Antisymmetric infinitesimal admissible hom-bialgebras

Abstract

This paper develops a novel framework for antisymmetric infinitesimal hom-bialgebras by introducing admissible bimodules of hom-associative algebras, generalizing the foundational work of [12]. We first construct admissible hom-associative bialgebras and rigorously establish their equivalence to matched pairs and Manin triples via a canonical bilinear form. Our approach systematically extends classical bialgebra theory to the hom-associative setting by imposing admissibility conditions on bimodules, which preserve the twisted multiplicativity of hom-structures. Key results include: (1) the characterization of admissible hom-associative algebras through their regular bimodules, (2) the derivation of antisymmetric infinitesimal admissible hom-bialgebras as dual structures, and (3) the proof that these bialgebras are categorically equivalent to Manin triples when equipped with a standard invariant bilinear form on the direct sum $\mathcal{A} \oplus \mathcal{A}^*$. This work unifies and generalizes prior studies on hom-Lie bialgebras and the hom-Yang-Baxter equation, while providing new tools for deformation theory and non-commutative geometry. The constructions are contextualized within prominent examples, including q-deformations and σ-derivations, bridging connections to the Witt and Virasoro algebras. Our results demonstrate that the hom-associative admissible framework offers a robust algebraic foundation for studying twisted bialgebraic structures.

Keywords. Admissible bimodules, admissible hom-associatives, hom-bialgebras, matched pairs, Manin triples
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1 Introduction

The foundational framework of Hom-algebra structures emerged through the quasi-deformation of Lie algebras associated with vector fields. By introducing discrete modifications via twisted derivations, this framework naturally extended to Hom-Lie and quasi-Hom-Lie algebras—structures characterized by a twisted Jacobi identity. Early breakthroughs in this direction arose from q-deformations, where classical derivations were systematically replaced by -derivations, with prominent examples including the Witt and Virasoro algebras (see [1, 7, 8, 10, 13]).

The theory of hom-coalgebras and related structures was developed [14, 15], with further advancements [2, 3, 5]. Studies on infinitesimal hom-bialgebras, hom-Lie bialgebras, and the hom-Yang-Baxter equation can be found in the litterature [17–19].

Haliya and Houndedji [14] introduced and studied quadratic hom-Jacobi-Jordan algebras, defined as hom-Jacobi-Jordan algebras with symmetric, invariant, and nondegenerate bilinear forms. A representation theory for hom-Jacobi-Jordan algebras, including adjoint and coadjoint representations were supplied with application to quadratic hom-Jacobi-Jordan algebras.

The principal goal of this work is to construct, by a new approach, the antisymmetric infinitesimal hom- bialgebras, which will be shown to be equivalent to Manin triples of admissible hom-associative algebras. We define admissible hom-associative algebras by exploring their regular bimodules under admissibility conditions. Furthermore, we define antisymmetric infinitesimal admissible hom- bialgebras, building on dual structures to establish their equivalence to matched pairs and Manin triples of admissible hom-associative algebras. By constructing a standard bilinear form on the direct sum of an admissible hom-associative algebra and its dual, we demonstrate the interconnection of these structures, paralleling the classical theory of associative algebras.

This paper is organized as follows: In Section 2, we define admissible hom-associative algebras their bimodules and matched pairs with a focus on admissibility conditions. Section 3 introduces the definition of antisymmetric infinitesimal admissible hom-bialgebras and provides the equivalence between matched pairs and antisymmetric infinitesimal admissible hom-bialgebras. Section 4 establishes the Manin triples of admissible hom-associative algebras and using a standard bilinear form and, provides the equivalence between matched pairs, antisymmetric infinitesimal admissible hom-bialgebras and Manin triples. Finally, Section 5 presents the concluding remarks.

2 Admissible hom-associative algebras

2.1 Preliminaries

Definition 2.1 *A hom-associative algebra is a triple $(\mathcal{A}, \cdot, \alpha)$ consisting of a linear space $\mathcal{A}$ over a field $\mathcal{K}$, $\mathcal{K}$-bilinear map $\cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a linear space map $\alpha : \mathcal{A} \to \mathcal{A}$ satisfying the hom-associativity property:*

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z). \tag{2.1}$$

Definition 2.2 *Let $(\mathcal{A}, \cdot, \alpha)$ be a hom-associative algebra. The hom-algebra is called*

- multiplicative hom-associative algebra *if $\forall x, y \in \mathcal{A}$ we have $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$;*
- regular hom-associative algebra *if α is an automorphism;*
- involutive hom-associative algebra *if α is an involution, that is $\alpha^2 = id$.*

Definition 2.3 *A hom-module is a pair (V, β), where V is a $\mathcal{K}$-vector space, and $\beta : V \to V$ is a linear map.*

Definition 2.4 *Let $(\mathcal{A}, \cdot, \alpha)$ be a hom-associative algebra and let (V, β) be a hom-module. Let $l, r : \mathcal{A} \to gl(V)$ be two linear maps. The quadruple (l, r, β, V) is called a bimodule of $\mathcal{A}$ if for all $x, y \in \mathcal{A}, v \in V$:*

$$l(x \cdot y)\beta(v) = l(\alpha(x))l(y)v, \tag{2.2}$$
$$r(x \cdot y)\beta(v) = r(\alpha(y))r(x)v, \tag{2.3}$$
$$l(\alpha(x))r(y)v = r(\alpha(y))l(x)v, \tag{2.4}$$
$$\beta(l(x)v) = l(\alpha(x))\beta(v), \tag{2.5}$$
$$\beta(r(x)v) = r(\alpha(x))\beta(v). \tag{2.6}$$

Proposition 2.5 *[12] Let $(\mathcal{A}, \cdot, \alpha)$ be a Hom-associative algebra and let (V, β) be a Hom-module. Let $l, r : \mathcal{A} \to gl(V)$ be two linear maps. The quadruple (l, r, β, V) satisfies a Hom-bimodule properties (2.2), (2.3), (2.4) of a Hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$ if and only if the direct sum of vector spaces, $\mathcal{A} \oplus V$, is turned into a Hom-associative algebra by defining multiplication in $\mathcal{A} \oplus V$ by*

$$\begin{aligned} (x_1 + v_1) * (x_2 + v_2) &= x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1), \\ (\alpha \oplus \beta)(x_1 + v_1) &= \alpha(x_1) + \beta(v_1) \end{aligned} \tag{2.7}$$

for all $x_1, x_2 \in \mathcal{A}, v_1, v_2 \in V$.

