

Antisymmetric infinitesimal admissible hom-bialgebras

Abstract

This paper develops a novel framework for antisymmetric infinitesimal hom-bialgebras by introducing admissible bimodules of hom-associative algebras, generalizing the foundational work of [12]. We first construct admissible hom-associative bialgebras and rigorously establish their equivalence to matched pairs and Manin triples via a canonical bilinear form. Our approach systematically extends classical bialgebra theory to the hom-associative setting by imposing admissibility conditions on bimodules, which preserve the twisted multiplicativity of hom-structures. Key results include: (1) the characterization of admissible hom-associative algebras through their regular bimodules, (2) the derivation of antisymmetric infinitesimal admissible hom-bialgebras as dual structures, and (3) the proof that these bialgebras are categorically equivalent to Manin triples when equipped with a standard invariant bilinear form on the direct sum $\mathcal{A} \oplus \mathcal{A}^*$. This work unifies and generalizes prior studies on hom-Lie bialgebras and the hom-Yang-Baxter equation, while providing new tools for deformation theory and non-commutative geometry. The constructions are contextualized within prominent examples, including q -deformations and σ -derivations, bridging connections to the Witt and Virasoro algebras. Our results demonstrate that the hom-associative admissible framework offers a robust algebraic foundation for studying twisted bialgebraic structures.

Keywords. Admissible bimodules, admissible hom-associatives, hom-bialgebras, matched pairs, Manin triples

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1 Introduction

The foundational framework of Hom-algebra structures emerged through the quasi-deformation of Lie algebras associated with vector fields. By introducing discrete modifications via twisted derivations, this framework naturally extended to Hom-Lie and quasi-Hom-Lie algebras—structures characterized by a twisted Jacobi identity. Early breakthroughs in this direction arose from q -deformations, where classical derivations were systematically replaced by q -derivations, with prominent examples including the Witt and Virasoro algebras (see [1, 7, 8, 10, 13]).

The theory of hom-coalgebras and related structures was developed [14, 15], with further advancements [2, 3, 5]. Studies on infinitesimal hom-bialgebras, hom-Lie bialgebras, and the hom-Yang-Baxter equation can be found in the literature [17–19].

Haliya and Houndedji [14] introduced and studied quadratic hom-Jacobi-Jordan algebras, defined as hom-Jacobi-Jordan algebras with symmetric, invariant, and nondegenerate bilinear forms. A representation theory for hom-Jacobi-Jordan algebras, including adjoint and coadjoint representations were supplied with application to quadratic hom-Jacobi-Jordan algebras.

The principal goal of this work is to construct, by a new approach, the antisymmetric infinitesimal hom-bialgebras, which will be shown to be equivalent to Manin triples of admissible hom-associative algebras. We define admissible hom-associative algebras by exploring their regular bimodules under admissibility conditions. Furthermore, we define antisymmetric infinitesimal admissible hom-bialgebras, building on dual structures to establish their equivalence to matched

pairs and Manin triples of admissible hom-associative algebras. By constructing a standard bilinear form on the direct sum of an admissible hom-associative algebra and its dual, we demonstrate the interconnection of these structures, paralleling the classical theory of associative algebras.

This paper is organized as follows: In Section 2, we define admissible hom-associative algebras their bimodules and matched pairs with a focus on admissibility conditions. Section 3 introduces the definition of antisymmetric infinitesimal admissible hom-bialgebras and provides the equivalence between matched pairs and antisymmetric infinitesimal admissible hom-bialgebras. Section 4 establishes the Manin triples of admissible hom-associative algebras and using a standard bilinear form and, provides the equivalence between matched pairs, antisymmetric infinitesimal admissible hom-bialgebras and Manin triples. Finally, Section 5 presents the concluding remarks.

2 Admissible hom-associative algebras

2.1 Preliminaries

Definition 2.1 A hom-associative algebra is a triple $(\mathcal{A}, \cdot, \alpha)$ consisting of a linear space $\mathcal{A}$ over a field $\mathcal{K}$, $\mathcal{K}$ -bilinear map $\cdot : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and a linear space map $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ satisfying the hom-associativity property:

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z). \quad (2.1)$$

Definition 2.2 Let $(\mathcal{A}, \cdot, \alpha)$ be a hom-associative algebra. The hom-algebra is called

- multiplicative hom-associative algebra if $\forall x, y \in \mathcal{A}$ we have $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$;
- regular hom-associative algebra if α is an automorphism;
- involutive hom-associative algebra if α is an involution, that is $\alpha^2 = id$.

Definition 2.3 A hom-module is a pair (V, β) , where V is a $\mathcal{K}$ -vector space, and $\beta : V \rightarrow V$ is a linear map.

Definition 2.4 Let $(\mathcal{A}, \cdot, \alpha)$ be a hom-associative algebra and let (V, β) be a hom-module. Let $l, r : \mathcal{A} \rightarrow gl(V)$ be two linear maps. The quadruple (l, r, β, V) is called a bimodule of $\mathcal{A}$ if for all $x, y \in \mathcal{A}, v \in V$:

$$l(x \cdot y)\beta(v) = l(\alpha(x))l(y)v, \quad (2.2)$$

$$r(x \cdot y)\beta(v) = r(\alpha(y))r(x)v, \quad (2.3)$$

$$l(\alpha(x))r(y)v = r(\alpha(y))l(x)v, \quad (2.4)$$

$$\beta(l(x)v) = l(\alpha(x))\beta(v), \quad (2.5)$$

$$\beta(r(x)v) = r(\alpha(x))\beta(v). \quad (2.6)$$

Proposition 2.5 [12] Let $(\mathcal{A}, \cdot, \alpha)$ be a Hom-associative algebra and let (V, β) be a Hom-module. Let $l, r : \mathcal{A} \rightarrow gl(V)$ be two linear maps. The quadruple (l, r, β, V) satisfies a Hom-bimodule properties (2.2), (2.3), (2.4) of a Hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$ if and only if the direct sum of vector spaces, $\mathcal{A} \oplus V$, is turned into a Hom-associative algebra by defining multiplication in $\mathcal{A} \oplus V$ by

$$\begin{aligned} (x_1 + v_1) * (x_2 + v_2) &= x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1), \\ (\alpha \oplus \beta)(x_1 + v_1) &= \alpha(x_1) + \beta(v_1) \end{aligned} \quad (2.7)$$

for all $x_1, x_2 \in \mathcal{A}, v_1, v_2 \in V$.

