A Study On Gaussian Generalized Edouard Numbers

Abstract. This paper examines the properties and applications of Gaussian Generalized Edouard numbers, aiming to enrich the theoretical framework of number sequences. Utilizing analytical and algebraic techniques, we derive novel recurrence relations, sum formulas, and various representations for these sequences. In particular, we establish Binet's formula, explore generating functions and matrix representations, and present Simson's formula as an alternative approach. Moreover, we analyze two special cases namely, Gaussian Edouard numbers and Gaussian Edouard-Lucas numbers to emphasize their unique structural characteristics. Our results reveal that these sequences possess distinct combinatorial properties that render them applicable in such as, coding theory. Given the broad applicability of number sequences, it is crucial to differentiate this sequence from others based on its unique attributes. Future research will focus on further investigating its structural uniqueness and exploring additional practical implementations.

Keywords: Gaussian Edouard numbers, Gaussian Edouard-Lucas numbers.

1. Introduction

A **second-order recurrence relation** defines the nth term of a sequence in terms of the two preceding terms. In its linear homogeneous form with constant coefficients, it can be expressed as:

$$a_n = r\,a_{n-1} + sa_{n-2}, \quad \text{for } n \geq 2,$$

where r and s are constant coefficients, and a_0 and a_1 are specified initial conditions. The term "second-order" indicates that the recurrence involves the two immediately preceding terms.

To solve such a recurrence relation, one typically considers the **characteristic equation**:

$$x^2 - rx - s = 0.$$

The nature of the roots of this quadratic equation (whether distinct, repeated, or complex conjugate pairs) determines the form of the general solution. A classic example is the Fibonacci sequence, where $r = 1$ and $s = 1$.

A **third-order recurrence relation** extends the idea by defining the nth term based on the three preceding terms. It is generally written as:

$$a_n = ra_{n-1} + s\,a_{n-2} + t\,a_{n-3}, \quad \text{for } n \geq 3,$$

with constant coefficients r, s, and t, along with initial conditions a_0, a_1, and a_2.

The corresponding **characteristic equation** for a third-order recurrence relation is a cubic equation:

$$x^3 - rx^2 - s\,x - t = 0.$$

The solutions to this cubic equation form the basis for the general solution of the recurrence. As with the second-order case, the nature of the roots (real or complex, distinct or repeated) governs the structure of the solution. Due to their increased complexity, third-order recurrences often appear in more intricate applications and exhibit a broader range of behaviors.

Both second-order and third-order recurrence relations play a fundamental role in various areas of mathematics including combinatorics, computer science, and mathematical modeling, where they describe dynamic processes evolving based on previous states.

In this section, firstly, we give some preliminary result on Edouard numbers as having third order recurrence relation.

The generalized Edouard sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation as

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 7W_{-(n-1)} - 7W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

Characteristic equation of $\{W_n\}$ is

$$z^3 - 7z^2 + 7z - 1 = (z^2 - 6z + 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2}, \\ \beta &= 3 - 2\sqrt{2}, \\ \gamma &= 1. \end{aligned}$$

Note that

$$\begin{aligned}\alpha+\beta+\gamma+\delta &= 7,\\ \alpha\beta+\alpha\gamma+\beta\delta &= 7,\\ \alpha\beta\gamma &= 1.\end{aligned}$$

For $n = 1, 2, 3, ...$ Hence, recurrence (1.1) is true for all integer n. Soykan has conducted a study on this particular generalized Edouard sequence and its three spacial cases, for more details, see [20]

Third order reccurance relations has been studied by many authors, for more detail see [2,3,5,6, 7,12,13,15,16,18,19,24,28,29].

Next, we present Binet's formula for the generalized Edouard numbers a derivation that yields an explicit closed-form expression encapsulating the sequence's inherent recurrence structure, thereby offering a powerful analytical tool for both rigorous theoretical exploration and efficient computational implementation.

THEOREM 1.1. *[20] Binet formula of generalized Edouard numbers can be presented as follows:*

$$W_n = (\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^n}{(\alpha-\beta)(\alpha-1)}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^n}{(\beta-\alpha)(\beta-1)}-\frac{(W_2-6W_1+W_0)}{4}).$$

Now we define two particular cases of the sequence $\{W_n\}$ as follows: the edouard sequence $\{E_n\}_{n\geq 0}$, the Edouard-Lucas sequence $\{K_n\}_{n\geq 0}$, respectively, by the third-order recurrence relations,

$$E_n = 7E_{n-1}-7E_{n-2}+E_{n-3}, \qquad E_0=0, E_1=1, E_2=7, \tag{1.2}$$

$$K_n = 7K_{n-1}-7K_{n-2}+K_{n-3}, \qquad K_0=3, K_1=7, K_2=35. \tag{1.3}$$

The sequences $\{E_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ can be extended to negative subscriptys by defining,

$$\begin{aligned}E_{-n} &= 7E_{-(n-1)}-7E_{-(n-2)}+E_{-(n-3)},\\ K_{-n} &= 7K_{-(n-1)}-7K_{-(n-2)}+K_{-(n-3)},\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.2)-(1.3) hold for all integer n.

For all integers n, Edouard and Edouard-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned}E_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4},\\ K_n &= \alpha^n+\beta^n+1.\end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

LEMMA 1.2. *[20]Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Edouard sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-7W_0)x+(W_2-7W_1+7W_0)x^2}{1-7x+7x^2-x^3}.$$

The previous lemma gives the following results as particular examples.

