Nonparametric method for Laplace transform estimating from empirical data using a symmetric kernel:Gaussian kernel case.

Abstract

We introduce a nonparametric estimation method for Laplace transform probability density, based on use a Gaussian kernel. The proposed estimator is studied from a theoretical perspective: we establish its convergence rate and analyze its asymptotic mean integrated squared error (AMISE), which is expressed directly in terms of the Laplace transform of the target density. Numerical simulations illustrate the performance of the method, which we also validate on real data to demonstrate its practical applicability.

Keywords:Laplace transform, Gaussian kernel, Integral estimator, Bandwidth .

2020MSC Classification: 60E10 ;62G05; 62G07

1 Introduction and notations

The analytical computation of the Laplace transform of certain probability densities often encounters difficulties due to the complexity of the functions involved.

In many situations for example, when the Laplace transform of the claim size distribution, the claim size distribution itself, or the inter-claim time distribution are not explicit or are unknown several authors have proposed estimation and approximation methods (see notably Goffard et al. [9], Touazi [16], Nitiema et al. [7], Harfouche and Bareche [20], Shimizu [19], Dussap [6]).

Mnatsakanov et al. [13] studied a nonparametric estimation of the Laplace transform of the survival probability in classical compound Poisson risk model, where the claim size distribution is unknown and is estimated using a discrete Laplace transform. Furthermore, to model situations in which claims can occur in large numbers over a short period, Zhang et al. [22] proposed a nonparametric estimation of the unknown density f_X using continuous kernel methods, relying on the Pollaczek-Khinchine formula to express the ruin probability.

More recently, Goffard et al. [11] introduced a kernel estimator of the Laplace transform of a density, with applications to goodness-of-fit tests for exponential and gamma distributions.

Mnatsakanov et al. [14] also proposed a nonparametric procedure based on an approximation of the Laplace transform inversion, allowing the estimation of the density of a positive random variable using a moment-reconstruction approach. They show that the mean squared error (MSE) of the estimated density is bounded, with a convergence rate of order $o\left(\frac{\sqrt{\alpha}}{n}\right)$, where $\alpha \sim n^{2/5}$.

In this paper, we propose a nonparametric estimation of the Laplace transform $\varphi(s)$, considered as a function of a complex variable s, of an unknown density $f_X(x)$, based on random observations and using the method of symmetric continuous kernels.

Our contribution lies in estimating the Laplace transform as a continuous function of the complex variable s, along with an analysis of the bias, variance, smoothing parameter, and the asymptotic mean integrated squared error (AMISE), all expressed in terms of the Laplace transform.

Assuming we are given a sample of independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots, X_n$, with mean $m = \mathbb{E}(X)$, and following an unknown density f_X, and under certain technical assumptions on the complex-variable function $\varphi(s)$, we define an empirical estimator of the Laplace transform, denoted $\hat{\varphi}_n(s)$. The paper is organized as follows:

Section 2 introduces the definition of the estimator $\hat{\varphi}_n(s)$ of the Laplace transform of the unknown density, and provides an expression for it, leading to the derivation of the optimal smoothing parameter as a function of the Laplace transform.

Section 3 is devoted to simulations, where the MISE is computed from the theoretical Laplace transform and the estimated one. For real data, the optimal smoothing parameter is selected using cross-validation based on the Laplace transform of the Gaussian kernel.

Section 4 concludes the paper.

Let us define:

- $K(\cdot)$: a continuous symmetric kernel with support $\mathbb{S}$, bounded on at least one side;
- $\mathbb{R}^+$: the set of positive real numbers;
- $\mathbb{E}$: the expectation operator;
- $\mathbb{V}ar$: the variance operator;
- COV: the covariance operator;

- $\mathbb{C}$: the set of complex numbers;
- $\mathbb{T}_{\mathbb{C}_+}$: a domain in $\mathbb{C}$ such that $\mathrm{Re}(s) > 0$ (strictly positive real part);
- L: the set of integrable functions;
- L^2: the set of square-integrable functions.

The theoretical results are based on the following assumptions:

H_1: f_X is Laplace-transformable;

H_2: $\varphi(s) \in L(\mathbb{T}_{\mathbb{C}_+})$, where $\varphi(s)$ denotes the Laplace transform of the density f_X;

H_3: $\varphi(s) \in L^2(\mathbb{T}_{\mathbb{C}_+})$, with $\mathbb{T}_{\mathbb{C}_+} \subset \mathbb{C}$;

H_4: $\int_{\mathbb{T}_{\mathbb{C}_+}} \varphi(s)\, ds \leqslant 1$;

H_5: The estimators $\hat{\varphi}_1(s), \hat{\varphi}_2(s), \dots, \hat{\varphi}_n(s)$ are assumed to be independent and identically distributed (i.i.d.).