Proof: Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. The left-hand side and right-hand side of Hom-associativity of $(\mathcal{A} \oplus V, *, \alpha \oplus \beta)$ are expanded as follows:

$$\begin{aligned}
 & ((x_1 + v_1) * (x_2 + v_2)) * (\alpha \oplus \beta)(x_3 + v_3) \\
 &= ((x_1 + v_1) * (x_2 + v_2)) * (\alpha(x_3) + \beta(v_3)) \\
 &= (x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1)) * (\alpha(x_3) + \beta(v_3)) \\
 &= (x_1 \cdot x_2) \cdot \alpha(x_3) + (l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))(l(x_1)v_2 + r(x_2)v_1)) \\
 &= (x_1 \cdot x_2) \cdot \alpha(x_3) + (l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1) \\
 & (\alpha \oplus \beta)(x_1 + v_1) * ((x_2 + v_2) * (x_3 + v_3)) \\
 &= (\alpha(x_1) + \beta(v_1)) * ((x_2 + v_2) * (x_3 + v_3)) \\
 &= (\alpha(x_1) + \beta(v_1)) * (x_2 \cdot x_3 + (l(x_2)v_3 + r(x_3)v_2)) \\
 &= \alpha(x_1) \cdot (x_2 \cdot x_3) + l(\alpha(x_1))(l(x_2)v_3 + r(x_3)v_2) + r(x_2 \cdot x_3)\beta(v_1) \\
 &= \alpha(x_1) \cdot (x_2 \cdot x_3) + (l(\alpha(x_1))l(x_2)v_3 + l(\alpha(x_1))r(x_3)v_2 + r(x_2 \cdot x_3)\beta(v_1))
 \end{aligned}$$

These elements of $\mathcal{A} \oplus V$ are equal if and only if

$$\begin{aligned}
 \alpha(x_1) \cdot (x_2 \cdot x_3) &= (x_1 \cdot x_2) \cdot \alpha(x_3) \\
 l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1 \\
 &= l(\alpha(x_1))l(x_2)v_3 + l(\alpha(x_1))r(x_3)v_2 + r(x_2 \cdot x_3)\beta(v_1)
 \end{aligned}$$

for all $x_1, x_2, x_3 \in \mathcal{A}, v_1, v_2, v_3 \in V$. This holds if and only if the hom-associativity holds, and for each $j = 1, 2, 3$ the respective V terms involving $v_j \in V$ are equal. If the terms are equal then the sums are equal. If the sums are equal then the terms should be equal if one specifies all or two of v_1, v_2, v_3 to zero element of V and using that linear transformations map zero to zero. Since,

$$\begin{aligned}
 \text{Hom-associativity} &\Leftrightarrow \alpha(x_1) \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot \alpha(x_3), \quad \forall x_1, x_2, x_3 \in \mathcal{A} \\
 (2.2) &\Leftrightarrow l(x_1 \cdot x_2)\beta(v_3) = l(\alpha(x_1))l(x_2)v_3, \quad \forall x_1, x_2, x_3 \in \mathcal{A} \\
 (2.4) &\Leftrightarrow l(\alpha(x_1))r(x_3)v_2 = r(\alpha(x_3))l(x_1)v_2, \quad \forall x_1, x_2, x_3 \in \mathcal{A} \\
 (2.3) &\Leftrightarrow r(\alpha(x_3))r(x_2)v_1 = r(x_2 \cdot x_3)\beta(v_1), \quad \forall x_1, x_2, x_3 \in \mathcal{A}.
 \end{aligned}$$

the proof is complete. $\square$

Example 2.6 Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative Hom-associative algebra. Let $L_{\cdot x}$ and $R_{\cdot x}$ denote the left and right multiplication operators, respectively, i. e. $L_{\cdot x}(y) = x \cdot y, R_{\cdot x}(y) = y \cdot x$ for any $x, y \in \mathcal{A}$. Let $L : \mathcal{A} \rightarrow gl(\mathcal{A})$ with $x \mapsto L_{\cdot x}$ and $R : \mathcal{A} \rightarrow gl(\mathcal{A})$ with $x \mapsto R_{\cdot x}$ (for every $x \in \mathcal{A}$) be two linear maps. Then, the triples $(L, 0, \alpha), (0, R, \alpha)$ and (L, R, α) are bimodules of $(\mathcal{A}, \cdot, \alpha)$.

Remark 2.7 (L, R, α) is called the **regular bimodule** of $(\mathcal{A}, \cdot, \alpha)$.

Theorem 2.8 [12] Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two Hom-associative algebras. Suppose there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \rightarrow gl(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \rightarrow gl(\mathcal{A})$ such that the quadruple $(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta, \mathcal{B})$ is a bimodule of $\mathcal{A}$, and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha, \mathcal{A})$ is a bimodule of $\mathcal{B}$, satisfying, for any $x, y \in \mathcal{A}, a, b \in \mathcal{B}$, the following conditions:

$$l_{\mathcal{A}}(\alpha(x))(a \circ b) = l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)\beta(b) + (l_{\mathcal{A}}(x)a) \circ \beta(b), \quad (2.8)$$

$$r_{\mathcal{A}}(\alpha(x))(a \circ b) = r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)\beta(a) + \beta(a) \circ (r_{\mathcal{A}}(x)b), \quad (2.9)$$

$$l_{\mathcal{B}}(\beta(a))(x \cdot y) = l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)\alpha(y) + (l_{\mathcal{B}}(a)x) \cdot \alpha(y), \quad (2.10)$$

$$r_{\mathcal{B}}(\beta(a))(x \cdot y) = r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)\alpha(x) + \alpha(x) \cdot (r_{\mathcal{B}}(a)y), \quad (2.11)$$

$$l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)\beta(b) + (r_{\mathcal{A}}(x)a) \circ \beta(b) - r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)\beta(a) - \beta(a) \circ (l_{\mathcal{A}}(x)b) = 0, \quad (2.12)$$

$$l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)\alpha(y) + (r_{\mathcal{B}}(a)x) \cdot \alpha(y) - r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)\alpha(x) - \alpha(x) \cdot (l_{\mathcal{B}}(a)y) = 0. \quad (2.13)$$

Then, there is a Hom-associative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{B}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$ given by

$$\begin{aligned} (x+a) * (y+b) &= (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a), \\ (\alpha \oplus \beta)(x+a) &= \alpha(x) + \beta(a) \end{aligned} \quad (2.14)$$

for all $x, y \in \mathcal{A}, a, b \in \mathcal{B}$.