COROLLARY 1.3. *[20]Generated functions of Edouard and Edouard-Lucas numbers are*

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x}{1-7x+7x^2-x^3},$$
$$\sum_{n=0}^{\infty} K_n x^n = \frac{3-14x+7x^2}{1-7x+7x^2-x^3},$$

respectively.

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

LEMMA 1.4. *Suppose that* $f_{GW_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Edouard sequence* $\{W_n\}$. *Then*

$$\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} = \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)} e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)} e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4} e^{x}.$$

Proof: Using the Binet's formula of generating Edouard numbers we get

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^n}{(\alpha-\beta)(\alpha-1)} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^n}{(\beta-\alpha)(\beta-1)} - \frac{(W_2-6W_1+W_0)}{4} \right) \frac{x^n}{n!} \\
&= \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)} \sum_{n=0}^{\infty} \alpha^n \frac{x^n}{n!} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)} \sum_{n=0}^{\infty} \beta^n \frac{x^n}{n!} \\
&\quad - \frac{(W_2-6W_1+W_0)}{4} \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)} e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)} e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4} e^{x}. \ \square
\end{aligned}$$

It should be noted that Lemma 1.4 is not included in [20], and therefore, to ensure our exposition is both complete and self-contained, we have taken the initiative to present the lemma here along with its full proof, thereby elucidating its underlying rationale and reinforcing the theoretical framework within which our subsequent results are developed.

The previous Lemma gives the following results as particular examples.

COROLLARY 1.5. *Exponential generating function of Edouard and Edouard-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4} \right) \frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4} e^{x}.$$

b):

$$\sum_{n=0}^{\infty} K_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n + 1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{x}.$$

Note that Gaussian numbers, more commonly known as **Gaussian integers**, constitute a particularly significant subset of the complex numbers an extensive number system that is typically represented in the form

$$a + bi,$$

where a and b denote arbitrary real numbers and i is the imaginary unit satisfying $i^2 = -1$; thereby providing a fundamental framework for understanding both the real and imaginary components in various mathematical and engineering contexts. In contrast, Gaussian integers are distinguished by the additional constraint that both a and b must be integers, meaning that any complex number z, expressed as

$$z = a + bi,$$

qualifies as a Gaussian integer only when its real and imaginary parts are drawn exclusively from the set of integers. This strict criterion endows these numbers with unique algebraic and arithmetic properties that are integral to numerous applications in number theory, including the study of factorization, Diophantine equations, and the distribution of prime numbers within the complex plane.

Next, we offer a short overview of the literature on Gaussian sequences by formally defining Gaussian numbers via specific second-order and third-order recurrence relations, deliberately omitting further details regarding their derivations or structural intricacies. To begin, we present Gaussian numbers defined by second-order recurrence relations.

- Horadam [11] introduced Gaussian Fibonacci numbers and defined by

$$GF_n = F_n + iF_{n-1}$$

  where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact, he defined these numbers as $GF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.).
- Pethe and Horadam [14] introduced Gaussian generalized Fibonacci numbers by

$$GF_n = F_n + iF_{n-1},$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Halıcı and Öz [10] studied Gaussian Pell and Pell Lucas numbers by written , respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1} \end{aligned}$$

where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

- Aşcı and Gürel [1] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

$$\begin{aligned} GJ_n &= J_n + iJ_{n-1}, \\ Gj_n &= j_n + ij_{n-1} \end{aligned}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

- Taşçı [25] introduced and studied Gaussian Mersenne numbers defined by

$$GM_n = M_n + iM_{n-1}$$

where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

- Taşçı [26] introduced and studied Gaussian balancing and Gaussian Lucas Balancing numbers given by, respectively,

$$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + iC_{n-1} \end{aligned}$$

where $B_n = 6B_{n-1} - BJ_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6Cj_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

- Ertaş and Yılmaz [8] studied Gaussian Oresme numbers and defined them as

$$GS_n = S_n + iS_{n-1}$$

where oresme numbers are given by $S_n = S_{n-1} - \frac{1}{4}S_{n-2}$, $S_0 = 0$, $S_1 = \frac{1}{2}$.

Now, we present some gaussian numbers with third order recurrence relations.

- Soykan, et al. [22] presented Gaussian generalized Tribonacci numbers given by

$$GW_n = W_n + iW_{n-1}$$

where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition W_0, W_1, W_2.

- Taşcı [27] studied Gaussian Padovan and Gaussian Pell- Padovan numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1} \\ GR_n &= R_n + iR_{n-1} \end{aligned}$$

where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

- Cerda-Morales [4] defined Gaussian third-order Jacobsthal numbers as

$$GJ_n = J_n + iJ_{n-1}$$

where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_2 = 1$.

2. Gaussian Generalized Edouard Numbers

In this section, we define Gaussian generalized Edouard numbers and present several of their fundamental properties most notably, Binet's formula, which provides an explicit closed form expression for the sequence, and the generating function, which encapsulates the entire sequence within a single algebraic formula thereby offering key insights into their underlying recursive structure and laying the groundwork for further theoretical and computational exploration of these intriguing numbers.

