) = P\big(\{0.30, 0.45\}\big) = \{\ \emptyset,\ \{0.30\},\ \{0.45\},\ \{0.30, 0.45\}\}.$$

Finally, the 3-SuperHyperProbability is

$$\text{SHP}^{(3)}(A) = P\big(\text{SHP}^{(2)}(A)\big) = P\big(\{\emptyset, \{0.30\}, \{0.45\}, \{0.30, 0.45\}\}\big),$$

which yields $2^4 = 16$ subsets. Concretely,

$$\text{SHP}^{(3)}(A) = \big\{\emptyset,\ \{\emptyset\},\ \{\{0.30\}\},\ \{\{0.45\}\},\ \{\{0.30, 0.45\}\},\ \{\emptyset, \{0.30\}\},\ \{\emptyset, \{0.45\}\},\ \{\emptyset, \{0.30, 0.45\}\},$$
$$\{\{0.30\}, \{0.45\}\},\ \{\{0.30\}, \{0.30, 0.45\}\},\ \{\{0.45\}, \{0.30, 0.45\}\},\ \{\emptyset, \{0.30\}, \{0.45\}\},\ \{\emptyset, \{0.30\}, \{0.30, 0.45\}\},$$
$$\{\emptyset, \{0.45\}, \{0.30, 0.45\}\},\ \{\{0.30\}, \{0.45\}, \{0.30, 0.45\}\},\ \{\emptyset, \{0.30\}, \{0.45\}, \{0.30, 0.45\}\}\big\}.$$

Interpretation.

- $\emptyset$: reject all IDS outputs (maximal caution).
- $\{\{0.30\}\}$: trust only Snort's detection probability.
- $\{\{0.45\}\}$: trust only Suricata's detection probability.
- $\{\{0.30, 0.45\}\}$: require consensus—both IDS agree.
- $\{\emptyset, \{0.45\}, \{0.30, 0.45\}\}$: allow either full rejection, trust Suricata alone, or consensus.
- $\{\emptyset, \{0.30\}, \{0.45\}, \{0.30, 0.45\}\}$: permit any first-order decision strategy.

This 3-SuperHyperProbability captures first-order uncertainty (IDS estimates), second-order choices of which IDS outputs to trust, and third-order combinations of those choices for robust intrusion response.

Example 4.7 (4-SuperHyperProbability in Disaster Response)**.** Disaster response is the coordinated effort to manage and mitigate the impact of natural or man-made disasters on affected populations and infrastructure (cf. [14, 59, 76]). Let

$$\Omega = \{\text{NeedAid, NoAid}\}, \quad \mathcal{F} = 2^{\Omega}, \quad A = \{\text{NeedAid}\}.$$

Two relief organizations—UN and Red Cross—estimate the probability that a region will need urgent aid within 24 hours:

$$p_{\text{UN}}(A) = 0.60, \quad p_{\text{RedCross}}(A) = 0.80.$$

Hence the Hyper-Probability is

$$\text{HP}(A) = \{0.60,\ 0.80\}.$$

By Definition 4.1:

$$\text{SHP}^{(2)}(A) = P\big(\{0.60, 0.80\}\big) = \{\ \emptyset,\ \{0.60\},\ \{0.80\},\ \{0.60, 0.80\}\}.$$

$$\text{SHP}^{(3)}(A) = P\big(\text{SHP}^{(2)}(A)\big),$$

which has $\left|\mathrm{SHP}^{(2)}(A)\right| = 4$ elements and thus $\left|\mathrm{SHP}^{(3)}(A)\right| = 2^4 = 16$. For brevity, one lists just a few of the 16 subsets:

$$\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\{0.60\},\{0.80\}\}, \quad \mathrm{SHP}^{(2)}(A).$$

Finally,

$$\mathrm{SHP}^{(4)}(A) = P\big(\mathrm{SHP}^{(3)}(A)\big),$$

which yields $2^{16} = 65536$ subsets of the 16 third-order sets. Representative 4-th-order elements include:

$$\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\{\{0.60\}\},\{\{0.80\}\}\}, \quad \mathrm{SHP}^{(3)}(A).$$

Interpretation.