2 Estimation of the Unknown Density via Laplace Transform

Inspired by the work of Zhang et al. [22], particularly their formulation of the Fourier transform applied to an unknown density, we propose an estimation of the Laplace transform of this density using a continuous symmetric kernel.

Definition 2.1. *Let $h > 0$ be the smoothing bandwidth and K a continuous symmetric kernel function with support $\mathbb{S}$. The pointwise continuous symmetric kernel estimator of the Laplace transform $\varphi(s)$, defined on a complex domain $\mathbb{T}_{\mathbb{C}_+}$, for an unknown density f_X with support $\mathbb{T} \subseteq \mathbb{R}^+$, is given at a point $x \in \mathbb{T}$ by:*

$$\hat{\varphi}_n(x, s) := \frac{1}{nh} \sum_{i=1}^{n} e^{-xs}\, K\left(\frac{x - X_i}{h}\right), \tag{1}$$

where x is the target point, X_i the i-th random variable, and $s \in \mathbb{T}_{\mathbb{C}_+}$ such that $\mathrm{Re}(s) > 0$.

We then give the definition of Laplace transform integral estimator for the unknown density at $x \in \mathbb{T}$:

Definition 2.2. *The continuous symmetric kernel integral estimator of the Laplace transform $\varphi(s)$ of the unknown density f_X with support $\mathbb{T}$ is defined for all $x \in \mathbb{T} \subseteq \mathbb{R}^+$ at a fixed point $s \in \mathbb{T}_{\mathbb{C}_+}$ by:*

$$\hat{\varphi}_n(s) := \int_{\mathbb{T}} \hat{\varphi}_n(x,s)dx, \tag{2}$$

where $\hat{\varphi}_n(x,s)$ is the pointwise estimator at a given point x defined by: (1).

The following lemma provides the bias and variance of the **integral estimator** $\hat{\varphi}_n(s)$ with continuous symmetric kernel K.

Lemma 2.1. *Let $h > 0$ a real number, K the continuous symmetric kernel with support $\mathbb{S}$, f_X the unknown density with support $\mathbb{T}$. Let $\varphi(x,s)$ a function integrable with respect to variables $x \in \mathbb{T}$, $s \in \mathbb{T}_{\mathbb{C}_+}$ with first and second derivatives bounded in $x \in \mathbb{T}$. Let $\hat{\varphi}_n(s)$ with support $\mathbb{T}_{\mathbb{C}_+}$ the integral estimator of Laplace transform of f_X,*

$$Biais\left\{\hat{\varphi}_n(s)\right\} = \frac{h^2}{2}\sigma_K^2 \int_{\mathbb{T}} \varphi''(x,s)\,dx + o\left(h^2\right),$$

$$\mathbb{V}ar\left\{\hat{\varphi}_n(s)\right\} = \frac{\theta_K}{nh}\left\{\int_{\mathbb{T}} e^{-xs}\varphi(x,s)dx\right\} + o\left(\frac{1}{nh}\right),$$

where

$$\sigma_K^2 = \int_{\mathbb{T}} w^2 K(w)\,dw < \infty,$$

$$\theta_K = \int_{\mathbb{T}} K^2(w)\,dw < \infty,$$

$$\varphi(x,s) = e^{-xs} f_X(x),$$

$$\varphi''(x,s) = \frac{\partial^2}{\partial x^2}\varphi(x,s),$$

Proof 2.1. *For a fixed point $x \in \mathbb{T}$ and fixed complex point $s \in \mathbb{T}_{\mathbb{C}}$, starting from the **pointwise estimator** $\hat{\varphi}_n(x,s)$ given by Definition 2.1, we have:*

$$\begin{aligned}
\mathbb{E}\left\{\hat{\varphi}_n(x,s)\right\} &= \mathbb{E}\left\{\frac{1}{nh}\sum_{i=1}^{n} e^{-xs}K\left(\frac{x-X_i}{h}\right)\right\} \\
&= \frac{1}{nh}\mathbb{E}\left\{\sum_{i=1}^{n} e^{-xs}K\left(\frac{x-X_i}{h}\right)\right\} \\
&= \frac{1}{nh}n\mathbb{E}\left\{e^{-xs}K\left(\frac{x-X_1}{h}\right)\right\} \\
&= \frac{1}{h}\mathbb{E}\left\{e^{-xs}K\left(\frac{x-X_1}{h}\right)\right\} \\
&= \frac{1}{h}\int_{\mathbb{T}} e^{-xs}K\left(\frac{x-z}{h}\right) f_X(z)dz,
\end{aligned} \tag{3}$$