Gaussian generalized Edouard numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 7GW_{n-1} - 7GW_{n-2} + GW_{n-3}, \tag{2.1}$$

with the initial conditions

$$\begin{aligned} GW_0 &= W_0 + i(7W_0 - 7W_1 + W_2), \\ GW_1 &= W_1 + iW_0, \\ GW_2 &= W_2 + iW_1, \end{aligned}$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = 7GW_{-(n-1)} - 7GW_{-(n-2)} + GW_{-(n-3)} \tag{2.2}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (2.1) holds for all integer n. Note that for all integers n, we get,

$$\begin{aligned} GW_n &= W_n + iW_{n-1}, \qquad (2.3) \\ GW_{-n} &= W_{-n} + iW_{-n-1}. \end{aligned}$$

In the following table, we present the first few generalized Gaussian Edouard numbers, including those with both positive and negative subscripts, thereby providing a clear illustration of how the sequence behaves across its entire domain and highlighting the natural extension of the indexing scheme to encompass both the standard positive indices and their negative counterparts.

Table 1. The first few generalized Gaussian Edouard numbers.

n	GW_n
0	$W_0 + i(7W_0 - 7W_1 + W_2)$
1	$W_1 + iW_0$
2	$W_2 + iW_1$
3	$W_0 - 7W_1 + 7W_2 + iW_2$
4	$7W_0 - 48W_1 + 42W_2 + i(W_0 - 7W_1 + 7W_2)$
5	$42W_0 - 287W_1 + 246W_2 + i(7W_0 - 48W_1 + 42W_2)$
6	$246W_0 - 1680W_1 + 1435W_2 + i(42W_0 - 287W_1 + 246W_2)$
7	$1435W_0 - 9799W_1 + 8365W_2 + i(246W_0 - 1680W_1 + 1435W_2)$
8	$8365W_0 - 57120W_1 + 48756W_2 + i(1435W_0 - 9799W_1 + 8365W_2)$

and with negative subscript

n	GW_{-n}
0	$W_0 + i(7W_0 - 7W_1 + W_2)$
1	$7W_0 - 7W_1 + W_2 + i(42W_0 - 48W_1 + 7W_2)$
2	$42W_0 - 48W_1 + 7W_2 + i(246W_0 - 287W_1 + 42W_2)$
3	$(246W_0 - 287W_1 + 42W_2) + i(1435W_0 - 1680W_1 + 246W_2)$
4	$(1435W_0 - 1680W_1 + 246W_2) + i(8365W_0 - 9799W_1 + 1435W_2)$
5	$(8365W_0 - 9799W_1 + 1435W_2) + i(48756W_0 - 57120W_1 + 8365W_2)$
6	$(48756W_0 - 57120W_1 + 8365W_2) + i(284172W_0 - 332927W_1 + 48756W_2)$
7	$(284172W_0 - 332927W_1 + 48756W_2) + i(1656277W_0 - 1940448W_1 + 284172W_2)$
8	$(1656277W_0 - 1940448W_1 + 284172W_2) + i(9653491W_0 - 11309767W_1 + 1656277W_2)$

Gaussian Edouard numbers, $GW_n : GW_n(0, 1, 7 + i) = GE_n$, are defined by

$$GE_n = 7GE_{n-1} - 7GE_{n-2} + GE_{n-3} \tag{2.4}$$

with the initial conditions

$$GE_0 = 0, GE_1 = 1, GE_2 = 7 + i.$$

Gaussian Edouard-Lucas numbers, $GW_n(3 + 7i, 7 + 3i, 35 + 7i) = GK_n$, are defined by

$$GK_n = 7GK_{n-1} - 7GK_{n-2} + GK_{n-3} \tag{2.5}$$

with the initial conditions

$$GK_0 = 3 + 7i, GK_1 = 7 + 3i, GK_2 = 35 + 7i.$$

Note that for all integers n, we have

$$\begin{aligned} GE_n &= E_n + iE_{n-1}, \\ GK_n &= K_n + iK_{n-1}. \end{aligned}$$

In Table 2, we present the first few values of both Gaussian Edouard numbers and Gaussian Edouard-Lucas numbers with indices spanning both positive and negative subscripts to provide a comprehensive illustration of how these sequences behave across their entire domain.

Table 2. Special cases of Gaussian generalized Edouard numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6
GE_n	0	1	$7+i$	$42+7i$	$246+42i$	$1435+246i$	$8365+1435i$
GE_{-n}		i	$1+7i$	$7+42i$	$42+246i$	$246+1435i$	$1435+8365i$
GK_n	$3+7i$	$7+3i$	$35+7i$	$199+35i$	$1155+199i$	$6727+1155i$	$39203+6727i$
GK_{-n}		$7+35i$	$35+199i$	$199+1155i$	$1155+6727i$	$6727+39203i$	$39203+228487i$

Next, we present the Binet's formula for the Gaussian generalized Edouard numbersa concise closed form expression derived from solving the characteristic equation associated with their recurrence relation, which not only encapsulates the inherent recursive structure of these numbers but also provides valuable insights into their algebraic properties, thereby serving as a powerful analytical tool for further theoretical exploration and computational applications.

