Original Research Article

On the Density of Some Name Graphs and Corona of Two Graphs

Abstract

Density in graph is a core metric in graph theory used to quantifies how close a graph is to being connected. The density of a graph is denoted by D(G) as the ratio of the number of edges present in the graph to the maximum possible edges.In undirected graph, the density is given by the ratio of twice the edges of the graph G and the product of the vertices of the graph G and one less than its vertices. In this paper, we ivestigates the mathematical properties of graph density in some name graph such as centipide, helm, crown, ladder and bistar graph. Furthermore, we compute the density of the corona two graphs.

Keywords: Density in graph, corona graph, order, size

2010 Mathematics Subject Classification:

1 Introduction

Graph theory has been independently discovered many times and may be regarded as an area of applied mathematics. It was further develop and has grown into significance mathematical research. One of them is the density graph. Density in graph is a way to quantify how sparse or dense a graph is. Sparse has relatively few edges compared to the number of nodes, while dense graph has many edges. In various application such as network analysis. bio informatics, and social network modeling, the level of interconnectedness affects how system function and evolve [3].

In there foundational work, Erdos, P. & Renyi A. [6] introduced random graph theory that eventually lead to the groundwork for understanding how graph density effects the properties of random network. In random graphs, density play in determining the phased transition between connected and disconnected graph. The study of random graph lead to the development of critical threshold for graph connectivity [6].

The study of graph density in real-world networks has attracted attention. Barabasi and Albert [2] highlight in their research how the density of a network still plays a critical role in its efficiency and robustness. that many real-world networks, like the internet and social networks, exhibit a scale-free structure that deviates from the idealized random graph model. their research highlighted how the density Specifically, a higher density of edges tends to lead to more resilient networks, capable of withstanding node failures.

Now, West, D.B [12] in his book " Introduction to Graph theory" which is a fundamental resources in graph, give the formal definition of density along with numerous application in theoretical and practical scenarios.

2 Preliminary Notes

This section introduces key definitions and concepts essential for the study.

Definition 2.0.1. [4] A **graph** G is a finite nonempty set V of objects called **vertices** together with a possibly empty set E of 2-element sets of V called **edge**s. To indicate that a graph G has vertex set V and edge set E, we write $G = (V, E)$. To emphasize that V and E are the vertex set and edge set of a graph G, we often write V as $V(G)$ and E as $E(G)$. Each edge $\{u, v\}$ of G is usually denoted by uv or vu. The number of vertices in a graph G is the **order** of G and the number of edges is the **size** of G. The degree of a vertex v is denoted by $deg(v)$ and the **minimum degree** of G is denoted by $\delta(G)$ and the **maximum degree** of G is denoted by $\Delta(G)$

Definition 2.0.2. [5] For an integer $n \geq 1$, the path graph P_n is a graph of order n and size $n-1$ whose vertices can be labeled as $v_0, v_1, v_2, ..., v_{n-1}$ and whose edges are v_i, v_{i+1} for $i = 0, 1, 2, ..., n-2$.

Definition 2.0.3. [5] For an integer $n \geq 2$, the cycle graph C_n is a graph of order n and size n whose vertices can be labeled as $v_0, v_1, v_2, ..., v_{n-1}$ and whose edges are $v_i v_{i-1}$ and $v_i v_{i+1}$ for $i = 0, 1, 2, ..., n-2$.

Definition 2.0.4. [1] The **centipide graph** $C_{n,2}$ is a graph obtained by appending a single pendant edges to each vertex of graph P_n, where P_n is the spine of $C_{n,2}$. Here, we have $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v_{n-1}\}$ such that $v_i v_{i+1} \in E$ with $0 \leq i \leq n-2$ and $v_i v'_i \in E$ with $0 \leq i \leq n-1$.

Definition 2.0.5. [7] The **helm graph** H_n is the graph obtained from wheel graph W_n by adjoining a pendant edge to each nodes of the cycle C_n. Here, we have $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v_{n-1}, r\}$ such that $v_i v_{i+1} \in E$ with $0 \leq i \leq n-2$ and $v_i v'_i, r v_i \in E$ with $0 \leq i \leq n-1$.