by setting $\varphi(z,s) = e^{-xs} f_X(z)$, $-w = \frac{x-z}{h}$, $K(-w) = K(w)$ *thus* $z = wh + x$ *and* (3) *becomes:*

$$\mathbb{E}\left\{\hat{\varphi}_n(x,s)\right\} = \int_{\mathbb{T}} K\left(w\right) \varphi(x + wh, s) dw \tag{4}$$

and the second-order Taylor expansion of $\varphi(x + wh, s)$ *yields:*

$$\begin{aligned}
\mathbb{E}\left\{\hat{\varphi}_n(x,s)\right\} &= \int_{\mathbb{T}} K\left(w\right) \varphi\left(x + wh, s\right) dw \\
&= \int_{\mathbb{T}} K\left(w\right) \left\{ \varphi\left(x,s\right) + wh\varphi'\left(x,s\right) + \frac{1}{2}\left(wh\right)^2 \varphi''\left(x,s\right) + o\left(w^2h^2\right) \right\} dw \\
&= \int_{\mathbb{T}} K\left(w\right) \varphi\left(x,s\right) dw + h\varphi'(x,s) \int_{\mathbb{T}} wK\left(w\right) dw \\
&\quad + \frac{1}{2} h^2 \varphi''(x,s) \int_{\mathbb{T}} w^2 K\left(w\right) dw + o\left(h^2\right) \\
&= \varphi\left(x,s\right) \int_{\mathbb{T}} K\left(w\right) dw + h\varphi'\left(x,s\right) \int_{\mathbb{T}} wK\left(w\right) dw \\
&\quad + \frac{1}{2} h^2 \varphi''\left(x,s\right) \int_{\mathbb{T}} w^2 K\left(w\right) dw + o\left(h^2\right);
\end{aligned}$$

where

$$\varphi'(x,s) = \frac{\partial}{\partial x}\varphi(x,s),$$

$$\varphi''(x,s) = \frac{\partial^2}{\partial x^2}\varphi(x,s),$$

Let $\sigma_K^2 = \int_{\mathbb{T}} w^2 K\left(w\right) dw$ *and* $\theta_K = \int_{\mathbb{T}} K^2\left(w\right) dw$ *the kernel constants. From the properties of the continuous symmetric kernel, we obtain:*

$$\begin{aligned}
Biais\left\{\hat{\varphi}_n(x,s)\right\} &= \mathbb{E}\left\{\hat{\varphi}_n(x,s)\right\} - \varphi\left(x,s\right) \\
&= \frac{h^2}{2}\sigma_K^2 \varphi''\left(x,s\right) + o\left(h^2\right),
\end{aligned} \tag{5}$$

By integrating both sides of equality (5), *we obtain the bias of the* ***integral estimator*** $\hat{\varphi}_n(s)$*:*

$$\begin{aligned}
Biais\left\{\hat{\varphi}_n(s)\right\} &= Bias\left\{\int_{\mathbb{T}} \hat{\varphi}_n(x,s) dx\right\} \\
\text{(using independence)} &= \int_{\mathbb{T}} Bias\left\{\hat{\varphi}_n(x,s)\right\} dx \\
&= \int_{\mathbb{T}} \left\{ \frac{h^2}{2}\sigma_K^2 \varphi''\left(x,s\right) + o\left(h^2\right) \right\} dx \\
&= \frac{h^2}{2}\sigma_K^2 \int_{\mathbb{T}} \varphi''\left(x,s\right) dx + o\left(h^2\right)
\end{aligned}$$

where

$$\begin{aligned}\int_{\mathbb{T}} o\left(h^2\right) dx &= \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} o\left(h^2\right) dx \\ &= \sum_{i=1}^{n} o\left(h^2\right) \Delta x \\ &= \left|\mathbb{T}\right| .o\left(h^2\right) \\ &= o\left(h^2\right), \qquad (6)\end{aligned}$$

