An iterative algorithm for min-max programming problem with addition-min fuzzy relational inequalities in P2P file sharing systems

Abstract: P2P file sharing systems can be represented using fuzzy relation inequalities. In this paper, some properties of fuzzy relation inequalities are discussed. Using these properties and greedy approach, we generate an algorithm for reducing the traffic in P2P file sharing systems by modeling it as a min-max optimization problem subject to addition-min fuzzy relation inequalities. The sufficient condition for the optimality of the solution is given. The method is illustrated using some examples.

Keywords: Fuzzy relation inequalities, Addition-min composition, Min-max optimization problem, Greedy algorithms, P2P file sharing system, Monotonicity of fuzzy relation equations.

1. Introduction

In peer-to-peer (P2P) networking, a group of computers are linked together with equal permissions and responsibilities for processing data. In other words, P2P is a file-sharing technology in the computer network that allow users to access mainly the multimedia files like videos, music, e-books, games, etc. The individual users in this network are referred to as peers. The peers request files from other peers by establishing transmission control protocol (TCP) or user datagram protocol (UDP) connections. A P2P network allows computer hardware and software to communicate without the need for a server. Unlike client-server architecture, there is no central server for processing requests in a P2P architecture. When one peer makes a request, multiple peers may have a copy of that requested file. The problem is how to get the IP address of all those peers. This is decided by the underlying architecture supported by the P2P systems. Using one of these methods, the client peer can get to know all the peers which have the requested file and the file transfer takes place directly between these two peers.

A P2P file sharing system can be represented by a system of addition-min fuzzy relation inequalities (FRI). In this study, we address the best practices for data transmission and administration in P2P file sharing systems that resemble Bit-Torrent. Assume n users are using a P2P file sharing system to download files at the same time. It is ensured that every user will receive the file data from every other user. We now look into the circumstances surrounding the user's receipt of the file data from the other user. Let $A_1, A_2, A_3, ..., A_n$ denotes n users and a_{ij} be the bandwidth between user A_i and A_j. Also assume that the quality level for the i^{th} user A_i to send the file data is x_i. Since there are bandwidth limitations, so the network traffic that A_j receives from A_i is actually $x_i \otimes a_{ij} = \min(x_i, a_{ij})$; $i \neq j$ and $i, j \in \{1, 2, 3, ..., n\}$. If b_j is the download traffic quality required for the j^{th} user A_j, then the constraint for the j^{th} user A_j for receiving file data from the remaining $n-1$ users can be represented as follows:

$$x_1 \wedge a_{1j} + x_2 \wedge a_{2j} + ... + x_{j-1} \wedge a_{j-1,j} + x_j \wedge 0 + x_{j+1} \wedge a_{j+1,j} ... + x_n \wedge a_{n,j} \geq b_j$$

Thus the system of constraints for the n users for receiving the file data is represented as

$$\left.\begin{array}{l} x_1 \wedge 0 + x_2 \wedge a_{21} + x_3 \wedge a_{31} + ... + x_j \wedge a_{j,1} ... + x_n \wedge a_{n1} \geq b_1 \\ x_1 \wedge a_{12} + x_2 \wedge 0 + x_3 \wedge a_{32} + ... + x_j \wedge a_{j,2} + ... + x_n \wedge a_{n2} \geq b_2 \\ x_1 \wedge a_{13} + x_2 \wedge a_{23} + x_3 \wedge 0 + ... + x_j \wedge a_{j,3} + ... + x_n \wedge a_{n3} \geq b_3 \\ \vdots \\ x_1 \wedge a_{1j} + x_2 \wedge a_{2j} + x_3 \wedge a_{3j} + ... + x_j \wedge 0 + ... + x_n \wedge a_{nj} \geq b_j \\ \vdots \\ x_1 \wedge a_{1n} + x_2 \wedge a_{2n} + x_3 \wedge a_{3n} + ... + x_j \wedge a_{j,n} + ... + x_n \wedge 0 \geq b_n \end{array}\right\} \quad (1)$$

The system (1) was firstly introduced by Li and Yang [5] for describing BitTorrent-like P2P file sharing system. To reduce the network congestion, the quality levels of the data send out by the users, i.e. $x_1, x_2, x_3, ..., x_n$ should be minimized. But it is impossible to minimize all the variables simultaneously. For this some of the researchers introduced the concept of lexicographic maximum solution introduced by Yang et al. [22] to solve system of inequalities (1). They found that the unique lexicographic minimum solution can be selected from the set of minimal solutions. A circulation algorithm was proposed by them to find the unique lexicographic minimum solution. Yang [20] worked on random line fault between two terminals of a P2P network system. They proposed random-term-absent (RTA) addition-min FRI and introduced their lexicographic minimum solution. Qiu et al. [10] introduced an arbitrary-term-absent (ATA) max-product FRI. They introduced such inequalities considering arbitrary line fault in the wireless communication station system. Using the concept based on lexicographic ordering, an efficient algorithm was introduced for obtaining the unique lexicographic minimal solution of the max-product ATA system. But the solution formed using the concept of lexicographic maximum solution has a priority sequence over the variables.