Proof: Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. The left-hand side and right-hand side of Hom-associativity of $(\mathcal{A} \oplus V, *, \alpha \oplus \beta)$ are expended as follows:

$$\begin{aligned}
&((x_1 + v_1) * (x_2 + v_2)) * (\alpha \oplus \beta)(x_3 + v_3)\\
&\quad = ((x_1 + v_1) * (x_2 + v_2)) * (\alpha(x_3) + \beta(v_3))\\
&\quad = (x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1)) * (\alpha(x_3) + \beta(v_3))\\
&\quad = (x_1 \cdot x_2) \cdot \alpha(x_3) + (l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))(l(x_1)v_2 + r(x_2)v_1)))\\
&\quad = (x_1 \cdot x_2) \cdot \alpha(x_3) + (l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1)\\
&(\alpha \oplus \beta)(x_1 + v_1) * ((x_2 + v_2) * (x_3 + v_3))\\
&\quad = (\alpha(x_1) + \beta(v_1)) * ((x_2 + v_2) * (x_3 + v_3))\\
&\quad = (\alpha(x_1) + \beta(v_1)) * (x_2 \cdot x_3 + (l(x_2)v_3 + r(x_3)v_2))\\
&\quad = \alpha(x_1) \cdot (x_2 \cdot x_3) + l(\alpha(x_1))(l(x_2)v_3 + r(x_3)v_2)) + r(x_2 \cdot x_3)\beta(v_1)\\
&\quad = \alpha(x_1) \cdot (x_2 \cdot x_3) + (l(\alpha(x_1))l(x_2)v_3 + l(\alpha(x_1))r(x_3)v_2 + r(x_2 \cdot x_3)\beta(v_1))
\end{aligned}$$

These elements of $\mathcal{A} \oplus V$ are equal if and only if

$$\begin{aligned}
&\alpha(x_1) \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot \alpha(x_3)\\
&l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1\\
&\qquad\qquad = l(\alpha(x_1))l(x_2)v_3 + l(\alpha(x_1))r(x_3)v_2 + r(x_2 \cdot x_3)\beta(v_1)
\end{aligned}$$

for all $x_1, x_2, x_3 \in \mathcal{A}, v_1, v_2, v_3 \in V$. This holds if and only if the hom-associativity holds, and for each $j = 1, 2, 3$ the respective V terms involving $v_j \in V$ are equal. If the terms are equal then the sums are equal. If the summs are equal then the terms should be equal if one specifies all or two of v_1, v_2, v_3 to zero element of V and using that linear transformations map zero to zero. Since,

$$\begin{aligned}
\text{Hom-associativity} \Leftrightarrow &\quad \alpha(x_1) \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot \alpha(x_3),\ \forall x_1, x_2, x_3 \in \mathcal{A}\\
(2.2) \Leftrightarrow &\quad l(x_1 \cdot x_2)\beta(v_3) = l(\alpha(x_1))l(x_2)v_3,\ \forall x_1, x_2, x_3 \in \mathcal{A}\\
(2.4) \Leftrightarrow &\quad l(\alpha(x_1))r(x_3)v_2 = r(\alpha(x_3))l(x_1)v_2,\ \forall x_1, x_2, x_3 \in \mathcal{A}\\
(2.3) \Leftrightarrow &\quad r(\alpha(x_3))r(x_2)v_1 = r(x_2 \cdot x_3)\beta(v_1),\ \forall x_1, x_2, x_3 \in \mathcal{A}.
\end{aligned}$$

the proof is complete. □

Example 2.6 *Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative Hom-associative algebra. Let $L_{\cdot x}$ and $R_{\cdot x}$ denote the left and right multiplication operators, respectively, i. e. $L_{\cdot x}(y) = x \cdot y, R_{\cdot x}(y) = y \cdot x$ for any $x, y \in \mathcal{A}$. Let $L_{\cdot} : \mathcal{A} \to gl(\mathcal{A})$ with $x \mapsto L_{\cdot x}$ and $R_{\cdot} : \mathcal{A} \to gl(\mathcal{A})$ with $x \mapsto R_{\cdot x}$ (for every $x \in \mathcal{A}$) be two linear maps. Then, the triples $(L_{\cdot}, 0, \alpha), (0, R_{\cdot}, \alpha)$ and $(L_{\cdot}, R_{\cdot}, \alpha)$ are bimodules of $(\mathcal{A}, \cdot, \alpha)$.*

Remark 2.7 *$(L_{\cdot}, R_{\cdot}, \alpha)$ is called the **regular bimodule** of $(\mathcal{A}, \cdot, \alpha)$.*

Theorem 2.8 *[12] Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two Hom-associative algebras. Suppose there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \to gl(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \to gl(\mathcal{A})$ such that the quadruple $(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta, \mathcal{B})$ is a bimodule of $\mathcal{A}$, and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha, \mathcal{A})$ is a bimodule of $\mathcal{B}$, satisfying, for any $x, y \in \mathcal{A}, a, b \in \mathcal{B}$, the following conditions:*

$$l_{\mathcal{A}}(\alpha(x))(a \circ b) = l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)\beta(b) + (l_{\mathcal{A}}(x)a) \circ \beta(b), \tag{2.8}$$

$$r_{\mathcal{A}}(\alpha(x))(a \circ b) = r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)\beta(a) + \beta(a) \circ (r_{\mathcal{A}}(x)b), \tag{2.9}$$

$$l_{\mathcal{B}}(\beta(a))(x \cdot y) = l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)\alpha(y) + (l_{\mathcal{B}}(a)x) \cdot \alpha(y), \tag{2.10}$$

$$r_{\mathcal{B}}(\beta(a))(x \cdot y) = r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)\alpha(x) + \alpha(x) \cdot (r_{\mathcal{B}}(a)y), \tag{2.11}$$

$$l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)\beta(b)+(r_{\mathcal{A}}(x)a)\circ\beta(b)-r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)\beta(a)-\beta(a)\circ(l_{\mathcal{A}}(x)b)=0, \tag{2.12}$$

$$l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)\alpha(y)+(r_{\mathcal{B}}(a)x)\cdot\alpha(y)-r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)\alpha(x)-\alpha(x)\cdot(l_{\mathcal{B}}(a)y)=0. \tag{2.13}$$

Then, there is a Hom-associative algebra structure on the direct sum $\mathcal{A}\oplus\mathcal{B}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$ given by

$$\begin{array}{rcl}(x+a)*(y+b) &=& (x\cdot y+l_{\mathcal{B}}(a)y+r_{\mathcal{B}}(b)x)+(a\circ b+l_{\mathcal{A}}(x)b+r_{\mathcal{A}}(y)a),\\ (\alpha\oplus\beta)(x+a) &=& \alpha(x)+\beta(a)\end{array} \tag{2.14}$$

for all $x,y\in\mathcal{A}, a,b\in\mathcal{B}$.