$$\begin{aligned}\mathbb{V}ar\left\{\hat{\varphi}_n(x,s)\right\} &= \mathbb{V}ar\left\{\frac{1}{nh}\sum_{i=1}^{n} e^{-xs} K\left(\frac{x-X_i}{h}\right)\right\} \\ &= \frac{1}{n^2h^2}\mathbb{V}ar\left\{\sum_{i=1}^{n} e^{-xs} K\left(\frac{x-X_i}{h}\right)\right\} \\ &= \frac{1}{n^2h^2}\left\{\sum_{i=1}^{n} V\left\{e^{-xs} K\left(\frac{x-X_i}{h}\right)\right\}\right\} \\ &\quad + \frac{1}{n^2h^2}\left\{2\sum_{1\leqslant i<j\leqslant n} \underbrace{COV\left\{e^{-xs}K\left(\frac{x-X_i}{h}\right); e^{-xs}K\left(\frac{x-X_j}{h}\right)\right\}}_{=\ 0,\text{using independence}}\right\} \\ &= \frac{1}{n^2h^2}\sum_{i=1}^{n} \mathbb{V}ar\left\{e^{-xs} K\left(\frac{x-X_i}{h}\right)\right\} \\ &= \frac{n}{n^2h^2}\mathbb{V}\left\{e^{-xs} K\left(\frac{x-X_1}{h}\right)\right\} \\ &= \frac{1}{nh^2}\mathbb{E}\left\{e^{-xs} K\left(\frac{x-X_1}{h}\right)\right\}^2 - \frac{1}{n^2h^2}\left\{E\left\{e^{-xs} K\left(\frac{x-X_1}{h}\right)\right\}\right\}^2 \\ &= \frac{1}{nh^2}\int_{\mathbb{T}}\left\{e^{-xs} K\left(\frac{x-z}{h}\right)\right\}^2 f_X\left(z\right) dz \\ &\quad -\frac{1}{n}\left\{\frac{1}{h}\int_{\mathbb{T}} e^{-xs} K\left(\frac{x-z}{h}\right) f_X\left(z\right) dz\right\}^2, \qquad (7)\end{aligned}$$

Setting $p(z,s) = e^{-xs}\varphi(z,s)$, $-w = \frac{x-z}{h}$, $K(-w) = K(w)$ *thus* $z = wh + x$, *equation* (7) *becomes:*

$$\begin{aligned}\mathbb{V}ar\left\{\hat{\varphi}_n(x,s)\right\} &= \frac{1}{nh^2}\int_{\mathbb{T}}\left\{K\left(\frac{x-z}{h}\right)\right\}^2 p(z,s)dz - \frac{1}{n}\left\{\frac{1}{h}\int_{\mathbb{T}} K\left(\frac{x-z}{h}\right)\varphi\left(z,s\right)dz\right\}^2 \\ &= \frac{1}{nh}\int_{\mathbb{T}}\left\{K\left(w\right)\right\}^2 p\left(x+wh,s\right)dw - \frac{1}{n}\left\{E\left\{\hat{\varphi}_n(x,s)\right\}\right\}^2 \qquad (8)\end{aligned}$$

A second-order Taylor expansion of $p\left(x+wh,s\right)$ *yields:*

$$p\left(x+wh,s\right)=p\left(x,s\right)+whp'\left(x,s\right)+\frac{1}{2}\left(wh\right)^{2}p''\left(x,s\right)+o\left(h^{2}\right), \tag{9}$$

where

$$p'(x,s)=\frac{\partial}{\partial x}p(x,s),$$

$$p''(x,s)=\frac{\partial^{2}}{\partial x^{2}}p(x,s),$$

By substituting the Taylor expansion of $p\left(x+wh,s\right)$ *into equation* (8)*, we obtain:*

$$\begin{aligned}\mathbb{V}ar\left\{\varphi_{n}(x,s)\right\} &= \frac{1}{nh}p\left(x,s\right)\int_{\mathbb{T}}K^{2}\left(w\right)dw+\frac{1}{n}p'\left(x,s\right)\int_{\mathbb{T}}wK^{2}\left(w\right)dw\\ &\quad -\frac{1}{n}\left[p\left(x,s\right)+Biais\left\{\hat{\varphi}_{n}(x,s)\right\}\right]^{2}+o\left(\frac{1}{nh}\right).\end{aligned} \tag{10}$$

As $n\longrightarrow\infty$ *then:*

$$\frac{1}{n}p'\left(x,s\right)\int_{\mathbb{T}}wK^{2}\left(w\right)dw\longrightarrow 0, \tag{11}$$

$$\frac{1}{n}\left[p\left(x,s\right)+Biais\left\{\hat{\varphi}_{n}(x,s)\right\}\right]^{2}\longrightarrow 0, \tag{12}$$

and

$$\begin{aligned}\mathbb{V}ar\left\{\hat{\varphi}_{n}(x,s)\right\} &= \frac{1}{nh}p\left(x,s\right)\int_{\mathbb{T}}K^{2}\left(w\right)dw+o\left(\frac{1}{nh}\right)\\ &= \frac{1}{nh}e^{-xs}\varphi\left(x,s\right)\int_{\mathbb{T}}K^{2}\left(w\right)dw+o\left(\frac{1}{nh}\right),\end{aligned} \tag{13}$$