In P2P file sharing system, network congestion may appear in case of large scale data transmission. The data transmission can be more stable and the network congestion can be avoided if the data transmission quality levels are low. Thus to avoid network congestion and keep the data transmission stable, [17] considered minimizing the sum of variables $x_1, x_2, x_3, ..., x_n$. But in this, it might be the case that one variable, say x_j is much bigger than the other ones. Hence, network congestion will appear for the j^{th} user. Therefore, Yang et al. [23] considered the problem of minimizing the biggest quality levels and introduced mini-max optimization problem with addition-min FRI as follows:

$$\min Z = \max(x_1, x_2, x_3, ..., x_n)$$

$$\left.\begin{array}{l} x_1 \wedge a_{11} + x_2 \wedge a_{21} + x_3 \wedge a_{31} + ... + x_j \wedge a_{j,1} + ... + x_n \wedge a_{n1} \geq b_1 \\ x_1 \wedge a_{12} + x_2 \wedge a_{22} + x_3 \wedge a_{32} + ... + x_j \wedge a_{j,2} + ... + x_n \wedge a_{n2} \geq b_2 \\ x_1 \wedge a_{13} + x_2 \wedge a_{23} + x_3 \wedge a_{33} + ... + x_j \wedge a_{j,3} + ... + x_n \wedge a_{n3} \geq b_3 \\ ... \\ ... \\ x_1 \wedge a_{1j} + x_2 \wedge a_{2j} + x_3 \wedge a_{3j} + ... + x_j \wedge a_{j,j} + ... + x_n \wedge a_{nj} \geq b_j \\ ... \\ ... \\ x_1 \wedge a_{1n} + x_2 \wedge a_{2n} + x_3 \wedge a_{3n} ... + x_j \wedge a_{j,n} + ... + x_n \wedge a_{nn} \geq b_n \end{array}\right\} \quad (2)$$

According to Ma et al. [9] it was more appropriate to consider the largest download traffic instead of total download traffic of a terminal in P2P network system. They first introduced max-min FRI and its lexicographic minimum solution for P2P network system. Yang [19] characterized P2P network system as addition-min FRI and defined the concepts of minimum zero point and min-max value for obtaining leximax minimum solution. Chen et al. [1] described P2P educational information resource sharing system using max-min FRI. They were first to define and study the widest interval solution of max-min FRI.

Li and Wang [7] found the set of minimal solutions for FRI with addition-min composition. Some necessary and sufficient conditions for a solution to be minimal were given and an algorithm was proposed for finding all minimal solutions. More work on finding minimal solutions of addition-min FRI can be found in [6].

Yang and Zheng [21] investigated the P2P network system represented with addition-min FRI with some line fault. They focused on number of arbitrary line faults that a P2P system can bear. Some algorithms were proposed for searching the maximum number of tolerant arbitrary and specific line faults in a given P2P network system.

Qiu and Yang [11] modeled P2P file sharing system using addition-min-product FRI. To reduce the network congestion and to improve data transmission stability, a min-max programming problem was established with such FRI. More work in this regard can be found in [14]. For the stability of data transmission and network congestion, Wu et al. [16] studied a min-max programming problem subject to addition min FRI. They proposed binding variable approach for solving such system representing a BitTorrent like P2P file sharing system. This approach reduced the effort required to find minimal optimal solution by directly determining the value of some components of minimal optimal solution. More work in this regard can be found in [13].

Guo et al. [2] modeled pricing relation in a supply chain system using min-product FRI and characterized its complete solution set. Lin et al. [8] investigated a max-min programming problem with supply chain system represented as min-product FRI. More work in this regard can be found in [15], [18], [24].

Guu et al. [4] presented a two phase approach for finding a better managerial solution by integrating two objectives, namely, system congestion and other managerial considerations with addition-min FRI. More work on minimizing a linear function subject to addition-min FRI can be found in [3].