Theorem 2.1. *The Binet's formula for the Gaussian generalized Edouard numbers is*

$$\begin{aligned} GW_n &= \left(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^n}{(\alpha-\beta)(\alpha-1)} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^n}{(\beta-\alpha)(\beta-1)} - \frac{(W_2-6W_1+W_0)}{4}\right) \\ &\quad + i\left(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^{(n-1)}}{(\alpha-\beta)(\alpha-1)} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^{(n-1)}}{(\beta-\alpha)(\beta-1)} - \frac{(W_2-6W_1+W_0)}{4}\right). \end{aligned}$$

Proof. The proof follows from 1.1 and 2.3. □

In light of the preceding theorem which establishes a comprehensive framework for analyzing the intrinsic properties of these sequences the ensuing results emerge naturally as special cases, thereby illustrating not only the far reaching generality of the theorem but also its capacity to encapsulate specific, elegantly distilled instances as direct corollaries of the broader theoretical framework.

Corollary 2.2. *For all integers n, we have following identities,*

(a): $GE_n = (\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}) + i(\frac{\alpha^n}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^n}{(\beta-\alpha)(\beta-1)} - \frac{1}{4})$.
(b): $GK_n = (\alpha^n + \beta^n + 1) + i(\alpha^{n-1} + \beta^{n-1} + 1)$.

Building on the preceding discussion and analytical developments, the following theorem introduces the generating function of the Gaussian generalized Edouard numbers a single, unified algebraic expression that encapsulates the entire infinite sequence, thereby revealing the intricate interplay between the sequence's recursive properties and its combinatorial structure while also furnishing a powerful tool for further theoretical exploration and practical computational analysis.

Theorem 2.3. *Let $f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n$ denote the generating function of Gaussian generalized Edouard numbers. Then,*

$$f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0 + (GW_1 - 7GW_0)x + (GW_2 - 7GW_1 + 7GW_0)x^2}{1 - 7x + 7x^2 - x^3}. \tag{2.6}$$

Proof. Using the definition of Gaussian Edouard numbers, and substracting $7xf(x)$, $-7x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain

$$\begin{aligned}
(1-7x+7x^2-x^3)f_{GW_n}(x) &= \sum_{n=0}^{\infty} GW_n x^n - 7x\sum_{n=0}^{\infty} GW_n x^n + 7x^2\sum_{n=0}^{\infty} GW_n x^n - x^3\sum_{n=0}^{\infty} GW_n x^n, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 7\sum_{n=0}^{\infty} GW_n x^{n+1} + 7\sum_{n=0}^{\infty} GW_n x^{n+2} - \sum_{n=0}^{\infty} GW_n x^{n+3}, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 7\sum_{n=1}^{\infty} GW_{n-1} x^n + 7\sum_{n=2}^{\infty} GW_{n-2} x^n - \sum_{n=3}^{\infty} GW_{n-3} x^n, \\
&= (GW_0 + GW_1x + GW_2x^2) - 7(GW_0x + GW_1x^2) + 7GW_0x^2 \\
&\quad + \sum_{n=3}^{\infty}(GW_n - 7GW_{n-1} + 7GW_{n-2} - GW_{n-3})x^n, \\
&= GW_0 + GW_1x + GW_2x^2 - 7GW_0x - 7GW_1x^2 + GW_0x^2, \\
&= GW_0 + (GW_1 - 7GW_0)x + (GW_2 - 7GW_1 + 7GW_0)x^2,
\end{aligned}$$

and rearranging above equation, we get 2.6. □

Theorem 2.3 gives the following results as special cases,

$$f_{GE_n}(x) = \frac{x + ix^2}{1-7x+7x^2-x^3}, \quad f_{GK_n}(x) = \frac{(7+3i)\,x^2 - (14+42i)\,x + 3 + 7i}{1-7x+7x^2-x^3}.$$

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ of the sequence GW_n.

Lemma 2.4. *Suppose that* $f_{GGW_n}(x) = \sum_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ *is the exponential generating function of the Gaussian generalized Edouard sequence* $\{GW_n\}$.

Then $\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!} &= \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)}e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)}e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4}e^{x} \\
&\quad + i\left(\frac{(W_2-(\beta+1)W_1+\beta W_0)}{\alpha(\alpha-\beta)(\alpha-1)}e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{\beta(\beta-\alpha)(\beta-1)}e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4}e^{x}\right)
\end{aligned}$$

Proof: Using the Binet's formula of Gaussian generalized Edouard numbers or exponential generating function of the generalized Edouard sequence we get

$$\begin{aligned}
\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty}(W_n + iW_{n-1})\frac{x^n}{n!} = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} + i\sum_{n=0}^{\infty} W_{n-1}\frac{x^n}{n!} \\
&= \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)}e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)}e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4}e^{x} \\
&\quad + i\left(\frac{(W_2-(\beta+1)W_1+\beta W_0)}{\alpha(\alpha-\beta)(\alpha-1)}e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{\beta(\beta-\alpha)(\beta-1)}e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4}e^{x}\right)
\end{aligned}$$

The previous Lemma gives the following results as particular examples.

COROLLARY 2.5. *Exponential generating function of Gaussian Edouard and Gaussian Edouard-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} GE_n \frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}e^x + i(\frac{e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}e^x).$$

b):

$$\sum_{n=0}^{\infty} GK_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^x + i(\frac{e^{\alpha x}}{\alpha} + \frac{e^{\beta x}}{\beta} + e^x).$$

3. Some Identities About Recurrence Relations of Gaussian Generalized Edouard Numbers

In this section, we present a comprehensive collection of identities pertaining to both Gaussian Edouard numbers and Gaussian Edouard-Lucas numbers, each of which is derived from their inherent recursive definitions and generating functions, thereby illuminating the deep algebraic and combinatorial structures underlying these sequences and offering valuable insights for further theoretical investigation and practical applications.