Finally, by integrating both sides of (13) *with respect to* x *over* $\mathbb{T}$*, we obtain the variance of* ***integral estimator*** $\hat{\varphi}_{n}(s)$*:*

$$\begin{aligned}\mathbb{V}ar\left\{\hat{\varphi}_{n}(s)\right\} &= \mathbb{V}ar\left\{\int_{\mathbb{T}}\hat{\varphi}_{n}(x,s)dx\right\}\\ \textit{(using independence)} &= \int_{\mathbb{T}}\mathbb{V}ar\left\{\hat{\varphi}_{n}(x,s)\right\}dx\\ &= \int_{\mathbb{T}}\left\{\frac{\theta_{K}}{nh}p\left(x,s\right)+o\left(\frac{1}{nh}\right)\right\}dx\\ &= \frac{\theta_{K}}{nh}\int_{\mathbb{T}}e^{-xs}\varphi\left(x,s\right)dx+o\left(\frac{1}{nh}\right);\end{aligned}$$

where

$$\begin{aligned}\int_{\mathbb{T}} o\left(\frac{1}{nh}\right) dx &= \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} o\left(\frac{1}{nh}\right) dx \\ &= \sum_{i=1}^{n} o\left(\frac{1}{nh}\right) \Delta x \\ &= |\mathbb{T}| \, .o\left(\frac{1}{nh}\right) \\ &= o\left(\frac{1}{nh}\right). \end{aligned} \tag{14}$$

Remark 2.1. *Unlike the bias and variance of estimator for the unknown density (which are expressed in terms of the real target x), the bias and variance of Laplace transform estimator for unknown density are expressed in terms of Laplace variable s.*

Through the following theorem, we establish the quadratic approximation of integrated mean squared error (integral) for the Laplace transform estimator of the unknown density and its convergence rate h in case of a continuous symmetric kernel estimator.

Theorem 2.1. *Let $\varphi(s)$ with complex domain $\mathbb{T}_{\mathbb{C}_+}$ be the Laplace transform of the unknown density f_X, and K a continuous symmetric kernel. Let $\varphi(x,s)$ be a function integrable with respect to variables $x \in \mathbb{T}$, $s \in \mathbb{T}_{\mathbb{C}_+}$ with first and second derivatives bounded in $x \in \mathbb{T}$. Then AMISE and the optimal bandwidth minimizing AMISE are:*

$$\begin{aligned} AMISE(h) &= \frac{\theta_K}{nh} \int_{\mathbb{T}_{\mathbb{C}_+}} \left(\int_{\mathbb{T}} e^{-xs} \varphi(x,s)\, dx \right) ds \\ &+ \frac{\sigma_K^4 h^4}{4} \int_{\mathbb{T}_{\mathbb{C}_+}} \left(\int_{\mathbb{T}} \varphi''(x,s)\, dx \right)^2 ds, \end{aligned} \tag{15}$$

$$h_{opt} = n^{-\frac{1}{5}} \left\{ \frac{\theta_K \int_{\mathbb{T}_{\mathbb{C}_+}} \left(\int_{\mathbb{T}} e^{-xs} \varphi(x,s)\, dx \right) ds}{\sigma_K^4 \int_{\mathbb{T}_{\mathbb{C}_+}} \left(\int_{\mathbb{T}} \varphi''(x,s)\, dx \right)^2 ds} \right\}^{\frac{1}{5}}, \tag{16}$$

where θ_K, σ_K, $\varphi(x,s)$, and $\varphi''(x,s)$ are given in Lemma 2.1.

Proof 2.2.

$$
\begin{aligned}
AMSE(h,s) &= \mathbb{E}\left[\int_0^\infty (\hat{\varphi}_n(x,s)-\varphi(x,s))^2\,dx\right]\\
&= \mathbb{E}\left[(\hat{\varphi}_n(s)-\varphi(s))^2\right]\\
&= \mathbb{V}ar\left\{\hat{\varphi}_n(s)\right\}+\left\{Bias\hat{\varphi}_n(s)\right\}^2\\
&= \frac{\theta_K}{nh}\int_{\mathbb{T}} e^{-xs}\varphi\left(x,s\right)dx+\frac{h^4}{4}\sigma_K^4\left(\int_{\mathbb{T}}\varphi''\left(x,s\right)dx\right)^2, \qquad (17)
\end{aligned}
$$

after integrating with respect to complex variable s in (17), we obtain the asymptotic approximation of integrated mean squared error, denoted by $AMISE(h)$:

$$
\begin{aligned}
AMISE(h) &= \int_{\mathbb{T}_{\mathbb{C}_+}} AMSE(h,s)ds\\
&= \frac{\theta_K}{nh}\int_{\mathbb{T}_{\mathbb{C}_+}}\left(\int_{\mathbb{T}} e^{-xs}\varphi\left(x,s\right)dx\right)ds\\
&\quad+\frac{h^4}{4}\sigma_K^4\int_{\mathbb{T}_{\mathbb{C}_+}}\left(\int_{\mathbb{T}}\varphi''\left(x,s\right)dx\right)^2 ds, \qquad (18)
\end{aligned}
$$

by setting

$$\frac{\partial}{\partial h}AMISE(h)=0,$$

and after some calculations and simplifications, we obtain the result.