2. System of addition-min FRI

Let $A=[a_{ij}]$ be a matrix of dimension $n\times n$ with $0\le a_{ij}\le 1,\ \forall i\in I, \forall j\in J$ where $I=\{1,2,\ldots,n\}$, $J=\{1,2,\ldots,n\}$ and $b=[b_j]$ be an $n-$dimensional vector such that $b_j>0,\ \forall j\in J.$ The system of addition-min FRI is represented as follows:

$$\left.\begin{array}{l} x_1\wedge a_{11}+x_2\wedge a_{21}+x_3\wedge a_{31}+\ldots+x_j\wedge a_{j,1}+\ldots+x_n\wedge a_{n1}\ge b_1 \\ x_1\wedge a_{12}+x_2\wedge a_{22}+x_3\wedge a_{32}+\ldots+x_j\wedge a_{j,2}+\ldots+x_n\wedge a_{n2}\ge b_2 \\ x_1\wedge a_{13}+x_2\wedge a_{23}+x_3\wedge a_{33}+\ldots+x_j\wedge a_{j,3}+\ldots+x_n\wedge a_{n3}\ge b_3 \\ \ldots \\ \ldots \\ x_1\wedge a_{1j}+x_2\wedge a_{2j}+x_3\wedge a_{3j}+\ldots+x_j\wedge a_{jj}+\ldots+x_n\wedge a_{nj}\ge b_j \\ \ldots \\ \ldots \\ x_1\wedge a_{1n}+x_2\wedge a_{2n}+x_3\wedge a_{3n}\ldots+x_j\wedge a_{j,n}+\ldots+x_n\wedge a_{nn}\ge b_n \end{array}\right\} \quad (3)$$

The system (3) can be represented as

$$\sum_{i\in I} x_i\wedge a_{ij}\ge b_j, \forall j\in J \quad (4)$$

The resolution of FRI is to find an $n-$dimensional vector $x=[x_i]$, where $0\le x_i\le 1,\ \forall i\in I$ such that (4) holds. The complete solution set of the system (4) is denoted by $X(A,b)$ and is defined as $X(A,b)=\{x\in[0,1]^n \mid x\circ A\ge b\}.$ The system (4) is said to be consistent if $X(A,b)\ne\phi.$

Definition 2.1. [12] Let $x=[x_1,\ x_2,\ldots,\ x_n]$ and $y=[y_1,\ y_2,\ldots,\ y_n]$ be the two vectors. Then $x<y$ if $x_i<y_i, \forall\ i\in I.$ Moreover, $x\le y$ if $x_i\le y_i, \forall i\in I.$

A solution $\hat{x}\in X(A,b)$ is said to be the maximum solution of the system (4) if $\hat{x}\ge x, \forall x\in X(A,b)$ and $\breve{x}\in X(A,b)$ is said to be minimal solution of the system (4) if there does not exist any $x\in X(A,b)$ satisfying $x\le\breve{x}$ and $x\ne\breve{x}.$ The complete solution set of the system of FRI (4) can be characterized by a unique maximum solution and all the minimal solutions. If the system of FRI (4) is consistent, then it has been verified that $\hat{x}=[1,1,1,\ldots,1]$ is always the maximum solution.

Lemma 2.1. (Monotonicity of composition)[12] Given $A\in[0,1]^{n\times n}$, for $x^1, x^2\in[0,1]^n$, if $x^1\le x^2$ then $x^1\circ A\le x^2\circ A.$

Proof. The result is a direct implication of the property of monotonicity of t-norms.

Theorem 2.1. For the system of FRI (4), $X(A,b) \neq \phi$ if and only if $\sum_{i\in I} a_{ij} \geq b_j, \forall\, j \in J$.

Proof. Let $X(A,b) \neq \phi$. This implies that $\sum_{i\in I} \min(x_i, a_{ij}) \geq b_j, \forall\, j \in J$. Moreover, for any $j \in J, a_{ij} \geq \min(x_i, a_{ij}), \forall i \in I$, i.e. for any $j \in J$, $\sum_{i\in I} a_{ij} \geq \sum_{i\in I} \min(x_i, a_{ij})$. Thus for any $j \in J$, $\sum_{i\in I} a_{ij} \geq b_j$.

Conversely, let $\sum_{i\in I} a_{ij} \geq b_j, \forall\, j \in J$. Using boundary condition of t-norms, $\sum_{i\in I} \min(1, a_{ij}) \geq b_j, \forall\, j \in J$. This implies that $[1,1,...,1]$ is a solution of the system of FRI (4). Thus $X(A,b) \neq \phi$.

Lemma 2.2. For $x \in X(A,b)$, $x_k \geq \max_{j\in J}[b_j - \sum_{\substack{i\in I\\ i\neq k}} a_{ij}]; \forall k \in I$.

Proof. For $x \in X(A,b)$, $\sum_{i\in I} \min(x_i, a_{ij}) \geq b_j, \forall j \in J$.