Proof: Let $v_1,v_2,v_3\in V$ and $x_1,x_2,x_3\in\mathcal{A}$. Set

$$[(x_1+v_1)*(x_2+v_2)]*(\alpha(x_3)+\beta(v_3))=(\alpha(x_1)+\beta(v_1))*[(x_2+v_2)*(x_3+v_3)],$$

which is developed to obtain (2.8)-(2.13). Then, using the relations

$$\beta(l_{\mathcal{A}}(x)a)=l_{\mathcal{A}}(\alpha(x))\beta(a),\ \beta(r_{\mathcal{A}}(x)a)=r_{\mathcal{A}}(\alpha(x))\beta(a),$$

$$\alpha(l_{\mathcal{B}}(a)x)=l_{\mathcal{B}}(\beta(a))\alpha(x),\ \alpha(r_{\mathcal{B}}(a)x)=r_{\mathcal{B}}(\beta(a))\alpha(x),$$

we show that $*$ is a Hom-associative algebra. □

Definition 2.9 *Let $(\mathcal{A},\cdot,\alpha)$ and $(\mathcal{B},\circ,\beta)$ be two Hom-associative algebras. Suppose that there are linear maps $l_{\mathcal{A}},r_{\mathcal{A}}:\mathcal{A}\to gl(\mathcal{B})$ and $l_{\mathcal{B}},r_{\mathcal{B}}:\mathcal{B}\to gl(\mathcal{A})$ such that $(l_{\mathcal{A}},r_{\mathcal{A}},\beta)$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}},r_{\mathcal{B}},\alpha)$ is a bimodule of $\mathcal{B}$. If the conditions (2.8) - (2.13) are satisfied, then, $(\mathcal{A},\mathcal{B},l_{\mathcal{A}},r_{\mathcal{A}},\beta,l_{\mathcal{B}},r_{\mathcal{B}},\alpha)$ is called a **matched pair of hom-associative algebras**.*

2.2 Bimodules of admissible hom-associative algebras

In this sequel, we consider a multiplicative hom-associative algebra $(\mathcal{A},\cdot,\alpha)$. Let (l,r,β,V) be a bimodule of the multiplicative hom-associative algebra $(\mathcal{A},\cdot,\alpha)$, and let $l^*,r^*:\mathcal{A}\to gl(V^*)$ be the linear maps given for all $x\in\mathcal{A}$, $a\in V^*$, $v\in V$, by

$$\langle l^*(x)a,v\rangle:=\langle l(x)v,a\rangle, \langle r^*(x)a,v\rangle:=\langle r(x)v,a\rangle. \tag{2.15}$$

It is clair that (r^*,l^*,β^*,V^*) is not a bimodule of $(\mathcal{A},\cdot,\alpha)$ in general.

Remark 2.10 *If α is involutive, then (r^*,l^*,β^*,V^*) is a bimodule of $(\mathcal{A},\cdot,\alpha)$. This case was considered in [12].*

Lemma 2.11 *Let (l,r,β,V) be a bimodule of $(\mathcal{A},\cdot,\alpha)$. Then, (r^*,l^*,β^*,V^*) is a bimodule of $(\mathcal{A},\cdot,\alpha)$ if and only if the following equations hold:*

$$r(\alpha^2(y))r(\alpha(x))v=r(y)r(\alpha(x))v; \tag{2.16}$$

$$l(\alpha^2(x))l(\alpha(y))v=l(x)l(\alpha(y))v; \tag{2.17}$$

$$l(y)r(\alpha(y))v=r(x)l(\alpha(y))v; \tag{2.18}$$

$$r(x)\beta(v)=\beta(r(\alpha(x))v);l(x)\beta(v)=\beta(l(\alpha(x))v) \tag{2.19}$$

Proof: We have:

$$\begin{aligned}\langle r^*(x\cdot y)\beta^*(a), v\rangle &= \langle \beta(r(x\cdot y)v), a\rangle = \langle r(\alpha(x\cdot y))\beta(v), a\rangle\\ &= \langle r(\alpha(x)\cdot\alpha(y))\beta(v), a\rangle = \langle r(\alpha^2(y))r(\alpha(x))v, a\rangle\\ &= \langle (r(y)r(\alpha(x)))^*a, v\rangle = \langle r^*(\alpha(x))r^*(y)a, v\rangle;\end{aligned}$$

$$\begin{aligned}\langle l^*(x\cdot y)\beta^*(a), v\rangle &= \langle \beta(l(x\cdot y)(v)), a\rangle = \langle l(\alpha(x\cdot y))\beta(v), a\rangle\\ &= \langle l(\alpha(x)\cdot\alpha(y))\beta(v), a\rangle = \langle l(\alpha^2(x))l(\alpha(y))\beta(v), a\rangle\\ &= \langle (l(x)l(\alpha(y)))^*a, v\rangle = \langle l^*(\alpha(y))l^*(x)a, v\rangle;\end{aligned}$$

$$\begin{aligned}\langle r^*(\alpha(x))l^*(y)a, v\rangle &= \langle l(y)r(\alpha(x))v, a\rangle\\ &= \langle r(x)l(\alpha(y))v, a\rangle = \langle l^*(\alpha(y))r^*(x)a, v\rangle;\end{aligned}$$

$$\begin{aligned}\langle \beta^*(r^*(x))a, v\rangle &= \langle r(x)(\beta(v)), a\rangle\\ &= \langle \beta(r(\alpha(x)))v, a\rangle = \langle r^*(\alpha(x))\beta^*(a), v\rangle;\end{aligned}$$

$$\begin{aligned}\langle \beta^*(l^*(x))a, v\rangle &= \langle l(x)(\beta(v)), a\rangle\\ &= \langle \beta(l(\alpha(x)))v, a\rangle = \langle l^*(\alpha(x))\beta^*(a), v\rangle;\end{aligned}$$

which leads to the conclusion. □

Definition 2.12 *A bimodule (l, r, β, V) of $(\mathcal{A}, \cdot, \alpha)$ is called* ***admissible*** *if (r^*, l^*, β^*, V^*) is a bimodule of $(\mathcal{A}, \cdot, \alpha)$, i.e. the conditions 2.16-2.19 in the lemma2.11 are satisfied.*