3 Simulations

To facilitate numerical simulations and interpretation of results, the complex variable s of Laplace transform is restricted to positive real values, as has also been adopted in several recent works such as Todorov et al. [17], Curato et al. [3], Mnatsakanov et al. [14], and Gadallah et al. [10].

We consider a known theoretical density, compute the optimal smoothing parameter h_{opt} given in (16) which will be used to evaluate the MISE after N replications, and compute the variance of the integral estimator $\hat{\varphi}_n$ given in (2). For each known distribution, we generate random variables following this distribution which are then estimated using the Gaussian kernel. The Laplace variable s is... All integrals over x and s are approximated using Euler's method (constant step discretization) implemented in MATLAB R2022b.

n	$X \sim$ Gauss $(0, 1)$			$X \sim$Gamma $(2, 1/2)$			$X \sim$ Weibull $(1, 3)$		
	h_{opt}	MISE	$\mathbb{V}$ar	h_{opt}	MISE	$\mathbb{V}$ar	h_{opt}	MISE	$\mathbb{V}$ar
50	0.0654	0.0101	1.8860e-04	0.0265	0.0072	6.1256e-05	0.3875	0.9878	0.1218
100	0.0570	0.0031	1.3833e-05	0.0231	0.0055	3.8449e-05	0.3373	0.7111	0.0565
500	0.0413	0.0011	2.4427e-06	0.0167	0.0032	7.7349e-06	0.2445	0.4413	0.0032
1000	0.0359	0.0006	5.8952e-07	0.0146	0.0016	2.2051e-06	0.2128	0.4169	0.0012
3000	0.0288	0.0002	6.6949e-08	0.0117	0.0004	1.4986e-07	0.1709	0.4117	0.0002
5000	0.0260	0.0001	1.5428e-08	0.0106	0.0004	9.4548e-08	0.1543	0.4229	0.0001

Table 1: Estimation of $\hat{\varphi}_n(s)$ for different probability densities, with $N = 50$ replications.

Comment 3.1. *Table 1 presents the performance of Laplace transform estimator $\hat{\varphi}_n(s)$ for different distributions and sample sizes. We observe that the mean integrated squared error (MISE) and variance systematically decrease with n, indicating better accuracy for larger samples. The estimation is significantly more stable for the Gaussian and Gamma distributions compared to Weibull distribution, which initially shows higher variance.*