Proof: Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. Set

$$[(x_1 + v_1) * (x_2 + v_2)] * (\alpha(x_3) + \beta(v_3)) = (\alpha(x_1) + \beta(v_1)) * [(x_2 + v_2) * (x_3 + v_3)],$$

which is developed to obtain (2.8)-(2.13). Then, using the relations

$$\beta(l_{\mathcal{A}}(x)a) = l_{\mathcal{A}}(\alpha(x))\beta(a), \quad \beta(r_{\mathcal{A}}(x)a) = r_{\mathcal{A}}(\alpha(x))\beta(a),$$

$$\alpha(l_{\mathcal{B}}(a)x) = l_{\mathcal{B}}(\beta(a))\alpha(x), \quad \alpha(r_{\mathcal{B}}(a)x) = r_{\mathcal{B}}(\beta(a))\alpha(x),$$

we show that $*$ is a Hom-associative algebra. $\square$

Definition 2.9 Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two Hom-associative algebras. Suppose that there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \rightarrow gl(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \rightarrow gl(\mathcal{A})$ such that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta)$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha)$ is a bimodule of $\mathcal{B}$. If the conditions (2.8) - (2.13) are satisfied, then, $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, \beta, l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha)$ is called a **matched pair of hom-associative algebras**.

2.2 Bimodules of admissible hom-associative algebras

In this sequel, we consider a multiplicative hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$. Let (l, r, β, V) be a bimodule of the multiplicative hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$, and let $l^*, r^* : \mathcal{A} \rightarrow gl(V^*)$ be the linear maps given for all $x \in \mathcal{A}, a \in V^*, v \in V$, by

$$\langle l^*(x)a, v \rangle := \langle l(x)v, a \rangle, \quad \langle r^*(x)a, v \rangle := \langle r(x)v, a \rangle. \quad (2.15)$$

It is clear that (r^*, l^*, β^*, V^*) is not a bimodule of $(\mathcal{A}, \cdot, \alpha)$ in general.

Remark 2.10 If α is involutive, then (r^*, l^*, β^*, V^*) is a bimodule of $(\mathcal{A}, \cdot, \alpha)$. This case was considered in [12].

Lemma 2.11 Let (l, r, β, V) be a bimodule of $(\mathcal{A}, \cdot, \alpha)$. Then, (r^*, l^*, β^*, V^*) is a bimodule of $(\mathcal{A}, \cdot, \alpha)$ if and only if the following equations hold:

$$r(\alpha^2(y))r(\alpha(x))v = r(y)r(\alpha(x))v; \quad (2.16)$$

$$l(\alpha^2(x))l(\alpha(y))v = l(x)l(\alpha(y))v; \quad (2.17)$$

$$l(y)r(\alpha(y))v = r(x)l(\alpha(y))v; \quad (2.18)$$

$$r(x)\beta(v) = \beta(r(\alpha(x))v); \quad l(x)\beta(v) = \beta(l(\alpha(x))v) \quad (2.19)$$

Proof: We have:

$$\begin{aligned}
 \langle r^*(x \cdot y)\beta^*(a), v \rangle &= \langle \beta(r(x \cdot y)v), a \rangle = \langle r(\alpha(x \cdot y))\beta(v), a \rangle \\
 &= \langle r(\alpha(x) \cdot \alpha(y))\beta(v), a \rangle = \langle r(\alpha^2(y))r(\alpha(x))v, a \rangle \\
 &= \langle (r(y)r(\alpha(x)))^*a, v \rangle = \langle r^*(\alpha(x))r^*(y)a, v \rangle; \\
 \\
 \langle l^*(x \cdot y)\beta^*(a), v \rangle &= \langle \beta(l(x \cdot y)(v)), a \rangle = \langle l(\alpha(x \cdot y))\beta(v), a \rangle \\
 &= \langle l(\alpha(x) \cdot \alpha(y))\beta(v), a \rangle = \langle l(\alpha^2(x))l(\alpha(y))\beta(v), a \rangle \\
 &= \langle (l(x)l(\alpha(y)))^*a, v \rangle = \langle l^*(\alpha(y))l^*(x)a, v \rangle; \\
 \\
 \langle r^*(\alpha(x))l^*(y)a, v \rangle &= \langle l(y)r(\alpha(x))v, a \rangle \\
 &= \langle r(x)l(\alpha(y))v, a \rangle = \langle l^*(\alpha(y))r^*(x)a, v \rangle; \\
 \\
 \langle \beta^*(r^*(x))a, v \rangle &= \langle r(x)(\beta(v)), a \rangle \\
 &= \langle \beta(r(\alpha(x)))v, a \rangle = \langle r^*(\alpha(x))\beta^*(a), v \rangle; \\
 \\
 \langle \beta^*(l^*(x))a, v \rangle &= \langle l(x)(\beta(v)), a \rangle \\
 &= \langle \beta(l(\alpha(x)))v, a \rangle = \langle l^*(\alpha(x))\beta^*(a), v \rangle;
 \end{aligned}$$

which leads to the conclusion. $\square$

Definition 2.12 A bimodule (l, r, β, V) of $(\mathcal{A}, \cdot, \alpha)$ is called **admissible** if (r^*, l^*, β^*, V^*) is a bimodule of $(\mathcal{A}, \cdot, \alpha)$, i.e. the conditions 2.16-2.19 in the lemma 2.11 are satisfied.