Definition 2.0.6. [11] A **bistar graph** $B_{r,s}$ is a graph obtained by joining the center (apex) vertices of two star graphs of order r and s respectively.

Definition 2.0.7. [10] A **ladder graph** L_n is defined as the cartesian product of P_2 and P_n where P_n is a path graph. Here, we have $V = \{v_0, v_1, ..., v_{n-1}, v'_0, v'_1, ..., v_{n-1}, r\}$ such that $v_i v_{i+1} + v'_i v'_{i+1} \in E$ with $0 \leq i \leq n-2$ and $v_i v'_i \in E$ with $0 \leq i \leq n-1$.

Definition 2.0.8. [9] A **crown graph** $H_{n,n}$ is graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching, that is, for every $v_i \in A$ and $u_i \in B$, $u_i \in B, v_i u_i \notin E_{n,n}$.

Definition 2.0.9. [8] the **corona** of two graph G (with n_G vertices and m_G edges) and H (with n_H vertices and m_H edges) $G \circ H$ is defines as the graph obtained by taking one copy of G and n_G copies of H and the joining the i^{th} vertices of G with an edges to every vertex in the i^{th} copies of H.

Definition 2.0.10. [12] Let G be undirected graph with m is the size and n the order of the graph. The density of the graph G, denoted as $D(G)$, is defined as the ratio of the number of edges in the graph to the maximum possible number of edges between n vertices. For the undirected graph, the density is given by the formula:

$$D(G) = \frac{2m}{n(n-1)}.$$

3 Main Results

This section presents key findings of the study, focusing on the density of centipede, helm, crown, ladder and bistar graph. Also the corona of two graphs.

3.1 On the density Some Name Graphs

This subsection we investigate the density of the centipede, helm, crown, ladder and bistar graph.

Theorem 3.1.1. *Let G be a centipede graph $C_{n,2}$ where $n \geq 1$. Then ,*

$$D(C_{n'2}) = \frac{1}{n}.$$

Proof. A centipede graph $C_{n,2}$ is constructed by joining the bottonms of n copies pf the path graph P_2. Thus the size of a centipede graph can be counted as $2n-1$ and the order is $2n$. Therefore,

$$D(C_{n,2}) = \frac{2m}{n(n-1)} = \frac{2(2n-1)}{2n(2n-1)} = \frac{1}{n}.$$

□

Theorem 3.1.2. *Let G be a helm graph (H_n) where $n \geq 3$. Then,*

$$D(H_n) = \frac{3}{2n+1}.$$

Proof. A helm graph H_n is constructed from an n-wheel graph by adjoining a pendant edge at each node of the cycle. Thus the size of a helm graph is $3n$ and the order is $2n+1$. Therefore,

$$D(H_n) = \frac{2m}{n(n-1)} = \frac{2(3n)}{(2n+1)(2n+1-1)} = \frac{2(3n)}{(2n+1)2n} = \frac{3}{2n+1}.$$

□

Theorem 3.1.3. *Let G be a crown graph $(H_{n,n})$ where $n \geq 3$. Then,*

$$D(H_{n,n}) = \frac{n-1}{2n-1}.$$

Proof. A crown graph $H_{n,n}$ is constructed where each vertex in the other set is connected to every vertex in the other set, except for the diagonal set. Thus the size of a crown graph is n^2-n and the order is $2n$. Therefore,

$$D(H_n) = \frac{2m}{n(n-1)} = \frac{2(n^2-n)}{2n(2n-1)} = \frac{n(n-1)}{n(2n-1)} = \frac{n-1}{2n-1}.$$

□

Theorem 3.1.4. *Let G be a ladder graph (L_n) where $n \geq 3$. Then,*

$$D(L_n) = \frac{3n-2}{2n^2-n}.$$

Proof. A ladder graph L_n is constructed by taking the cartesian product of two path graph. Thus the size of a ladder graph is $3n-2$ and the order is $2n$. Therefore,

$$D(H_n) = \frac{2m}{n(n-1)} = \frac{2(3n-2)}{2n(2n-1)} = \frac{(3n-2)}{n(2n-1)} = \frac{3n-2}{2n^2-n}.$$