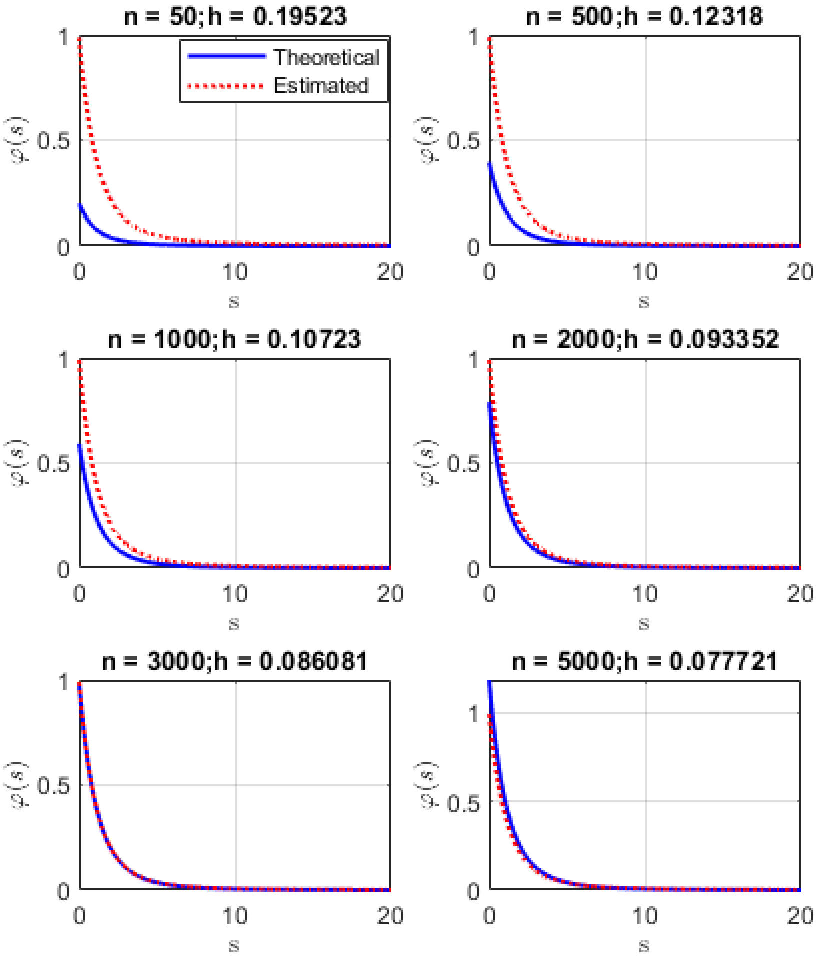

Figure 1: **Estimation of the Laplace transform for the Weibull(1, 3) density.**

Comment 3.2. *Figure 1 illustrates the convergence of Laplace transform estimator towards its theoretical value as the sample size n increases. The gradual decrease of the smoothing parameter h improves the estimation accuracy while controlling variance. For $n \geq 1000$, estimator shows excellent stability and becomes particularly reliable.*

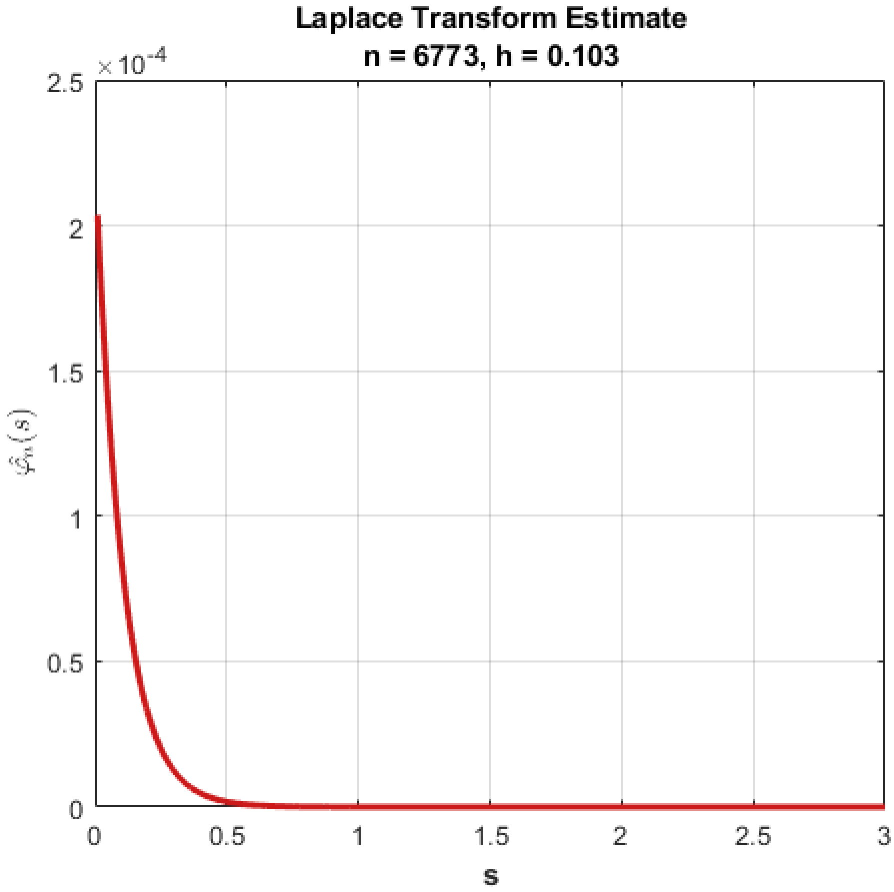

Figure 2: Evolution of $\hat{\varphi}_n(s)$ for the random claim severity function **AutoClaims**.

Comment 3.3. *Figure 2 shows evolution of Laplace transform estimator* $\hat{\varphi}_n(s)$ *as a function of variable* s*, for a sample of size* $n = 6773$ *and an optimal smoothing parameter* h*. In this case, MISE cannot be computed since the true density is both unknown and random. AutoClaims dataset is available in* `CASdatasets` *package of R software.*

4 Conclusion

We have proposed a nonparametric estimation method for Laplace transform of an unknown density based on a sample of independent and identically distributed random variables. This approach, founded on the use of continuous symmetric kernels, enables analytical expressions for the bias, variance, and asymptotic mean integrated squared error (AMISE) in terms of Laplace transform. It thus provides a flexible and robust framework for analyzing situations where explicit form of density, or its Laplace transform, is difficult or even impossible to obtain.