When we focus on the regular bimodule (L, R, α) , we get:

Corollary 2.13 Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative hom-associative algebra. The regular bimodule (L, R, α) is admissible if and only if the following equations hold:

$$x \cdot y = \alpha^2(x) \cdot y; \quad y \cdot x = y \cdot \alpha^2(x); \quad (2.20)$$

$$x \cdot (z \cdot \alpha(y)) = (\alpha(x) \cdot z) \cdot y. \quad (2.21)$$

Definition 2.14 A hom-associative algebra $(A, \cdot, \alpha)$ is **admissible** if its regular bimodule is admissible.

Corollary 2.15 Let $(A, \cdot, \alpha)$ be an admissible hom-associative algebra. Then,

$$L^*(x)[(\alpha^*)^2(a)] = L^*(x)a; R^*(x)[(\alpha^*)^2(a)] = R^*(x)a, \quad (2.22)$$

$\forall x \in \mathcal{A}, a \in \mathcal{A}^*$.

Proof: $(A, \cdot, \alpha)$ is an admissible hom-associative algebra, then we have $\alpha^2(x)\alpha^2(y) = x \cdot \alpha^2(y) = x \cdot y$. Hence,

$$\begin{aligned}
 \langle L^*(x)[(\alpha^*)^2(a)], y \rangle &= \langle \alpha^2(L(x)y), a \rangle = \langle \alpha^2(x \cdot y), a \rangle = \langle \alpha^2(x) \cdot \alpha^2(y), a \rangle \\
 &= \langle x \cdot y, a \rangle = \langle L(x)y, a \rangle = \langle L^*(x)a, y \rangle.
 \end{aligned}$$

And in the same way, we show that $\langle R^*(x)[(\alpha^*)^2(a)], y \rangle = \langle R^*(x)a, y \rangle$. $\square$

Proposition 2.16 Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ be two admissible hom-associative algebras. Then, for all $x, y \in \mathcal{A}, a, b \in \mathcal{A}^*$ we get

$$b \circ L^*(\alpha(x))a = (\alpha^*)^2(b) \circ L^*(\alpha(x))a, b \circ R^*(\alpha(x))a = (\alpha^*)^2(b) \circ R^*(\alpha(x))a$$

Proof: We have:

$$\begin{aligned}\langle b \circ R^*(\alpha(x))a, y \rangle &= \langle R^*(\alpha(x))a, L_\circ^*(b)y \rangle = \langle a, L_\circ^*(b)y \cdot \alpha(x) \rangle = \langle a, \alpha^2(L_\circ^*(b)y) \cdot \alpha(x) \rangle \\ &= \langle a, R_\circ(\alpha(x))[\alpha^2(L_\circ^*(b)y)] \rangle = \langle b \circ (\alpha^*)^2[R_\circ^*(\alpha(x))a], y \rangle\end{aligned}$$

It follows that $b \circ R^*(\alpha(x))a = b \circ (\alpha^*)^2[R_\circ^*(\alpha(x))a]$. In view of $(\mathcal{A}^*, \circ, \alpha^*)$ also being admissible, we have:

$$b \circ R^*(\alpha(x))a = b \circ (\alpha^*)^2[R_\circ^*(\alpha(x))a] = (\alpha^*)^2(b) \circ (\alpha^*)^2[R_\circ^*(\alpha(x))a] = (\alpha^*)^2(b) \circ R^*(\alpha(x))a.$$

Analogously, we show that $b \circ L^*(\alpha(x))a = (\alpha^*)^2(b) \circ L^*(\alpha(x))a$. $\square$

Theorem 2.17 *Let $(\mathcal{A}, \cdot, \alpha)$ be an admissible hom-associative algebra. Suppose that there is an admissible hom-associative algebra structure " $\circ$ " on its dual space $(\mathcal{A}^*, \alpha^*)$. Then,*

$(\mathcal{A}, \mathcal{A}^, R^*, L^*, \alpha^*, R_\circ^*, L_\circ^*, \alpha)$ is a matched pair of admissible hom-associative algebras if and only if, for any $x, y \in \mathcal{A}$ and $a, b \in \mathcal{A}^*$,*

$$R_\circ^*(\alpha(x))(a \circ b) = R_\circ^*(L_\circ^*(a)x)\alpha^*(b) + (R_\circ^*(x)a) \circ \alpha^*(b), \quad (2.23)$$

$$R_\circ^*(R_\circ^*(a)x)\alpha^*(b) + L_\circ^*(x)a \circ \alpha^*(b) = L_\circ^*(L_\circ^*(b)x)\alpha^*(a) + \alpha^*(a) \circ (R_\circ^*(x)b). \quad (2.24)$$

Proof: This follows the same proof of the Theorem3 in [12]. Obviously, (2.23) gives (2.8), and (2.24) reduces to (2.12) when $l_{\mathcal{A}} = R_\circ^*, r_{\mathcal{A}} = L_\circ^*$, $l_B = l_{\mathcal{A}^*} = R_\circ^*, r_B = r_{\mathcal{A}^*} = L_\circ^*$. Now, show that

$$\begin{aligned}(2.8) &\iff (2.9) \iff (2.10) \iff (2.11) \\ &\text{and } (2.12) \iff (2.13).\end{aligned}$$

Suppose (2.23) and (2.24) are satisfied and show that one has

$$\begin{aligned}L_\circ^*(\alpha(x))(a^* \circ b^*) &= L_\circ^*(R_\circ^*(b^*)x)\alpha^*(a^*) + \alpha^*(a^*) \circ (L_\circ^*(x)b^*) \\ R_\circ^*(\alpha^*(a^*))(x \cdot y) &= R_\circ^*(L_\circ^*(x)a^*)\alpha(y) + (R_\circ^*(a^*)x) \cdot \alpha(y) \\ L_\circ^*(\alpha^*(a^*))(x \cdot y) &= L_\circ^*(R_\circ^*(y)a^*)\alpha(x) + \alpha(x) \cdot (L_\circ^*(a^*)y) \\ R_\circ^*(R_\circ^*(x)a^*)\alpha(y) + (L_\circ^*(a^*)x) \cdot \alpha(y) &- L_\circ^*(L_\circ^*(y)a^*)\alpha(x) - \alpha(x) \cdot (R_\circ^*(a^*)y) = 0.\end{aligned}$$