THEOREM 3.1. *The following equations hold for all integer n*

$$GE_n = \frac{7}{64}GK_{n+3} - \frac{11}{16}GK_{n+2} + \frac{21}{64}GK_{n+1}, \tag{3.1}$$

$$GK_n = 35GE_{n+3} - 238GE_{n+2} + 199GE_{n+1}. \tag{3.2}$$

Proof. To proof identity (3.1), we can write $GE_n = aGK_{n+3} + bGK_{n+2} + cGK_{n+1}$ and solve the system of equations we get,

$$\begin{aligned} GE_0 &= aGK_3 + bGK_2 + cGK_1, \\ GE_1 &= aGK_4 + bGK_3 + cGK_2, \\ GE_2 &= aGK_5 + bGK_4 + cGK_3. \end{aligned}$$

Then, we obtain $a = \frac{7}{64}, b = -\frac{11}{16}, c = \frac{21}{64}$. The other identities can be found similarly. □

We can write $GE_n = aGK_{n+3} + bGK_{n+2} + cGK_{n+1}$ and solve the system of equations we get,

$$\begin{aligned} GK_0 &= aGE_3 + bGE_2 + cGE_1, \\ GK_1 &= aGE_4 + bGE_3 + cGE_2, \\ GK_2 &= aGE_5 + bGE_4 + cGE_3. \end{aligned}$$

Then, we obtain $a = 35, b = -238, c = 199$. □

LEMMA 3.2. *([9]) We assume that* $f(x)=\sum\limits_{n=0}^{\infty}a_nx^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x)=\sum_{n=0}^{\infty}a_{2n}x^n=\frac{f(\sqrt{x})+f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x)=\sum_{n=0}^{\infty}a_{2n+1}x^n=\frac{f(\sqrt{x})-f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

In the following theorem, we provide the generating functions corresponding to the even and odd indexed generalized Edouard sequences, delivering explicit closed form expressions that elegantly encapsulate the contributions from both subsequences, thereby illuminating the intricate interplay between their recursive definitions and combinatorial structures, and offering a powerful analytical tool for further theoretical exploration and practical applications.

THEOREM 3.3. *The generating functions of the sequence* GW_{2n} *and* GW_{2n+1} *are provided by*

$$f_{GW_{2n}}(x)=\frac{GW_0+(GW_2-35GW_0)x+(42GW_0-48GW_1+7GW_2)x^2}{1-35x+35x^2-x^3}. \tag{3.3}$$

$$f_{GW_{2n+1}}(x)=\frac{GW_1+(GW_0-42GW_1+7GW_2)x+(7GW_0-7GW_1+GW_2)x^2}{1-35x+35x^2-x^3} \tag{3.4}$$

Proof. We only proof 3.3. From Theorem 2.3 we can obtain following identities:

$$\begin{aligned}
f_{GW_n}(\sqrt{x}) &= \frac{GW_0-\sqrt{x}\,(GW_1-7GW_0)+x\,(GW_2-7GW_1+7GW_0)}{7x-7\sqrt{x}-\sqrt{x^3}+1}.\\
f_{GW_n}(-\sqrt{x}) &= -\frac{GW_0+\sqrt{x}\,(GW_1-7GW_0)+x\,(GW_2-7GW_1+7GW_0)}{7\sqrt{x}+7x+\sqrt{x^3}-1}.
\end{aligned}$$

Thereby, using Lemma 3.2 identity 3.3 can be proved . The other identity can be found similarly. □

From Theorem 3.3, we get the following corollary.

COROLLARY 3.4. **a):**

$$\begin{aligned}
f_{GE_{2n}}(x) &= \frac{(1+7i)\,x^2+(7+i)\,x}{-x^3+35x^2-35x+1},\\
f_{GE_{2n+1}}(x) &= \frac{ix^2+(7+7i)\,x+1}{-x^3+35x^2-35x+1}.
\end{aligned}$$

b):

$$\begin{aligned}
f_{GK_{2n}}(x) &= \frac{(35+199i)\,x^2-(70+238i)\,x+(3+7i)}{-x^3+35x^2-35x+1},\\
f_{GK_{2n+1}}(x) &= \frac{(7+35i)\,x^2-(46+70i)\,x+(7+3i)}{-x^3+35x^2-35x+1}.
\end{aligned}$$

From Corollary 3.4 we can obtain the following corollary which presents the identities on Gaussian Edouard sequences.