References

[1] Albrecher, H. and Boxma, O. J. (2004). *A ruin model with dependence between claim sizes and claim intervals*. Insurance: Mathematics and Economics, 35(2), 245–254. https://doi.org/10.1016/j.insmatheco.2003.09.009

[2] Chauveau, D. E., van Rooij, A. C. M., and Ruymgaart, F. H. (1994). *Regularized Inversion of Noisy Laplace Transforms*. Advances in Applied Mathematics, 15, 186–201. https://doi.org/10.1006/aama.1994.1007

[3] Imma V. C., Maria E. M., Maria C. R.. (2018) , *Spot volatility estimation using the Laplace transform*. Econometrics and Statistics, Volume 6, 2018, Pages 22-43, ISSN 2452-3062, https://doi.org/10.1016/j.ecosta.2016.07.002.

[4] Lima, F. D. F., Garcia, J. M. A., and Egidio dos Reis, A. D. (2002). *Fourier/Laplace Transforms and Ruin Probabilities*. Astin Bulletin, 32, 91–105. https://doi.org/10.2143/AST.32.1.1017

[5] Libengué Dobélé Kopoka, F. G. B. (2013). *Méthode non paramétrique des noyaux associés mixtes et application*. https://dspace.univ-ouargla.dz/jspui/handle/123456789/16808

[6] Dussap, F. (2022). *New Insights on Inverse Problems: Multidimensional Strategies for Deconvolution or Regression, and Ruin Probability Estimation*. Université Paris Cité. https://hal.science/tel-03726223

[7] Nitiema, P. C. et al. (2016). *On the best quadrature formula with weight function*. Far East Journal of Applied Mathematics, 95(5), 373–385. http://dx.doi.org/10.17654/AM095050373

[8] Geurts, S. A. F. (2019). *The ruin probability of insurance models with linear and stochastic premium*. https://research.tue.nl/en/studentTheses/the-ruin-probability

[9] Goffard, P.-O., Loisel, S., and Pommeret, D. (2015). *Polynomial approximations for bivariate aggregate claims amount probability distributions*. https://hal.science/hal-01098336v3

[10] Gadallah, A. M., Mohammed, B. I., Al-Babtain, A. A., Khosa, S. K., Kilai, M., Yusuf, M., and Bakr, M. E. (2022). *Modeling various survival distributions*

using a nonparametric hypothesis testing based on Laplace transform approach with some real applications. Computational and Mathematical Methods in Medicine, 2022(1), 5075716.

[11] Jammalamadaka, S. R., and Meintanis, S. G. (2022). *Goodness-of-fit tests for compound distributions with applications in insurance.* Journal of Statistical Theory and Practice, 16(3), 52. https://doi.org/10.1007/s42519-022-00276-6

[12] Jarman, K. D. H. (2002). *Some Maple applications to risk theory models.* Texas Tech University. http://hdl.handle.net/2346/21719

[13] Mnatsakanov, R., Ruymgaart, L., and Ruymgaart, F. (2008). *Nonparametric estimation of ruin probabilities given a random sample of claims.* Mathematical Methods of Statistics, 17, 35–43. https://doi.org/10.3103/S1066530708010031

[14] Elmagbri, F. and Mnatsakanov, R. M. (2018). *Nonparametric density estimation based on the scaled Laplace transform inversion.* https://doi.org/10.1016/j.trmi.2018.09.003

[15] Ross, S. M. (2003). *Introduction to Probability Models* (9th ed.). Academic Press. https://www.fcfm.buap.mx/jzacarias/cursos/procesos/libros/book8pe.pdf

[16] Touazi, A. et al. (2014). *Méthode du noyau dans l'analyse de stabilité forte d'un modèle de risque*. Colloque International, USTHB. https://hal.science/hal-01086342

[17] Todorov, V., and Tauchen, G. (2012). *Realized Laplace transforms for pure-jump semimartingales.*

[18] Usabel, M. (1999). *Calculating multivariate ruin probabilities via Gaver–Stehfest inversion technique.* Insurance: Mathematics and Economics, 25(2), 133–142. https://doi.org/10.1016/S0167-6687(99)00029-3

[19] Yasutaka, S. (2012). *Non-parametric estimation of the Gerber-Shiu function for the Wiener-Poisson risk model.* Scandinavian Actuarial Journal, 2012(1), 56–69. https://doi.org/10.1080/03461238.2010.523515

[20] Harfouche, Z. and Bareche, A. (2021). *Semi-parametric approach for approximating the ruin probability of classical risk models with large claims.* Communications in Statistics - Simulation and Computation. https://doi.org/10.1080/03610918.2021.1992636

[21] Zhang, Z. and Yang, H. (2013). *Non parametric estimate of the ruin probability in a pure-jump Lévy risk model*. Insurance: Mathematics and Economics, 53(1), 24–35. https://doi.org/10.1016/j.insmatheco.2013.04.004

[22] Zhang, Z., Yang, H., and Yang, H. (2014). *On a nonparametric estimator for ruin probability in the classical risk model*. Scandinavian Actuarial Journal, 2014(4), 309–338.