We have

$$\begin{aligned}\langle R_\circ^*(x)a^*, y \rangle &= \langle L_\circ^*(y)a^*, x \rangle = \langle y \cdot x, a^* \rangle, \\ \langle R_\circ^*(b^*)x, a^* \rangle &= \langle L_\circ^*(a^*)x, b^* \rangle = \langle a^* \circ b^*, x \rangle, \\ \alpha^*(R_\circ^*(x)a^*) &= R_\circ^*(\alpha(x))\alpha^*(a^*), \quad \alpha^*(L_\circ^*(x)a^*) = L_\circ^*(\alpha(x))\alpha^*(a^*), \\ \alpha(R_\circ^*(a^*)x) &= R_\circ^*(\alpha^*(a^*))\alpha(x), \quad \alpha(L_\circ^*(a^*)x) = L_\circ^*(\alpha^*(a^*))\alpha(x),\end{aligned}$$

for all $x, y \in \mathcal{A}, a^*, b^* \in \mathcal{A}^*$. Set $\alpha(x) = z, \alpha(y) = t, \alpha^*(a^*) = c^*$ and $\alpha^*(b^*) = d^*$. Then

[label=()]the statement (2.8) $\iff$ (2.9) follows from

$$\begin{aligned}\langle R_\circ^*(\alpha(x))(a^* \circ b^*), y \rangle &= \langle y \cdot \alpha(x), a^* \circ b^* \rangle = \langle (L_\circ^*(y) \circ \alpha)x, a^* \circ b^* \rangle \\ &= \langle x, \alpha^*(L_\circ^*(y)(a^* \circ b^*)) \rangle = \langle L_\circ^*(\alpha(y))\alpha^*(a^* \circ b^*), x \rangle \\ &= \langle L_\circ^*(\alpha(y))(\alpha^*(a^*) \circ \alpha^*(b^*)), x \rangle \\ &= \langle L_\circ^*(\alpha(y))(c^* \circ d^*), x \rangle; \\ \langle R_\circ^*(L_\circ^*(a^*)x)\alpha(b^*), y \rangle &= \langle y \cdot L_\circ^*(a^*)x, \alpha^*(b^*) \rangle = \langle L_\circ^*(y)(\alpha^*(b^*)), L_\circ^*(a^*)x \rangle \\ &= \langle L_\circ^*(a^*)x, L_\circ^*(y)(\alpha^*(b^*)) \rangle \\ &= \langle a^* \circ (L_\circ^*(y)(\alpha^*(b^*))), x \rangle \\ &= \langle \alpha^*(c^*) \circ (L_\circ^*(y)(d^*)), x \rangle; \\ \langle (R_\circ^*(x)a^*) \circ \alpha^*(b^*), y \rangle &= \langle R_\circ^*(\alpha^*(b^*))y, R_\circ^*(x)a^* \rangle = \langle a^*, (R_\circ^*(\alpha^*(b^*))y) \cdot x \rangle \\ &= \langle L_\circ^*[R_\circ^*(\alpha^*(b^*))y]a^*, x \rangle = \langle L_\circ^*(R_\circ^*(d^*)y)\alpha^*(c^*), x \rangle;\end{aligned}$$

the statement (2.9) $\iff$ (2.10) follows from

$$\begin{aligned}
 \langle L^*(\alpha(x))(a^* \circ b^*), y \rangle &= \langle a^* \circ b^*, \alpha(x) \cdot y \rangle = \langle R_{\circ}^*(b^*)(\alpha(x) \cdot y), a^* \rangle \\
 &= \langle R_{\circ}^*(\alpha^*(d^*))(z \cdot y), a^* \rangle; \\
 \langle \alpha^*(a^*) \circ (L^*(x)b^*), y \rangle &= \langle \alpha^*(a^*), R_{\circ}^*(L^*(x)b^*)y \rangle = \langle a^*, \alpha[R_{\circ}^*(L^*(x)b^*)y] \rangle \\
 &= \langle a^*, R_{\circ}^*[\alpha^*(L^*(x)b^*)]\alpha(y) \rangle \\
 &= \langle a^*, R_{\circ}^*[L^*(\alpha(x))\alpha^*(b^*)]\alpha(y) \rangle \\
 &= \langle a^*, R_{\circ}^*(L^*(z)d^*)\alpha(y) \rangle; \\
 \langle L^*(R_{\circ}^*(b^*)x)\alpha^*(a^*), y \rangle &= \langle (R_{\circ}^*(b^*)x) \circ y, \alpha^*(a^*) \rangle = \langle \alpha[(R_{\circ}^*(b^*)x) \circ y], a^* \rangle \\
 &= \langle (R_{\circ}^*(\alpha^*(b^*))\alpha(x)) \circ \alpha(y), a^* \rangle \\
 &= \langle R_{\circ}^*(d^*)z \cdot \alpha(y), a^* \rangle;
 \end{aligned}$$

the statement (2.8) $\iff$ (2.11) follows from

$$\begin{aligned}
 \langle R^*(\alpha(x))(a^* \circ b^*), y \rangle &= \langle a^* \circ b^*, y \cdot \alpha(x) \rangle = \langle L^*(a^*)b^*, y \cdot z \rangle \\
 &= \langle L_{\circ}^*(a^*)(y \cdot z) \rangle = \langle L_{\circ}^*(\alpha^*(c^*))(y \cdot z) \rangle; \\
 \langle (R^*(x)a^*) \circ \alpha^*(b^*), y \rangle &= \langle \alpha^*(b^*), L^*(R^*(x)a^*)y \rangle = \langle b^*, \alpha^*[L^*(R^*(x)a^*)y] \rangle \\
 &= \langle b^*, L^*(R^*(\alpha(x))\alpha^*(a^*))\alpha(y) \rangle \\
 &= \langle b^*, L^*(R^*(z)c^*)\alpha(y) \rangle; \\
 \langle R^*(L_{\circ}^*(a^*)x)\alpha^*(b^*), y \rangle &= \langle y \cdot L_{\circ}^*(a^*)x, \alpha^*(b^*) \rangle = \langle \alpha(y) \cdot \alpha(L_{\circ}^*(a^*)x), b^* \rangle \\
 &= \langle \alpha(y) \cdot L_{\circ}^*(\alpha^*(a^*))\alpha(x), b^* \rangle = \langle \alpha(y) \cdot L_{\circ}^*(c^*)z, b^* \rangle;
 \end{aligned}$$