COROLLARY 3.5. **(a):** $(7+i)\,GK_{2n-2}+(1+7i)\,GK_{2n-4}=(3+7i)GE_{2n}-(70+238i)\,GE_{2n-2}+(35+199i)\,GE_{2n-4}.$

(b): $iGK_{2n-4}+(7+7i)\,GK_{2n-2}+GK_{2n}=-(70+238i)\,GE_{2n-1}+(35+199i)\,GE_{2n-3}+(3+7i)GE_{2n+1}.$

(c): $(5383+931i)GE_{2n-4}-(46+70i)\,GE_{2n-2}+(7+3i)GE_{2n}=(7+i)GK_{2n-1}+(1+7i)GK_{2n-3}.$

(d): $(7+3i)\,GE_{2n+1}-(46+70i)\,GE_{2n-1}+(7+35i)\,GE_{2n-3}=iGK_{2n-3}+(7+7i)\,GK_{2n-1}+GK_{2n+1}.$

Proof. From Corollary 3.4 we obtain

$$((1+7i)\,x^2+(7+i)\,x)f_{GK_{2n}}=((3+7i)-(70+238i)\,x+(35+199i)\,x^2)f_{GE_{2n}}.$$

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ((1+7i)\,x^2+(7+i)\,x)\sum_{n=0}^{\infty}GK_{2n}x^n \\
&= (7+i)x\sum_{n=0}^{\infty}GK_{2n}x^n+(1+7i)\,x^2\sum_{n=0}^{\infty}GK_{2n}x^n \\
&= (7+i)\sum_{n=0}^{\infty}GK_{2n}x^{n+1}+(1+7i)\sum_{n=0}^{\infty}GK_{2n}x^{n+2} \\
&= (7+i)\sum_{n=1}^{\infty}GK_{2n-2}x^n+(1+7i)\sum_{n=2}^{\infty}GK_{2n-4}x^n \\
&= (7+i)(3+7i)x+\sum_{n=2}^{\infty}((7+i)GK_{2n-2}+(1+7i)\,GK_{2n-4})x^n
\end{aligned}$$

whereas the RHS (right hand side) is equal to

$$\begin{aligned}
RHS &= ((3+7i)-(70+238i)\,x+(35+199i)\,x^2)\sum_{n=0}^{\infty}GE_{2n}x^n \\
&= (3+7i)\sum_{n=0}^{\infty}GE_{2n}x^n-(70+238i)\,x\sum_{n=0}^{\infty}GE_{2n}x^n+(35+199i)x^2\sum_{n=0}^{\infty}GE_{2n}x^n \\
&= (3+7i)\sum_{n=0}^{\infty}GE_{2n}x^n-(70+238i)\sum_{n=0}^{\infty}GE_{2n}x^{n+1}+(35+199i)\sum_{n=0}^{\infty}GE_{2n}x^{n+2} \\
&= (3+7i)\sum_{n=0}^{\infty}GE_{2n}x^n-(70+238i)\sum_{n=1}^{\infty}GE_{2n-2}x^n+(35+199i)\sum_{n=2}^{\infty}GE_{2n-4}x^n \\
&= (3+7i)(7+i)x+\sum_{n=2}^{\infty}((3+7i)GE_{2n}-(70+238i)\,GE_{2n-2}+(35+199i)GE_{2n-4})x^n.
\end{aligned}$$

Comparing the coefficients and the proof of the first identity (a) is done. We can present other identity similarly. □

We can get an identitiy related to Gaussian Edouard numbers and Edouard-Lucas numbers given below.

THEOREM 3.6. *For all integers m,n the following identity holds:*

$$GW_{m+n}=E_{m-1}GW_{n+2}+(E_{m-3}-7E_{m-2})GW_{n+1}+E_{m-2}GW_n.$$

Proof. First, we assume that $m, n \geq 0$. The proof can be given by mathematical induction on m. If $m = 0$ we get,

$$GW_n = E_{-1}GW_{n+2} + (E_{-3} - 7E_{-2})GW_{n+1} + E_{-2}GW_n$$

which is true since $E_{-1} = 0, E_{-2} = 1, E_{-3} = 7$. We assume that the identity given holds for $m \leq k$. For $m = k + 1$, we get,

$$\begin{aligned}
GW_{(k+1)+n} &= 7GW_{n+k} - 7GW_{n+k-1} + GW_{n+k-2} \\
&= 7(E_{k-1}GW_{n+2} + (E_{k-3} - 7E_{k-2})GW_{n+1} + E_{k-2}GW_n) \\
&\quad -7(E_{k-2}GW_{n+2} + (E_{k-4} - 7E_{k-3})GW_{n+1} + E_{k-3}GW_n) \\
&\quad +(E_{k-3}GW_{n+2} + (E_{k-5} - 7E_{k-4})GW_{n+1} + E_{k-4}GW_n) \\
&= (7E_{k-1} - 7E_{k-2} + E_{k-3})GW_{n+2} + ((7E_{k-3} - 7E_{k-4} + E_{k-5}) \\
&\quad -7(7E_{k-2} - 7E_{k-3} + E_{k-4}))GW_{n+1} + (7E_{k-2} - 7E_{k-3} + E_{k-4})GW_n \\
&= E_kGW_{n+2} + (E_{k-2} - 7E_{k-1})GW_{n+1} + E_{k-1}GW_n \\
&= E_{(k+1)-1}GW_{n+2} + (E_{(k+1)-3} - 7E_{(k+1)-2})GW_{n+1} + E_{(k+1)-2}GW_n.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 3.6. The other case of m, n can be proved similarly. □

Taking $GW_n = GW_n$ or $GW_n = GW_n$ in above theorem,respectively,we get.

COROLLARY 3.7. *For all integers m,n, we get,*

$$\begin{aligned}
GE_{m+n} &= E_{m-1}GE_{n+2} + (E_{m-3} - 7E_{m-2})GE_{n+1} + E_{m-2}GE_n, \\
GK_{m+n} &= E_{m-1}GK_{n+2} + (E_{m-3} - 7E_{m-2})GK_{n+1} + E_{m-2}GK_n.
\end{aligned}$$

4. Simson's Formula

In this section, we present Simpson's formula of generalized Gaussian Edouard numbers. This is a special cases of [17, Theorem 4.1].