When we focus on the regular bimodule$(L_., R_., \alpha)$, we get:

Corollary 2.13 *Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative hom-associative algebra. The regular bimodule$(L_., R_., \alpha)$ is admissible if and only if the following equations hold:*

$$x\cdot y = \alpha^2(x)\cdot y;\quad y\cdot x = y\cdot\alpha^2(x); \tag{2.20}$$

$$x\cdot(z\cdot\alpha(y)) = (\alpha(x)\cdot z)\cdot y. \tag{2.21}$$

Definition 2.14 *A hom-associative algebra $(A, \cdot, \alpha)$ is* ***admissible*** *if its regular bimodule is admissible.*

Corollary 2.15 *Let $(A, \cdot, \alpha)$ be an admissible hom-associative algebra. Then,*

$$L^*_.(x)[(\alpha^*)^2(a)] = L^*_.(x)a; R^*_.(x)[(\alpha^*)^2(a)] = R^*_.(x)a, \tag{2.22}$$

$\forall x \in \mathcal{A}, a \in \mathcal{A}^*$.

Proof: $(A, \cdot, \alpha)$ is an admissible hom-associative algebra, then we have
$\alpha^2(x)\alpha^2(y) = x\cdot\alpha^2(y) = x\cdot y$. Hence,

$$\begin{aligned}\langle L^*_.(x)[(\alpha^*)^2(a)], y\rangle &= \langle \alpha^2(L_.(x)y), a\rangle = \langle \alpha^2(x\cdot y), a\rangle = \langle \alpha^2(x)\cdot\alpha^2(y), a\rangle\\ &= \langle x\cdot y, a\rangle = \langle L_.(x)y, a\rangle = \langle L^*_.(x)a, y\rangle.\end{aligned}$$

And in the same way, we show that $\langle \mathrm{R}^*_.(x)[(\alpha^*)^2(a)], y\rangle = \langle R^*_.(x)a, y\rangle$. □

Proposition 2.16 *Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ be two admissible hom-associative algebras. Then, for all $x, y \in \mathcal{A}, a, b \in \mathcal{A}^*$ we get*

$$b\circ L^*(\alpha(x))a = (\alpha^*)^2(b)\circ L^*(\alpha(x))a, b\circ R^*(\alpha(x))a = (\alpha^*)^2(b)\circ R^*(\alpha(x))a$$

Proof: We have:

$$\begin{aligned}\langle b\circ R^*(\alpha(x))a, y\rangle &= \langle R^*(\alpha(x))a, L^*_\circ(b)y\rangle = \langle a, L^*_\circ(b)y\cdot\alpha(x)\rangle = \langle a, \alpha^2(L^*_\circ(b)y)\cdot\alpha(x)\rangle \\ &= \langle a, R_\cdot(\alpha(x))[\alpha^2(L^*_\circ(b)y)]\rangle = \langle b\circ(\alpha^*)^2[R^*_\cdot(\alpha(x))a], y\rangle\end{aligned}$$

It follows that $b\circ R^*(\alpha(x))a = b\circ(\alpha^*)^2[R^*_\cdot(\alpha(x))a]$. In view of $(\mathcal{A}^*, \circ, \alpha^*)$ aslo being admissible, we have:

$$b\circ R^*(\alpha(x))a = b\circ(\alpha^*)^2[R^*_\cdot(\alpha(x))a] = (\alpha^*)^2(b)\circ(\alpha^*)^2[R^*_\cdot(\alpha(x))a] = (\alpha^*)^2(b)\circ R^*_\cdot(\alpha(x))a.$$

Analogously, we show that $b\circ L^*(\alpha(x))a = (\alpha^*)^2(b)\circ L^*(\alpha(x))a$. □

Theorem 2.17 *Let $(\mathcal{A}, \cdot, \alpha)$ be an admissible hom-associative algebra. Suppose that there is an admissible hom-associative algebra structure " $\circ$ " on its dual space $(\mathcal{A}^*, \alpha^*)$. Then, $(\mathcal{A}, \mathcal{A}^*, R^*_\cdot, L^*_\cdot, \alpha^*, R^*_\circ, L^*_\circ, \alpha)$ is a matched pair of admissible hom-associative algebras if and only if, for any $x, y \in \mathcal{A}$ and $a, b \in \mathcal{A}^*$,*

$$R^*_\cdot(\alpha(x))(a\circ b) = R^*_\cdot(L^*_\circ(a)x)\alpha^*(b) + (R^*_\cdot(x)a)\circ\alpha^*(b), \tag{2.23}$$

$$R^*_\cdot(R^*_\circ(a)x)\alpha^*(b) + L^*_\cdot(x)a\circ\alpha^*(b) = L^*_\cdot(L^*_\circ(b)x)\alpha^*(a) + \alpha^*(a)\circ(R^*_\cdot(x)b). \tag{2.24}$$

Proof: This follows the same proof of the Theorem3 in [12]. Obviously, (2.23) gives (2.8), and (2.24) reduces to (2.12) when $l_{\mathcal{A}} = R^*_\cdot, r_{\mathcal{A}} = L^*_\cdot$, $l_B = l_{\mathcal{A}^*} = R^*_\circ, r_B = r_{\mathcal{A}^*} = L^*_\circ$. Now, show that

$$(2.8) \iff (2.9) \iff (2.10) \iff (2.11)$$
$$\text{and } (2.12) \iff (2.13).$$

Suppose (2.23) and (2.24) are satisfied and show that one has

$$\begin{aligned}L^*_\cdot(\alpha(x))(a^*\circ b^*) &= L^*_\cdot(R^*_\circ(b^*)x)\alpha^*(a^*) + \alpha^*(a^*)\circ(L^*_\cdot(x)b^*)\\ R^*_\circ(\alpha^*(a^*))(x\cdot y) &= R^*_\circ(L^*_\cdot(x)a^*)\alpha(y) + (R^*_\circ(a)x)\cdot\alpha(y)\\ L^*_\circ(\alpha^*(a^*))(x\cdot y) &= L^*_\circ(R^*_\cdot(y)a^*)\alpha(x) + \alpha(x)\cdot(L^*_\circ(a^*)y)\\ R^*_\circ(R^*_\cdot(x)a^*)\alpha(y) &+ (L^*_\circ(a^*)x)\cdot\alpha(y) - L^*_\circ(L_\cdot(y)a^*)\alpha(x) - \alpha(x)\cdot(R^*_\circ(a)y) = 0.\end{aligned}$$