- First order (HP): two probability estimates 0.60 and 0.80.
- Second order ($\mathrm{SHP}^{(2)}$): choices of which organization(s) to trust.
- Third order ($\mathrm{SHP}^{(3)}$): combinations of those trust-choices.
- Fourth order ($\mathrm{SHP}^{(4)}$): selections of which third-order strategies to adopt, capturing meta-uncertainty about decision policies.

Theorem 4.8 (Null and Certain Events)**.** *For every $n \geq 1$,*

$$\mathrm{SHP}^{(n)}(\emptyset) = P^n(\{0\}), \qquad \mathrm{SHP}^{(n)}(\Omega) = P^n(\{1\}).$$

In particular, these images depend only on the trivial Hyper-Probability values.

Proof. We argue by induction on n.

Base case $n = 1$. Since each p_k satisfies $p_k(\emptyset) = 0$ and $p_k(\Omega) = 1$,

$$\mathrm{HP}(\emptyset) = \{0\}, \quad \mathrm{HP}(\Omega) = \{1\}.$$

Inductive step. Suppose $\mathrm{SHP}^{(n-1)}(\emptyset) = P^{n-1}(\{0\})$. Then

$$\mathrm{SHP}^{(n)}(\emptyset) = P\big(\mathrm{SHP}^{(n-1)}(\emptyset)\big) = P\big(P^{n-1}(\{0\})\big) = P^n(\{0\}).$$

A similar argument gives $\mathrm{SHP}^{(n)}(\Omega) = P^n(\{1\})$. This completes the induction. □

Theorem 4.9 (Cardinality Growth)**.** *Let $m = \left|\mathrm{HP}(A)\right|$. Then for each $n \geq 1$,*

$$\left|\mathrm{SHP}^{(n)}(A)\right| = 2^{\left|\mathrm{SHP}^{(n-1)}(A)\right|}, \quad \left|\mathrm{SHP}^{(1)}(A)\right| = m.$$

In particular, $\left|\mathrm{SHP}^{(n)}(A)\right|$ is given by n–fold iterated exponentiation of 2 starting from m.

Proof. By definition, $\mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big)$. Since for any finite set X, $\left|P(X)\right| = 2^{|X|}$, it follows that

$$\left|\mathrm{SHP}^{(n)}(A)\right| = \left|P(\mathrm{SHP}^{(n-1)}(A))\right| = 2^{\left|\mathrm{SHP}^{(n-1)}(A)\right|}.$$

The base case $\left|\mathrm{SHP}^{(1)}(A)\right| = |\mathrm{HP}(A)| = m$ is immediate. □

Theorem 4.10 (Monotonicity)**.** *If $A, B \in \mathcal{F}$ satisfy $A \subseteq B$, then for every $n \geq 1$,*

$$\mathrm{SHP}^{(n)}(A) \subseteq \mathrm{SHP}^{(n)}(B).$$

Proof. We prove by induction on n.

Base case $n = 1$**.** If $A \subseteq B$, then for each k, $p_k(A) \leq p_k(B)$. Hence $\mathrm{HP}(A) = \{p_k(A)\} \subseteq \{p_k(B)\} = \mathrm{HP}(B)$.

Inductive step. Assume $\mathrm{SHP}^{(n-1)}(A) \subseteq \mathrm{SHP}^{(n-1)}(B)$. Since the powerset operator P is monotone (i.e. $X \subseteq Y$ implies $P(X) \subseteq P(Y)$), we have

$$\mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big) \;\subseteq\; P\big(\mathrm{SHP}^{(n-1)}(B)\big) = \mathrm{SHP}^{(n)}(B).$$

This completes the induction. □

Theorem 4.11 (Finite Additivity at Order n)**.** *If $A, B \in \mathcal{F}$ are disjoint events, then for every integer $n \geq 1$,*

$$\mathrm{SHP}^{(n)}(A \cup B) = P^n\big(\{\, p_k(A) + p_k(B) \mid k \in K\}\big).$$

Proof. We argue by induction on n.

Base case $n = 1$. By the finite additivity of each p_k,

$$p_k(A \cup B) = p_k(A) + p_k(B), \quad k \in K.$$

Hence $\mathrm{HP}(A \cup B) = \{p_k(A \cup B)\} = \{p_k(A) + p_k(B)\}$, proving the statement for $n = 1$.