the statement (2.12) $\iff$ (2.13) follows from

$$\begin{aligned}
 \langle L^*(L_{\circ}^*(b^*)x)\alpha^*(a^*), y \rangle &= \langle (L_{\circ}^*(b^*)x) \cdot y, \alpha^*(a^*) \rangle = \langle a^*, \alpha(L_{\circ}^*(b^*)x) \cdot \alpha(y) \rangle \\
 &= \langle a^*, L_{\circ}^*(\alpha^*(b^*))\alpha(x) \cdot \alpha(y) \rangle = \langle a^*, L_{\circ}^*(d^*)z \cdot \alpha(y) \rangle; \\
 \langle \alpha^*(a^*) \circ (R^*(x)b^*), y \rangle &= \langle R_{\circ}^*(R_{\circ}^*(x)b^*)y, \alpha^*(a^*) \rangle = \langle \alpha^*(a^*) \circ (R^*(x)b^*), y \rangle \\
 &= \langle \alpha[R_{\circ}^*(R_{\circ}^*(x)b^*)y], a^* \rangle \\
 &= \langle R_{\circ}^*[R_{\circ}^*(\alpha(x))\alpha^*(b^*)]\alpha(y), a^* \rangle \\
 &= \langle R_{\circ}^*(R^*(z)d^*)\alpha(y), a^* \rangle; \\
 \langle (L^*(x)a^*) \circ \alpha^*(b^*), y \rangle &= \langle R_{\circ}^*(\alpha^*(b^*))y, L^*(x)a^* \rangle = \langle x \cdot (R_{\circ}^*(d^*)y), a^* \rangle \\
 &= \langle \alpha(z) \cdot (R_{\circ}^*(d^*)y), a^* \rangle; \\
 \langle R^*(R_{\circ}^*(a^*)x)\alpha^*(b^*), y \rangle &= \langle y \cdot R_{\circ}^*(a^*)x, \alpha^*(b^*) \rangle = \langle \alpha^*(b^*), L^*(y)(R_{\circ}^*(a^*)x) \rangle \\
 &= \langle (L^*(y)(d^*), R_{\circ}^*(a^*)x) \rangle = \langle L^*(y)d^* \circ a^*, x \rangle \\
 &= \langle L_{\circ}^*(L^*(y)d^*)x, a^* \rangle = \langle L_{\circ}^*(L^*(y)d^*)\alpha(z), a^* \rangle
 \end{aligned}$$

which completes the proof. $\square$

3 Antisymmetric infinitesimal admissible hom-bialgebras



Definition 3.1 A pair of admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ is called an **antisymmetric infinitesimal admissible hom-bialgebra** if

$$\langle \Delta(x \cdot y), \alpha^*(a) \otimes b \rangle = \langle (\alpha \otimes L^*(\alpha(x)))\Delta(y), \alpha^*(a) \otimes b \rangle + \langle (R^*(\alpha(y)) \otimes \alpha)\Delta(y), \alpha^*(a) \otimes b \rangle \quad (3.1)$$

$$\begin{aligned}
 &\langle (L^*(\alpha(y)) \otimes \alpha), \alpha^*(a) \otimes b \rangle - \langle (\alpha \otimes R^*(\alpha(y)))\Delta(x), \alpha^*(a) \otimes b \rangle = \\
 &-\langle \sigma[(L^*(\alpha(x)) \otimes \alpha)\Delta(y)], \alpha^*(a) \otimes b \rangle + \langle \sigma[(\alpha \otimes R^*(\alpha(x)))\Delta(y)], \alpha^*(a) \otimes b \rangle \quad (3.2)
 \end{aligned}$$

Theorem 3.2 A pair of admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ is an antisymmetric infinitesimal admissible hom-bialgebras if and only if

$(\mathcal{A}, \mathcal{A}^*, R^*, L^*, \alpha^*, R_{\circ}^*, L_{\circ}^*, \alpha)$ is a matched pair of admissible hom-associative algebras.

Proof: On the one hand, we have:

$$\langle \Delta(x \cdot y), \alpha^*(a) \otimes b \rangle = \langle x \cdot y, \alpha^*(a) \circ b \rangle = \langle x \cdot y, L_{\circ}(\alpha^*(a))b \rangle = \langle L_{\circ}^*(\alpha^*(a))(x \cdot y), b \rangle;$$

$$\begin{aligned}
\langle \alpha(x) \cdot (L_\circ^*(a)y), b \rangle &= \langle L_\circ^*(a)y, L^*(\alpha(x))b \rangle = \langle y, a \circ L^*(\alpha(x))b \rangle = \langle y, (\alpha^*)^2(a) \circ L^*(\alpha(x))b \rangle \\
&= \langle y, \Delta^*[(\alpha^*)^2(a) \otimes L^*(\alpha(x))b] \rangle = \langle \Delta(y), (\alpha^*)^2(a) \otimes L^*(\alpha(x))b \rangle \\
&= \langle \Delta(y), (\alpha^* \otimes L^*(\alpha(x)))(\alpha^*(a) \otimes b) \rangle = \langle (\alpha \otimes L(\alpha(x))\Delta(y), \alpha^*(a) \otimes b) \rangle;
\end{aligned}$$

$$\begin{aligned}
\langle L_\circ^*(R^*(y)a)\alpha(x), b \rangle &= \langle \alpha(x), L_\circ(R^*(y)a)b \rangle = \langle \alpha(x), (R^*(y)a) \circ b \rangle = \langle x, \alpha^*(R^*(y)a) \circ \alpha^*(b) \rangle \\
&= \langle x, \Delta^*[\alpha^*(R^*(y)a) \otimes \alpha^*(b)] \rangle = \langle \Delta(x), \alpha^*(R^*(y)a) \otimes \alpha^*(b) \rangle \\
&= \langle \Delta(x), R^*(\alpha(y))\alpha^*(a) \otimes \alpha^*(b) \rangle = \langle \Delta(x), (R^*(\alpha(y)) \otimes \alpha^*)(\alpha^*(a) \otimes b) \rangle \\
&= \langle (R(\alpha(y)) \otimes \alpha)\Delta(y), \alpha^*(a) \otimes b \rangle.
\end{aligned}$$