We have

$$\begin{aligned}\langle R^*_\cdot(x)a^*, y\rangle &= \langle L^*_\cdot(y)a^*, x\rangle = \langle y\cdot x, a^*\rangle,\\ \langle R^*_\circ(b^*)x, a^*\rangle &= \langle L^*_\circ(a^*)x, b^*\rangle = \langle a^*\circ b^*, x\rangle,\\ \alpha^*(R^*_\cdot(x)a^*) &= R^*_\cdot(\alpha(x))\alpha^*(a^*), \quad \alpha^*(L^*_\cdot(x)a^*) = L^*_\cdot(\alpha(x))\alpha^*(a^*),\\ \alpha(R^*_\circ(a^*)x) &= R^*_\circ(\alpha^*(a^*))\alpha(x), \quad \alpha(L^*_\circ(a^*)x) = L^*_\circ(\alpha^*(a^*))\alpha(x),\end{aligned}$$

for all $x, y \in \mathcal{A}, a^*, b^* \in \mathcal{A}^*$. Set $\alpha(x) = z, \alpha(y) = t$, $\alpha^*(a^*) = c^*$ and $\alpha^*(b^*) = d^*$. Then

[label=()]the statement (2.8) $\iff$ (2.9) follows from

$$\begin{aligned}\langle R^*_\cdot(\alpha(x))(a^*\circ b^*), y\rangle &= \langle y\cdot\alpha(x), a^*\circ b^*\rangle = \langle (L_\cdot(y)\circ\alpha)x, a^*\circ b^*\rangle\\ &= \langle x, \alpha^*(L^*_\cdot(y)(a^*\circ b^*))\rangle = \langle L^*_\cdot(\alpha(y))\alpha^*(a^*\circ b^*), x\rangle\\ &= \langle L^*_\cdot(\alpha(y))(\alpha^*(a^*)\circ\alpha^*(b^*)), x\rangle\\ &= \langle L^*_\cdot(\alpha(y))(c^*\circ d^*), x\rangle;\\ \langle R^*_\cdot(L^*_\circ(a^*)x)\alpha(b^*), y\rangle &= \langle y\cdot L^*_\circ(a^*)x, \alpha^*(b^*)\rangle = \langle L^*_\cdot(y)(\alpha^*(b^*)), L^*_\circ(a^*)x\rangle\\ &= \langle L^*_\circ(a^*)x, L^*_\cdot(y)(\alpha^*(b^*))\rangle\\ &= \langle a^*\circ(L^*_\cdot(y)(\alpha^*(b^*))), x\rangle\\ &= \langle \alpha^*(c^*)\circ(L^*_\cdot(y)(d^*)), x\rangle;\\ \langle (R^*_\cdot(x)a^*)\circ\alpha^*(b^*), y\rangle &= \langle R^*_\circ(\alpha^*(b^*))y, R^*_\cdot(x)a^*\rangle = \langle a^*, (R^*_\circ(\alpha^*(b^*))y)\cdot x\rangle\\ &= \langle L^*_\cdot[R^*_\circ(\alpha^*(b^*))y]a^*, x\rangle = \langle L^*_\cdot(R^*_\circ(d^*)y)\alpha^*(c^*), x\rangle;\end{aligned}$$

the statement (2.9) $\Longleftrightarrow$ (2.10) follows from

$$\begin{array}{rcl}
\langle L^*(\alpha(x))(a^*\circ b^*), y\rangle &=& \langle a^*\circ b^*, \alpha(x)\cdot y\rangle = \langle R^*_\circ(b^*)(\alpha(x)\cdot y), a^*\rangle \\
&=& \langle R^*_\circ(\alpha^*(d^*))(z\cdot y), a^*\rangle; \\
\langle \alpha^*(a^*)\circ (L^*_\cdot(x)b^*), y\rangle &=& \langle \alpha^*(a^*), R^*_\circ(L^*_\cdot(x)b^*)y\rangle = \langle a^*, \alpha[R^*_\circ(L^*_\cdot(x)b^*)y]\rangle \\
&=& \langle a^*, R^*_\circ[\alpha^*(L^*_\cdot(x)b^*)]\alpha(y)\rangle \\
&=& \langle a^*, R^*_\circ[L^*_\cdot(\alpha(x))\alpha^*(b^*)]\alpha(y)\rangle \\
&=& \langle a^*, R^*_\circ(L^*_\cdot(z)d^*)\alpha(y)\rangle; \\
\langle L^*_\cdot(R^*_\circ(b^*)x)\alpha^*(a^*), y\rangle &=& \langle (R^*_\circ(b^*)x)\circ y, \alpha^*(a^*)\rangle = \langle \alpha[(R^*_\circ(b^*)x)\circ y], a^*\rangle \\
&=& \langle (R^*_\circ(\alpha^*(b^*))\alpha(x))\circ\alpha(y), a^*\rangle \\
&=& \langle R^*_\circ(d^*)z\cdot\alpha(y), a^*\rangle;
\end{array}$$

the statement (2.8) $\Longleftrightarrow$ (2.11) follows from

$$\begin{array}{rcl}
\langle R^*(\alpha(x))(a^*\circ b^*), y\rangle &=& \langle a^*\circ b^*, y\cdot\alpha(x)\rangle = \langle L_\cdot(a^*)b^*, y\cdot z\rangle \\
&=& \langle L^*_\circ(a^*)(y\cdot z)\rangle = \langle L^*_\circ(\alpha^*(c^*))(y\cdot z)\rangle; \\
\langle (R^*_\cdot(x)a^*)\circ\alpha^*(b^*), y\rangle &=& \langle \alpha^*(b^*), L^*_\cdot(R^*_\cdot(x)a^*)y\rangle = \langle b^*, \alpha^*[L^*_\cdot(R^*_\cdot(x)a^*)y]\rangle \\
&=& \langle b^*, L^*_\cdot(R^*_\cdot(\alpha(x))\alpha^*(a^*))\alpha(y)\rangle \\
&=& \langle b^*, L^*_\cdot(R^*_\cdot(z)c^*)\alpha(y)\rangle; \\
\langle R^*_\cdot(L^*_\circ(a^*)x)\alpha^*(b^*), y\rangle &=& \langle y\cdot L^*_\circ(a^*)x, \alpha^*(b^*)\rangle = \langle \alpha(y)\cdot\alpha(L^*_\circ(a^*)x), b^*\rangle \\
&=& \langle \alpha(y)\cdot L^*_\circ(\alpha^*(a^*))\alpha(x), b^*\rangle = \langle \alpha(y)\cdot L^*_\circ(c^*)z, b^*\rangle;
\end{array}$$