Inductive step. Assume the result holds for $n - 1$. Then

$$\mathrm{SHP}^{(n)}(A \cup B) = P\big(\mathrm{SHP}^{(n-1)}(A \cup B)\big) = P\Big(P^{n-1}\big(\{p_k(A) + p_k(B)\}\big)\Big) = P^n\big(\{p_k(A) + p_k(B)\}\big),$$

completing the induction. □

Theorem 4.12 (Existence of Extreme Elements)**.** *For any event A and any integer $n \geq 2$,*

$$\emptyset \;\in\; \mathrm{SHP}^{(n)}(A) \quad \textit{and} \quad \mathrm{SHP}^{(n-1)}(A) \;\in\; \mathrm{SHP}^{(n)}(A).$$

Proof. By definition $\mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big)$. The standard powerset $P\big(\mathrm{SHP}^{(n-1)}(A)\big)$ always contains two distinguished elements:

- The empty set $\emptyset$.
- The universal subset $\mathrm{SHP}^{(n-1)}(A)$ itself.

Thus both claims follow immediately from the construction of the powerset. □

Theorem 4.13 (Complement Mapping)**.** *Let $f : [0,1] \to [0,1]$ be the complement map $f(x) = 1 - x$, and extend f to the n-fold powerset by*

$$P^n(f) : P^n([0,1]) \longrightarrow P^n([0,1]).$$

Then for every event A and every $n \geq 1$,

$$\mathrm{SHP}^{(n)}(A^c) = P^n(f)\big(\mathrm{SHP}^{(n)}(A)\big).$$

Proof. We proceed by induction on n.

Base case $n = 1$. For each $k \in K$,

$$p_k(A^c) = 1 - p_k(A),$$

hence

$$\mathrm{HP}(A^c) = \{\, p_k(A^c)\} = \{\, 1 - p_k(A)\} = f\big(\{p_k(A)\}\big) = P^1(f)\big(\mathrm{HP}(A)\big).$$

Inductive step. Assume $\mathrm{SHP}^{(n-1)}(A^c) = P^{n-1}(f)\big(\mathrm{SHP}^{(n-1)}(A)\big)$. Then

$$\begin{aligned}\mathrm{SHP}^{(n)}(A^c) &= P\big(\mathrm{SHP}^{(n-1)}(A^c)\big)\\ &= P\Big(P^{n-1}(f)\big(\mathrm{SHP}^{(n-1)}(A)\big)\Big)\\ &= P^{n}(f)\big(\mathrm{SHP}^{(n-1)}(A)\big)\\ &= P^{n}(f)\big(\mathrm{SHP}^{(n)}(A)\big),\end{aligned}$$

where the third equality follows from the functoriality of the powerset operator P. This completes the induction. □

5 Conclusion and Future Works

In this paper, we revisited the fundamental properties of HyperProbability and SuperHyperProbability. Specifically, we examined their potential real-world applications and explored several of their mathematical characteristics. The author believes that HyperProbability and SuperHyperProbability can serve as a foundation for developing hierarchical probability theory. Since this study is based solely on theoretical exploration, future work is expected to include computational experiments and further practical investigation.

And as future work, we aim to investigate extensions of HyperProbability and SuperHyperProbability by incorporating advanced uncertainty-handling frameworks such as Fuzzy Sets [28, 29, 124, 125], Intuitionistic Fuzzy Sets [11, 12], Vague Sets [4, 51], Rough Sets [80, 81], Bipolar Fuzzy Sets [3], HyperFuzzy Sets [32, 63, 107], Picture Fuzzy Sets [21, 56], Hesitant Fuzzy Sets [110, 111], Neutrosophic Sets [50, 95, 106], Soft Sets [71, 74], Bipolar Neutrosophic Sets [1, 73], Quadripartitioned Neutrosophic Sets [44, 64, 123], Pentapartitioned Neutrosophic Sets [6, 22], Heptapartitioned Neutrosophic Sets [16, 75], HyperPlithogenic Sets [41–43], and Plithogenic Sets [38, 46, 47]. Such extensions may further enhance their expressiveness and applicability in modeling complex and hierarchical forms of uncertainty.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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