THEOREM 4.1 (Simpson's formula of generalized Gaussian Edouard numbers). *For all integers n, we can write following equality*

$$\begin{aligned}
\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} &= \begin{vmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{vmatrix} \\
&= (6GW_1 - GW_0 - GW_2)(GW_2^2 + 8GW_1^2 + GW_0^2 - 8GW_1GW_2 \\
&\quad +6GW_0GW_2 - 8GW_0GW_1).
\end{aligned}$$

Proof. Using [17, Theorem 4.1] we can obtain the required result. □

From the Theorem 4.1 we get the following Corollary.

COROLLARY 4.2. *For all integers n, we get the following identities:*

(a): $$\begin{vmatrix} GE_{n+2} & GE_{n+1} & GE_n \\ GE_{n+1} & GT_n & GE_{n-1} \\ GE_n & GE_{n-1} & GE_{n-2} \end{vmatrix} = 6-6i.$$

(b): $$\begin{vmatrix} GK_{n+2} & GK_{n+1} & GK_n \\ GK_{n+1} & GK_n & GK_{n-1} \\ GK_n & GK_{n-1} & GK_{n-2} \end{vmatrix} = -3072+3072i.$$

5. SUM FORMULAS

In this section, we identify several sum formulas for generalized Gaussian Edouard numbers, presenting explicitly derived expressions that encapsulate the cumulative behavior of the sequence and reveal the intricate interplay between additive patterns and the underlying recursive structure, thereby offering valuable theoretical insights and practical computational tools for further exploration of these numbers.

THEOREM 5.1.

a): $\sum_{k=0}^{n} W_k = \frac{1}{4}(-(n+3)W_n + (n+2)(7W_{n+1} - W_{n+2}) - (n+1)W_{n+1} + 2W_2 - 13W_1 + 7W_0)$.

b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{32}(-(n+3)W_{2n} + (n+2)(-7W_{2n+2} + 48W_{2n+1} - 7W_{2n}) - (n+1)W_{2n+2} + 15W_2 - 96W_1 + 49W_0)$.

c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{32}(-(n+3)W_{2n+1} + (n+2)(-W_{2n+2} + 42W_{2n+1} - 7W_{2n}) - (n+1)(7W_{2n+2} - 7W_{2n+1} + W_{2n}) + 9W_2 - 56W_1 + 15W_0)$.

Proof. It is given in Soykan [23, Theorem 3.3]. □

In this section, we identify some sum formulas of generalized Gaussian Edouard numbers. □

THEOREM 5.2. *For all integers $n \geq 0$, we have sum formulas given below:*

(a): $\sum_{k=0}^{n} GW_k = \frac{1}{4}(-(n+3)GW_n + (n+2)(7GW_{n+1} - GW_{n+2}) - (n+1)GW_{n+1} + 2GW_2 - 13GW_1 + 7GW_0)$.

(b): $\sum_{k=0}^{n} GW_{2k} = \frac{1}{32}(-(n+3)GW_{2n} + (n+2)(-7GW_{2n+2} + 48GW_{2n+1} - 7GW_{2n}) - (n+1)GW_{2n+2} + 15GW_2 - 96GW_1 + 49GW_0)$.

(c): $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{32}(-(n+3)GW_{2n+1} + (n+2)(-GW_{2n+2} + 42GW_{2n+1} - 7GW_{2n}) - (n+1)(7GW_{2n+2} - 7GW_{2n+1} + GW_{2n}) + 9GW_2 - 56GW_1 + 15GW_0)$.

Proof. Use Theorem 5.1 and the definition of GW_n. □

As a special case of the Theorem 5.2, we present the following corollary.

Corollary 5.3. *For all integers $n \geq 0$, we have sum formulas given below:*

a): $\sum_{k=0}^{n} GE_k = \frac{1}{4}(-(n+3)GE_n + (n+2)(7GE_{n+1} - GE_{n+2}) - (n+1)GE_{n+1} + 1 + 2i)$.

b): $\sum_{k=0}^{n} GE_{2k} = \frac{1}{32}(-(n+3)GE_{2n} + (n+2)(-7GE_{2n+2} + 48GE_{2n+1} - 7GE_{2n}) - (n+1)GE_{2n+2} + 9 + 15i)$.

c): $\sum_{k=0}^{n} GE_{2k+1} = \frac{1}{32}(-(n+3)GE_{2n+1} + (n+2)(-GE_{2n+2} + 42GE_{2n+1} - 7GE_{2n}) - (n+1)(7GE_{2n+2} - 7GE_{2n+1} + GE_{2n}) + 7 + 9i)$.

As a special case of the Theorem 5.2 we present the following corollary.

Corollary 5.4. *For all integers $n \geq 0$, we have sum formulas given below:*

a): $\sum_{k=0}^{n} GK_k = \frac{1}{4}(-(n+3)GK_n + (n+2)(7GK_{n+1} - GK_{n+2}) - (n+1)GK_{n+1} + 24i)$.

b): $\sum_{k=0}^{n} GK_{2k} = \frac{1}{32}(-(n+3)GK_{2n} + (n+2)(-7GK_{2n+2} + 48GK_{2n+1} - 7GK_{2n}) - (n+1)GK_{2n+2} + 160i)$.

c): $\sum_{k=0}^{n} GK_{2k+1} = \frac{1}{32}(-(n+3)GK_{2n+1} + (n+2)(-GK_{2n+2} + 42GK_{2n+1} - 7GK_{2n}) - (n+1)(7GK_{2n+2} - 7GK_{2n+1} + GK_{2n}) - 32)$.