the statement (2.12) $\Longleftrightarrow$ (2.13) follows from

$$\begin{array}{rcl}
\langle L^*_\cdot(L^*_\circ(b^*)x)\alpha^*(a^*), y\rangle &=& \langle (L^*_\circ(b^*)x)\cdot y, \alpha^*(a^*)\rangle = \langle a^*, \alpha(L^*_\circ(b^*)x)\cdot\alpha(y)\rangle \\
&=& \langle a^*, L^*_\circ(\alpha^*(b^*))\alpha(x)\cdot\alpha(y)\rangle = \langle a^*, L^*_\circ(d^*)z\cdot\alpha(y)\rangle; \\
\langle \alpha^*(a^*)\circ(R^*_\cdot(x)b^*), y\rangle &=& \langle R^*_\circ(R^*_\circ(x)b^*)y, \alpha^*(a^*)\rangle = \langle \alpha^*(a^*)\circ(R^*_\cdot(x)b^*), y\rangle \\
&=& \langle \alpha[R^*_\circ(R^*_\circ(x)b^*)y], a^*\rangle \\
&=& \langle R^*_\circ[R^*_\circ(\alpha(x))\alpha^*(b^*)]\alpha(y), a^*\rangle \\
&=& \langle R^*_\circ(R^*_\cdot(z)d^*)\alpha(y), a^*\rangle; \\
\langle (L^*_\cdot(x)a^*)\circ\alpha^*(b^*), y\rangle &=& \langle R^*_\circ(\alpha^*(b^*))y, L^*_\cdot(x)a^*\rangle = \langle x\cdot(R^*_\circ(d^*)y), a^*\rangle \\
&=& \langle \alpha(z)\cdot(R^*_\circ(d^*)y), a^*\rangle; \\
\langle R^*_\cdot(R^*_\circ(a^*)x)\alpha^*(b^*), y\rangle &=& \langle y\cdot R^*_\circ(a^*)x, \alpha^*(b^*)\rangle = \langle \alpha^*(b^*), L_\cdot(y)(R^*_\circ(a^*)x)\rangle \\
&=& \langle (L^*_\cdot(y)(d^*), R^*_\circ(a^*)x\rangle = \langle L^*_\cdot(y)d^*\circ a^*, x\rangle \\
&=& \langle L^*_\circ(L^*_\cdot(y)d^*)x, a^*\rangle = \langle L^*_\circ(L^*_\cdot(y)d^*)\alpha(z), a^*\rangle
\end{array}$$

which completes the proof. □

3 Antisymmetric infinitesimal admissible hom-bialgebras

3. **Definition 3.1** *A pair of admissible hom-associative algebras $(\mathcal{A},\cdot,\alpha)$ and $(\mathcal{A}^*,\circ,\alpha^*)$ is called an **antisymmetric infinitesimal admissible hom-bialgebra** if*

$$\langle\Delta(x\cdot y), \alpha^*(a)\otimes b\rangle = \langle(\alpha\otimes L_\cdot(\alpha(x)))\Delta(y), \alpha^*(a)\otimes b\rangle + \langle(R_\cdot(\alpha(y))\otimes\alpha)\Delta(y), \alpha^*(a)\otimes b\rangle \quad (3.1)$$

$$\begin{array}{l}
\langle(L_\cdot(\alpha(y))\otimes\alpha), \alpha^*(a)\otimes b\rangle - \langle(\alpha\otimes R_\cdot(\alpha(y)))\Delta(x), \alpha^*(a)\otimes b\rangle = \\
-\langle\sigma[(L_\cdot(\alpha(x))\otimes\alpha)\Delta(y)], \alpha^*(a)\otimes b\rangle + \langle\sigma[(\alpha\otimes R_\cdot(\alpha(x)))\Delta(y), \alpha^*(a)\otimes b]\rangle
\end{array} \quad (3.2)$$

Theorem 3.2 *A pair of admissible hom-associative algebras $(\mathcal{A},\cdot,\alpha)$ and $(\mathcal{A}^*,\circ,\alpha^*)$ is an antisymmetric infinitesimal admissible hom-bialgebras if and only if*
$(\mathcal{A},\mathcal{A}^*,R^*_\cdot,L^*_\cdot,\alpha^*,R^*_\circ,L^*_\circ,\alpha)$ is a matched pair of admissible hom-associative algebras.

Proof: On the one hand, we have:

$$\langle\Delta(x\cdot y), \alpha^*(a)\otimes b\rangle = \langle x\cdot y, \alpha^*(a)\circ b\rangle = \langle x\cdot y, L_\circ(\alpha^*(a))b\rangle = \langle L^*_\circ(\alpha^*(a))(x\cdot y), b\rangle;$$

$$\begin{aligned}\langle \alpha(x)\cdot(L_\circ^*(a)y), b\rangle &= \langle L_\circ^*(a)y, L_\cdot^*(\alpha(x))b\rangle = \langle y, a\circ L_\cdot^*(\alpha(x))b\rangle = \langle y, (\alpha^*)^2(a)\circ L_\cdot^*(\alpha(x))b\rangle\\ &= \langle y, \Delta^*[(\alpha^*)^2(a)\otimes L_\cdot^*(\alpha(x))b]\rangle = \langle \Delta(y), (\alpha^*)^2(a)\otimes L_\cdot^*(\alpha(x))b\rangle\\ &= \langle \Delta(y), (\alpha^*\otimes L_\cdot^*(\alpha(x))(\alpha^*(a)\otimes b)\rangle = \langle(\alpha\otimes L_\cdot(\alpha(x))\Delta(y), \alpha^*(a)\otimes b\rangle;\end{aligned}$$

$$\begin{aligned}\langle L_\circ^*(R_\cdot^*(y)a)\alpha(x), b\rangle &= \langle \alpha(x), L_\circ(R_\cdot^*(y)a)b\rangle = \langle \alpha(x), (R_\cdot^*(y)a)\circ b\rangle = \langle x, \alpha^*(R_\cdot^*(y)a))\circ\alpha^*(b)\rangle\\ &= \langle x, \Delta^*[\alpha^*(R_\cdot^*(y)a))\otimes\alpha^*(b)]\rangle = \langle \Delta(x), \alpha^*(R_\cdot^*(y)a))\otimes\alpha^*(b)\rangle\\ &= \langle \Delta(x), R_\cdot^*(\alpha(y))\alpha^*(a)\otimes\alpha^*(b)\rangle = \langle \Delta(x), (R_\cdot^*(\alpha(y))\otimes\alpha^*)(\alpha^*(a)\otimes b)\rangle\\ &= \langle (R_\cdot(\alpha(y))\otimes\alpha)\Delta(y), \alpha^*(a)\otimes b\rangle.\end{aligned}$$