Then, 3.1 $\iff$ 2.23. And on the other hand, we have:

$$\begin{aligned}
\langle L_\circ^*(L^*(y)a)\alpha(x), b \rangle &= \langle \alpha(x), L^*(y)a \circ b \rangle = \langle x, L^*(\alpha(y))\alpha^*(a) \circ \alpha^*(b) \rangle \\
&= \langle \Delta(x), (L^*(\alpha(y)) \otimes \alpha^*)(\alpha^*(a) \otimes b) \rangle \\
&= \langle (L(\alpha(y)) \otimes \alpha)\Delta(x), \alpha^*(a) \otimes b \rangle;
\end{aligned}$$

$$\begin{aligned}
\langle R(\alpha(y))L_\circ^*(a)x, b \rangle &= \langle L_\circ^*(a)x, R^*(\alpha(y))b \rangle = \langle x, L_\circ(a)[R^*(\alpha(y))b] \rangle = \langle x, a \circ [R^*(\alpha(y))b] \rangle \\
&= \langle x, (\alpha^*)^2(a) \circ [R^*(\alpha(y))b] \rangle = \langle \Delta(x), (\alpha^*)^2(a) \otimes [R^*(\alpha(y))b] \rangle \\
&= \langle \Delta(x), (\alpha^* \otimes R^*(\alpha(y)))(\alpha^*(a) \otimes b) \rangle = \langle (\alpha \otimes R(\alpha(y)))\Delta(x), \alpha^*(a) \otimes b \rangle;
\end{aligned}$$

$$\begin{aligned}
\langle \alpha(x) \cdot (R_\circ^*(a)y), b \rangle &= \langle L(\alpha(x))(R_\circ^*(a)y), b \rangle = \langle (R_\circ^*(a)y), L^*(\alpha(x))b \rangle = \langle y, L^*(\alpha(x))b \circ \alpha^*(a) \rangle \\
&= \langle y, L^*(\alpha(x))b \circ (\alpha^*)^2(a) \rangle = \langle \Delta(y), (L^*(\alpha(x)) \otimes \alpha^*)(b \otimes \alpha^*(a)) \rangle \\
&= \langle \Delta(y), (L^*(\alpha(x)) \otimes \alpha^*)\sigma^*(\alpha^*(a) \otimes b) \rangle \\
&= \langle \sigma[L(\alpha(x)) \otimes \alpha]\Delta(y), \alpha^*(a) \otimes b \rangle;
\end{aligned}$$

$$\begin{aligned}
\langle R_\circ^*(R^*(x)a)\alpha(y), b \rangle &= \langle \alpha(y), b \circ (R^*(x)a) \rangle = \langle y, \alpha^*(b) \circ (R^*(\alpha(x))\alpha^*(a)) \rangle \\
&= \langle \Delta(y), (\alpha^* \otimes R^*(\alpha(x)))(\sigma^*(\alpha^*(a) \otimes b)) \rangle \\
&= \langle \sigma(\alpha^* \otimes R^*(\alpha(x)))\Delta(y), \alpha^*(a) \otimes b \rangle.
\end{aligned}$$

Hence, 3.2 $\iff$ 2.24. □

4 Manin triple of admissible hom-associative algebras

Definition 4.1 A Manin triple of admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, *, \alpha')$ is a triple $(\mathcal{A} \oplus \mathcal{B}, \mathcal{A}, \mathcal{B})$, together with a nondegenerate symmetric bilinear form $S(\cdot, \cdot)$ on the admissible hom-associative algebra $(\mathcal{A} \oplus \mathcal{B}, \star, \alpha \oplus \alpha')$ such that

- S is invariant, i. e. for all $x, y, z \in \mathcal{A}$ and $a, b, c \in \mathcal{B}$,

$$\begin{aligned}
S((x+a) \star (y+b), (z+c)) &= S((x+a), (y+b) \star (z+c)), \\
S((\alpha \oplus \alpha')(x+a), y+b) &= S(x+a, (\alpha \oplus \alpha')(y+b)).
\end{aligned}$$

- The admissible hom-associative algebras $\mathcal{A}$ and $\mathcal{B}$ are isotropic admissible hom-associative algebras of $\mathcal{A} \oplus \mathcal{B}$.

Corollary 4.2 Let $(\mathcal{A}, \cdot, \alpha)$ be an admissible hom-associative algebra. Suppose there exists an admissible hom-associative algebra structure on its dual space $\mathcal{A}^*$ denoted by $(\mathcal{A}^*, \circ, \alpha^*)$. Then, there is $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ an admissible hom-associative algebra on the direct sum of the underlying vector space of $\mathcal{A}$ and its dual space $\mathcal{A}^*$ such that, $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the associated Manin triple with the invariant bilinear symmetric form given by

$$S_{\mathcal{A}}(x+a, y+b) = \langle x, b \rangle + \langle y, a \rangle, \quad (4.1)$$

for all $x, y \in \mathcal{A}$ and $a, b \in \mathcal{A}^*$.

Definition 4.3 $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is called the **standard Manin triple of the admissible hom-associative algebra $\mathcal{A}$ associated to $S_{\mathcal{A}}$** ; where $S_{\mathcal{A}}$ is a natural pairing between algebra and its dual space.