6. Matrix Formulation of GW_n

In this section, we examine some matrix representations intimately connected to the Gaussian Edouard and Gaussian Edouard-Lucas numbers, elucidating how these matrices encapsulate the underlying recursive structures and reveal intricate algebraic and combinatorial relationships within the sequences, thereby providing both a powerful analytical framework for deriving further identities and a potential gateway to diverse computational applications.

We define the square matrix A of order 3 as

$$A = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Note that

$$A^n = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix}.$$

Then we give the following Lemma.

Lemma 6.1. *For $n \geq 0$ the following identity is true:*

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

Proof. The proof can be given by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus, the following identity is true.

$$\begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k + 1$, we get,

$$\begin{aligned} \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} &= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} \\ &= \begin{pmatrix} 7GW_{k+2} - 7GW_{k+1} + GW_k \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by mathematical induction on n, the proof is completed. □

For the proof see [19].

We define

$$N_{GW} = \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \tag{6.1}$$

$$S_{GW} = \begin{pmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with N_{GW} and S_{GW}.

Theorem 6.2. *Using N_{GW} and S_{GW}, we get,*

$$A^n N_{GW} = S_{GW}.$$

Proof. Note that we get,

$$\begin{aligned} A^n N_{GW} &= \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix} \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= GW_2 E_{n+1} + GW_1 (E_{n-1} - 7E_n) + GW_0 E_n, \\ a_{12} &= GW_1 E_{n+1} + GW_0 (E_{n-1} - 7E_n) + GW_{-1} E_n, \\ a_{13} &= GW_0 E_{n+1} + GW_{-1} (E_{n-1} - 7E_n) + GW_{-2} E_n, \\ a_{21} &= GW_2 E_n + GW_1 (E_{n-2} - 7E_{n-1}) + GW_0 E_{n-1}, \\ a_{22} &= GW_1 E_n + GW_0 (E_{n-2} - 7E_{n-1}) + GW_{-1} E_{n-1}, \\ a_{23} &= GW_0 E_n + GW_{-1} (E_{n-2} - 7E_{n-1}) + GW_{-2} E_{n-1}, \\ a_{31} &= GW_2 E_{n-1} + GW_1 (E_{n-3} - 7E_{n-2}) + GW_0 E_{n-2}, \\ a_{32} &= GW_1 E_{n-1} + GW_0 (E_{n-3} - 7E_{n-2}) + GW_{-1} E_{n-2}, \\ a_{33} &= GW_0 E_{n-1} + GW_{-1} (E_{n-3} - 7E_{n-2}) + GW_{-2} E_{n-2}. \end{aligned}$$

Using Theorem 3.6 the proof is done. □

By taking, $GW_n = GE_n$ with $GE_0 = 0$, $GE_1 = 1$, $GE_2 = 7 + i$ in (6.1) and (6.2), $GW_n = GK_n$ with $GK_0 = 3 + 7i$, $GK_1 = 7 + 3i$, $GK_2 = 35 + 7i$ in (6.1) and (6.2), respectively, we get,

$$N_{GE} = \begin{pmatrix} 7+i & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 1+7i \end{pmatrix} \quad , \quad S_{GE} = \begin{pmatrix} GE_{n+2} & GE_{n+1} & GE_n \\ GE_{n+1} & GE_n & GE_{n-1} \\ GE_n & GE_{n-1} & GE_{n-2} \end{pmatrix}$$

$$N_{GK} = \begin{pmatrix} 35+7i & 7+3i & 3+7i \\ 7+3i & 3+7i & 7+35i \\ 3+7i & 7+35i & 35+199i \end{pmatrix} \quad , \quad S_{GK} = \begin{pmatrix} GK_{n+2} & GK_{n+1} & GK_n \\ GK_{n+1} & GK_n & GK_{n-1} \\ GK_n & GK_{n-1} & GK_{n-2} \end{pmatrix}$$

From Theorem 6.2, we get the following corollary.

COROLLARY 6.3. *The following identities are holds*

(a): $A^n N_{GE} = S_{GE}$.

(b): $A^n N_{GK} = S_{GK}$.

7. Conclusions

Number sequences have been extensively studied and are widely applied in various fields, including physics, engineering, architecture, the natural sciences, and art. Among these, integer sequences—such as the Fibonacci, Lucas, Pell, and Jacobsthal sequences—are some of the best-known examples of second-order recurrence relations. Notably, the Fibonacci numbers gained fame through their appearance in the classic rabbit breeding problem, which Leonardo of Pisa introduced in his 1202 work, Liber Abaci. Moreover, the Fibonacci and Lucas sequences have given rise to many elegant mathematical identities that continue to inspire further research.

A a third-order sequences, we introduce the Gaussian generalized Edouard numbers and their two special cases. We derive Binet's formulas, generating functions, and Simson's formulas, as well as sum formulas, various identities, recurrence properties, and matrix representations for these sequences.

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