Then, 3.1 $\iff$ 2.23. And on the other hand, we have:

$$\begin{aligned}\langle L_\circ^*(L_\cdot^*(y)a)\alpha(x), b\rangle &= \langle \alpha(x), L_\cdot^*(y)a\circ b\rangle = \langle x, L_\cdot^*(\alpha(y))\alpha^*(a)\circ\alpha^*(b)\rangle\\ &= \langle \Delta(x), (L_\cdot^*(\alpha(y))\otimes\alpha^*)(\alpha^*(a)\otimes b)\rangle\\ &= \langle (L_\cdot(\alpha(y))\otimes\alpha)\Delta(x), \alpha^*(a)\otimes b\rangle;\end{aligned}$$

$$\begin{aligned}\langle R_\cdot(\alpha(y))L_\circ^*(a)x, b\rangle &= \langle L_\circ^*(a)x, R_\cdot^*(\alpha(y))b\rangle = \langle x, L_\circ(a)[R_\cdot^*(\alpha(y))b]\rangle = \langle x, a\circ[R_\cdot^*(\alpha(y))b]\rangle\\ &= \langle x, (\alpha^*)^2(a)\circ[R_\cdot^*(\alpha(y))b]\rangle = \langle \Delta(x), (\alpha^*)^2(a)\otimes[R_\cdot^*(\alpha(y))b]\rangle\\ &= \langle \Delta(x), (\alpha^*\otimes R_\cdot^*(\alpha(y)))(\alpha^*(a)\otimes b)\rangle = \langle(\alpha\otimes R_\cdot(\alpha(y)))\Delta(x), \alpha^*(a)\otimes b\rangle;\end{aligned}$$

$$\begin{aligned}\langle \alpha(x)\cdot(R_\circ^*(a)y), b\rangle &= \langle L_\cdot(\alpha(x))(R_\circ^*(a)y), b\rangle = \langle (R_\circ^*(a)y), L_\cdot^*(\alpha(x))b\rangle = \langle y, L_\cdot^*(\alpha(x))b\circ a\rangle\\ &= \langle y, L_\cdot^*(\alpha(x))b\circ(\alpha^*)^2(a)\rangle = \langle \Delta(y), (L_\cdot^*(\alpha(x))\otimes\alpha^*)(b\otimes\alpha^*(a))\rangle\\ &= \langle \Delta(y), (L_\cdot^*(\alpha(x))\otimes\alpha^*)\sigma^*(\alpha^*(a)\otimes b)\rangle\\ &= \langle \sigma[L_\cdot(\alpha(x))\otimes\alpha]\Delta(y), \alpha^*(a)\otimes b\rangle;\end{aligned}$$

$$\begin{aligned}\langle R_\circ^*(R_\cdot^*(x)a)\alpha(y), b\rangle &= \langle \alpha(y), b\circ(R_\cdot^*(x)a)\rangle = \langle y, \alpha^(b)\circ(R_\cdot^*(\alpha(x))\alpha^*(a))\rangle\\ &= \langle \Delta(y), (\alpha^*\otimes R_\cdot^*(\alpha(x)))(\sigma^*(\alpha^*(a)\otimes b))\rangle\\ &= \langle \sigma(\alpha^*\otimes R_\cdot^*(\alpha(x)))\Delta(y), \alpha^*(a)\otimes b)\rangle.\end{aligned}$$

Hence, 3.2 $\iff$ 2.24. □

4 Manin triple of admissible hom-associative algebras

Definition 4.1 *A Manin triple of admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, *, \alpha')$ is a triple $(\mathcal{A}\oplus\mathcal{B}, \mathcal{A}, \mathcal{B})$, together with a nondegenerate symmetric bilinear form $S(\cdot, \cdot)$ on the admissible hom-associative algebra $(\mathcal{A}\oplus\mathcal{B}, \star, \alpha\oplus\alpha')$ such that*

- *S is invariant, i. e. for all $x, y, z \in \mathcal{A}$ and $a, b, c \in \mathcal{B}$,*

$$\begin{aligned}S((x+a)\star(y+b), (z+c)) &= S((x+a), (y+b)\star(z+c)),\\ S((\alpha\oplus\alpha')(x+a), y+b) &= S(x+a, (\alpha\oplus\alpha')(y+b)).\end{aligned}$$

- *The admissible hom-associative algebras $\mathcal{A}$ and $\mathcal{B}$ are isotropic admissible hom-associative algebras of $A\oplus\mathcal{B}$.*

Corollary 4.2 *Let $(\mathcal{A}, \cdot, \alpha)$ be an admissible hom-associative algebra. Suppose there exists an admissible hom-associative algebra algebra structure on its dual space $\mathcal{A}^*$ denoted by $(\mathcal{A}^*, \circ, \alpha^*)$. Then, there is $(\mathcal{A}\oplus\mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ an admissible hom-associative algebra on the direct sum of the underlying vector space of $\mathcal{A}$ and its dual space $\mathcal{A}^*$ such that, $(\mathcal{A}\oplus\mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the associated Manin triple with the invariant bilinear symmetric form given by*

$$S_{\mathcal{A}}(x+a, y+b) = < x, b > + < y, a >, \tag{4.1}$$

for all $x, y \in \mathcal{A}$ and $a, b \in \mathcal{A}^*$.