Proposition 4.4 Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ be two admissible hom-associative algebras. Then, $(\mathcal{A}, \mathcal{A}^*, R^*, L^*, R_{\circ}^*, L_{\circ}^*, \alpha, \alpha^*)$ is a matched pair of the admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ if and only if $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is a standard Manin triple.

Proof: • First, suppose that $(\mathcal{A}, \mathcal{A}^*, R^*, L^*, R_{\circ}^*, L_{\circ}^*, \alpha, \alpha^*)$ is a matched pair of the admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$. Then, $(\mathcal{A} \oplus \mathcal{A}^*, *, \alpha \oplus \alpha^*)$ is an admissible hom-associative algebra, with the product $*$ defined by 2.14. Consequently, let us compute and compare the following relations: $S_{\mathcal{A}}((x+a) * (y+b), (z+c))$ and $S_{\mathcal{A}}((x+a), (y+b) * (z+c)) \forall x, y, z \in \mathcal{A}$ and $\forall a, b, c \in \mathcal{A}^*$.

$$\begin{aligned} S_{\mathcal{A}}((x+a) * (y+b), (z+c)) &= S_{\mathcal{A}}(xy + R_{\circ}^*(a)y + L_{\circ}^*(b)x + a \circ b + R^*(x)b + L^*(y)a, z+c) \\ &= \langle xy + R_{\circ}^*(a)y + L_{\circ}^*(b)x, c \rangle + \langle z, a \circ b + R^*(x)b + L^*(y)a \rangle \\ &= \langle xy, c \rangle + \langle R_{\circ}^*(a)y, c \rangle + \langle L_{\circ}^*(b)x, c \rangle + \langle z, a \circ b \rangle + \langle z, R^*(x)b \rangle \\ &+ \langle z, L^*(y)a \rangle = \langle xy, c \rangle + \langle y, R_a(c) \rangle + \langle x, L_b(c) \rangle + \langle z, a \circ b \rangle \\ &+ \langle R_x(z), b \rangle + \langle L_y(z), a \rangle \\ &= \langle xy, c \rangle + \langle y, c \circ a \rangle + \langle x, b \circ c \rangle + \langle z, a \circ b \rangle + \langle zx, b \rangle + \langle yz, a \rangle. \end{aligned}$$

$$\begin{aligned} S_{\mathcal{A}}((x+a), (y+b) * (z+c)) &= S_{\mathcal{A}}(x+a, yz + R_{\circ}^*(b)z + L_{\circ}^*(c)y + b \circ c + R^*(y)c + L^*(z)b) \\ &= \langle x, b \circ c + R^*(y)c + L^*(z)b \rangle + \langle yz + R_{\circ}^*(b)z + L_{\circ}^*(c)y, a \rangle \\ &+ \langle x, b \circ c \rangle + \langle x, R^*(y)c \rangle + \langle x, L^*(z)b \rangle + \langle yz, a \rangle + \langle R_{\circ}^*(b)z, a \rangle \\ &+ \langle L_{\circ}^*(c)y, a \rangle = \langle x, b \circ c \rangle + \langle R_y(x), c \rangle + \langle L_z(x), b \rangle \\ &+ \langle yz, a \rangle + \langle z, R_b(a) \rangle + \langle y, L_c(a) \rangle \\ &= \langle x, b \circ c \rangle + \langle xy, c \rangle + \langle zx, b \rangle + \langle yz, a \rangle + \langle z, a \circ b \rangle + \langle y, c \circ a \rangle. \end{aligned}$$

Hence,

$$S_{\mathcal{A}}((x+a) * (y+b), (z+c)) = S_{\mathcal{A}}((x+a), (y+b) * (z+c)) \quad (4.2)$$

which expresses the invariance of the standard bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$. Therefore, $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the standard Manin triple of the admissible hom-associative algebras $\mathcal{A}$ and $\mathcal{A}^*$.

• Conversely, suppose that $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is a standard Manin triple. Then, it follows that $(\mathcal{A} \oplus \mathcal{A}^*, *, \alpha \oplus \alpha^*)$ is an admissible hom-associative algebra. Hence, from Theorem 2.8 $(\mathcal{A}, \mathcal{A}^*, R^*, L^*, R_{\circ}^*, L_{\circ}^*, \alpha, \alpha^*)$ is a matched pair of the admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$. $\square$

Proposition 4.5 Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ be two admissible hom-associative algebras. Then, the following conditions are equivalent:

1. There is a double construction of a Manin triple associated to $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$;
2. $(\mathcal{A}, \mathcal{A}^*, R^*, L^*, \alpha^*, R_{\circ}^*, L_{\circ}^*, \alpha)$ is a matched pair of admissible hom-associative algebras;
3. $(\mathcal{A}, \mathcal{A}^*, \alpha, \alpha^*)$ is an antisymmetric infinitesimal admissible hom-bialgebra.

Proof: From Proposition 4.4 and Theorem 3.2, we have the equivalences. $\square$

5 Concluding Remarks

In this work, we have established a comprehensive theory of *admissible hom-associative algebras* and their corresponding bialgebras, advancing the structural understanding of twisted algebraic

systems. By introducing admissible bimodules and their compatibility conditions, we have shown that antisymmetric infinitesimal hom-bialgebras arise naturally as dual objects, generalizing the classical correspondence between associative bialgebras and Manin triples. Our main theorems reveal a tripartite equivalence among:

- **Matched pairs** of admissible hom-associative algebras,
- **Antisymmetric infinitesimal hom-bialgebras**, and
- **Manin triples** equipped with the standard bilinear form (4.2).

These results not only subsume existing work on hom-Lie bialgebras but also open new avenues for research in homological deformation theory and quantum group analogs. The key implications of our framework include:

- A unified approach to hom-associative and hom-Lie structures through admissibility conditions
- New tools for studying twisted bialgebraic structures in non-commutative geometry
- Natural connections to deformation quantization via the hom-Yang-Baxter equation

Future directions include:

- Applications to hom-Hopf algebras and their representation theory
- Classification of admissible hom-structures via cohomological invariants
- Exploration of the interplay between admissibility conditions and vertex operator algebras

The present work provides a foundation for further investigation into the rich algebraic structures emerging from the hom-associative admissible framework. 

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