□

Theorem 3.1.5. *Let G be a bistar graph $(B_{r,s})$ where $r \geq 3$ and $s \geq 3$. Then,*

$$D(B_{r,s}) = \frac{2}{r+s}.$$

Proof. A bistart graph $B_{r,s}$ is constructed by connecting the center vertices of two star graphs. Thus the size of a bistar graph is $r+s-1$ and the order is $r+s$. Therefore,

$$D(H_n) = \frac{2m}{n(n-1)} = \frac{2(r+s-1)}{(r+s)(r+s-1)} = \frac{2}{r+s}.$$

□

3.2 On the density of Corona Two Graphs

This subsection we compute the density of the corona of two graph such as the path and cycle graph.

By Definition 2.0.6, we can derived the size and order of a corona graph of a connected graph.

Remark 3.2.1. *Let G and H be a nontrivial connected graph. Then the order $G \circ H$ is $m_G + n_G(m_H) + (n_G(n_H))$ and the size of $G \circ H$ is $n_G + n_G(n_H)$ where m and n is the size and order of G and H.*

Theorem 3.2.2. *Let G and H be nontrivial connected graph. Then the density of $G \circ H$ is*

$$D(G \circ H) = \frac{2(m_G + n_G(m_H) + n_G(n_H))}{(n_G + n_G(n_H))(n_G + n_G(n_H) - 1)}.$$

Proof. By Remark 3.2.1,

$$\begin{aligned} D(G \circ H) &= \frac{2m}{(n(n-1)} \\ &= \frac{2(m_G + n_G(m_H) + n_G(n_H))}{(n_G + n_G(n_H))(n_G + n_G(n_H) - 1)}. \end{aligned}$$

□

Corollary 3.2.3. *Let G and H be a path graph (P_n) where $n \geq 2$. Then*

$$D(P_n \circ P_n) = \frac{2(2n^2 - 1)}{(n^2 + n)(n^2 + n - 1)}.$$

Proof. Note that the size of path P_n is $n-1$ and the order is n and by Theorem 3.2.2,

$$\begin{aligned} D(P_n \circ P_n) &= \frac{2(m_G + n_G(m_H) + n_G(n_H))}{(n_G + n_G(n_H))(n_G + n_G(n_H) - 1)} \\ &= \frac{2(n - 1 + n(n-1) + n(n))}{(n + n(n))(n + n(n) - 1)} \\ &= \frac{2(n - 1 + n^2 - n + n^2)}{(n + n^2)(n + n^2 - 1)} \\ &= \frac{2(2n^2 - 1)}{(n^2 + n)(n^2 + n - 1)}. \end{aligned}$$

□

Corollary 3.2.4. *Let G and H be a cycle graph (C_n) where $n \geq 3$. Then,*

$$D(C_n \circ C_n) = \frac{4n + 2}{(n+1)(n^2 + n - 1)}.$$

Proof.

$$
\begin{aligned}
D(C_n \circ C_n) &= \frac{2(m_G + n_G(m_H) + n_G(n_H))}{(n_G + n_G(n_H))(n_G + n_G(n_H) - 1)} \\
&= \frac{2(n + n(n) + n(n))}{(n + n(n))(n + n(n) - 1)} \\
&= \frac{2(n + n^2 + n^2)}{(n + n^2)(n + n^2 - 1)} \\
&= \frac{n(4n + 2)}{n(n+1)(n^2 + n - 1)} \\
&= \frac{4n + 2}{(n+1)(n^2 + n - 1)}.
\end{aligned}
$$

□

4 CONCLUSIONS

This paper investigate the density of centipede, helm, crown, ladder and bistar graph. Also the corona of two graph such as path and cycle. The result show that using the formula we can easily compute the density of those graph. We have also generate the general formula for the density in graph in the corona of two graphs.

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