Definition 4.3 *$(\mathcal{A}\oplus\mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is called the* ***standard Manin triple of the admissible hom-associative algebra $\mathcal{A}$ associated to*** *$S_{\mathcal{A}}$; where $S_{\mathcal{A}}$ is a natural paring between algebra and its dual space.*

Proposition 4.4 *Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ be two admissible hom-associative algebras. Then, $(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ}, \alpha, \alpha^*)$ is a matched pair of the admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ if and only if $(\mathcal{A}\oplus\mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is a standard Manin triple.*

Proof: • First, suppose that $(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ}, \alpha, \alpha^*)$ is a matched pair of the admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha,)$ and $(\mathcal{A}^*, \circ, \alpha^*)$. Then, $(\mathcal{A}\oplus\mathcal{A}^*, *, \alpha\oplus\alpha^*)$ is an admissible hom-associative algebra, with the product $*$ defined by 2.14. Consequently, let us compute and compare the following relations: $S_{\mathcal{A}}\left((x+a)*(y+b),(z+c)\right)$ and $S_{\mathcal{A}}\left((x+a),(y+b)*(z+c)\right)$ $\forall x, y, z \in \mathcal{A}$ and $\forall a, b, c \in \mathcal{A}^*$.

$$\begin{aligned}
S_{\mathcal{A}}((x+a)*(y+b),(z+c)) &= S_{\mathcal{A}}(xy + R^*_{\circ}(a)y + L^*_{\circ}(b)x + a\circ b + R^*_{\cdot}(x)b + L^*_{\cdot}(y)a, z+c)\\
&= \langle xy + R^*_{\circ}(a)y + L^*_{\circ}(b)x, c\rangle + \langle z, a\circ b + R^*_{\cdot}(x)b + L^*_{\cdot}(y)a\rangle\\
&= \langle xy, c\rangle + \langle R^*_{\circ}(a)y, c\rangle + \langle L^*_{\circ}(b)x, c\rangle + \langle z, a\circ b\rangle + \langle z, R^*_{\cdot}(x)b\rangle\\
&+ \langle z, L^*_{\cdot}(y)a\rangle = \langle xy, c\rangle + \langle y, R_a(c)\rangle + \langle x, L_b(c)\rangle + \langle z, a\circ b\rangle\\
&+ \langle R_x(z), b\rangle + \langle L_y(z), a\rangle\\
&= \langle xy, c\rangle + \langle y, c\circ a\rangle + \langle x, b\circ c\rangle + \langle z, a\circ b\rangle + \langle zx, b\rangle + \langle yz, a\rangle\,.
\end{aligned}$$

$$\begin{aligned}
S_{\mathcal{A}}\left((x+a),(y+b)*(z+c)\right) &= S_{\mathcal{A}}(x+a, yz + R^*_{\circ}(b)z + L^*_{\circ}(c)y + b\circ c + R^*_{\cdot}(y)c + L^*_{\cdot}(z)b)\\
&= \langle x, b\circ c + R^*_{\cdot}(y)c + L^*_{\cdot}(z)b\rangle + \langle yz + R^*_{\circ}(b)z + L^*_{\circ}(c)y, a\rangle\\
&+ \langle x, b\circ c\rangle + \langle x, R^*_{\cdot}(y)c\rangle + \langle x, L^*_{\cdot}(z)b\rangle + \langle yz, a\rangle + \langle R^*_{\circ}(b)z, a\rangle\\
&+ \langle L^*_{\circ}(c)y, a\rangle = \langle x, b\circ c\rangle + \langle R_y(x), c\rangle + \langle L_z(x), b\rangle\\
&+ \langle yz, a\rangle + \langle z, R_b(a)\rangle + \langle y, L_c(a)\rangle\\
&= \langle x, b\circ c\rangle + \langle xy, c\rangle + \langle zx, b\rangle + \langle yz, a\rangle + \langle z, a\circ b\rangle + \langle y, c\circ a\rangle\,.
\end{aligned}$$

Hence,

$$S_{\mathcal{A}}((x+a)*(y+b),(z+c)) = S_{\mathcal{A}}\left((x+a),(y+b)*(z+c)\right) \tag{4.2}$$

which expresses the invariance of the standard bilinear form on $\mathcal{A}\oplus\mathcal{A}^*$. Therefore, $(A\oplus\mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the standard Manin triple of the admissible hom-associative algebras algebras $\mathcal{A}$ and $\mathcal{A}^*$.

• Conversely, suppose that $(\mathcal{A}\oplus\mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is a standard Manin triple. Then, it follows that $(\mathcal{A}\oplus\mathcal{A}^*, *, \alpha\oplus\alpha^*)$ is an admissible hom-associative algebra. Hence, from Theorem2.8 $(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ}, \alpha, \alpha^*)$ is a matched pair of the admissible hom-associative algebras $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$. □

Proposition 4.5 *Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ be two admissible hom-associative algebras. Then, the following conditions are equivalent:*

1. *There is a double construction of a Manin triple associated to $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$;*
2. *$(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, \alpha^*, R^*_{\circ}, L^*_{\circ}, \alpha)$ is a matched pair of admissible hom-associative algebras;*
3. *$(\mathcal{A}, \mathcal{A}^*, \alpha, \alpha^*)$ is an antisymmetric infinitesimal admissible hom-bialgebra.*

Proof: From Proposition4.4 and Theorem3.2, we have the equivalences. □

5 Concluding Remarks

In this work, we have established a comprehensive theory of *admissible hom-associative algebras* and their corresponding bialgebras, advancing the structural understanding of twisted algebraic systems. By introducing admissible bimodules and their compatibility conditions, we have shown that antisymmetric infinitesimal hom-bialgebras arise naturally as dual objects, generalizing the classical correspondence between associative bialgebras and Manin triples. Our main theorems reveal a tripartite equivalence among:

- **Matched pairs** of admissible hom-associative algebras,
- **Antisymmetric infinitesimal hom-bialgebras**, and
- **Manin triples** equipped with the standard bilinear form (4.2).

These results not only subsume existing work on hom-Lie bialgebras but also open new avenues for research in homological deformation theory and quantum group analogs. The key implications of our framework include:

- A unified approach to hom-associative and hom-Lie structures through admissibility conditions
- New tools for studying twisted bialgebraic structures in non-commutative geometry
- Natural connections to deformation quantization via the hom-Yang-Baxter equation

Future directions include:

- Applications to hom-Hopf algebras and their representation theory
- Classification of admissible hom-structures via cohomological invariants
- Exploration of the interplay between admissibility conditions and vertex operator algebras

The present work provides a foundation for further investigation into the rich algebraic structures emerging from the hom-associative admissible framework.

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