This implies that $\min(x_k, a_{kj}) + \sum_{\substack{i\in I\\ i\neq k}} \min(x_i, a_{ij}) \geq b_j, \forall j \in J$,

i.e. $\min(x_k, a_{kj}) \geq b_j - \sum_{\substack{i\in I\\ i\neq k}} \min(x_i, a_{ij}), \forall j \in J$.

Thus $x_k \geq b_j - \sum_{\substack{i\in I\\ i\neq k}} \min(x_i, a_{ij}), \forall j \in J$,

i.e. $x_k \geq \max_{j\in J}(b_j - \sum_{\substack{i\in I\\ i\neq k}} \min(x_i, a_{ij}))$. Since, for any $i \in I$, $\min(x_i, a_{ij}) \leq a_{ij}$, $\forall j \in J$, this implies that $\sum_{\substack{i\in I\\ i\neq k}} \min(x_i, a_{ij}) \leq \sum_{\substack{i\in I\\ i\neq k}} a_{ij}$, $\forall j \in J$, i.e. $b_j - \sum_{\substack{i\in I\\ i\neq k}} \min(x_i, a_{ij}) \geq b_j - \sum_{\substack{i\in I\\ i\neq k}} a_{ij}$, $\forall j \in J$.

Thus, $\max_{j\in J}[b_j - \sum_{\substack{i\in I\\ i\neq k}} \min(x_i, a_{ij})] \geq \max_{j\in J}[b_j - \sum_{\substack{i\in I\\ i\neq k}} a_{ij}]$. This implies that $x_k \geq \max_{j\in J}[b_j - \sum_{\substack{i\in I\\ i\neq k}} a_{ij}]$.

From Lemma 2.2, it is clear that $\max_{j\in J}[b_j - \sum_{\substack{i\in I\\ i\neq k}} a_{ij}]$ serves as the lower bound for the k^{th} component of any solution vector. We denote the lower bound of the k^{th} component of any solution vector by x_k^L, where $x_k^L = \max_{j\in J}[b_j - \sum_{\substack{i\in I\\ i\neq k}} a_{ij}]$.

Theorem 2.2. Let $x' \in X(A,b)$. If $x \in [x', \hat{x}]$, then $x \in X(A,b)$. Moreover, if $x'' \notin X(A,b)$, and $x < x''$, then $x \notin X(A,b)$.

Proof. Since $x' \in X(A,b)$, thus $x' \circ A \geq b$. If $x \geq x'$, then using monotonicity property of t-norms $x \circ A \geq x' \circ A$. This implies that $x \circ A \geq x' \circ A \geq b$, i.e. $x \in X(A,b)$.

If $x'' \notin X(A,b)$, this implies that there exists $j \in J$ such that $\sum_{i \in I} \min(x_i'', a_{ij}) < b_j$. Since $x < x''$ i.e. $x_i < x_i'', \forall i \in I$, thus by using monotonicity property of t-norms;
for $j \in J$, $\sum_{i \in I} \min(x_i, a_{ij}) < \sum_{i \in I} \min(x_i'', a_{ij}) < b_j$.
This implies that for $j \in J$, $\sum_{i \in I} \min(x_i, a_{ij}) < b_j$, i.e. $x \notin X(A,b)$.

3. The problem

We are interested in solving the following min-max optimization problem with addition-min FRI as constraints:

$$\min Z = \max(x_1, x_2, ..., x_n) \tag{5}$$

subject to

$$\sum_{i \in I} \min(x_i, a_{ij}) \geq b_j, \forall j \in J \tag{6}$$

where the symbols have their usual meaning as defined in the Section 2. Finding the optimal solution of the problem (5)-(6) means to find one or more solutions from $X(A,b)$ that minimizes (5).

Definition 3.1. Let $x = [x_i]_{i \in I}$ with $0 \leq x_i \leq 1, \forall i \in I$. Then x_i is said to be a binding variable for the system of FRI (6) if $x_i < a_{ij}$ holds for some constraint $j \in J$.
We denote the set of binding variables for the j^{th} constraint by I_j, where $I_j = \{i \in I \mid \min(x_i, a_{ij}) < a_{ij}\}$, $\forall j \in J$.
If x_i is a binding variable for the j^{th} constraint, then $\min(x_i, a_{ij}) < a_{ij}$. This means that the value of x_i cannot be greater than or equal to a_{ij}.

Definition 3.2. Let $x \in X(A,b)$. Define $J_B = \{j \in J \mid \sum_{i \in I} \min(x_i, a_{ij}) = b_j\}$, where J_B denotes the set of constraints which are satisfied at the equality level.

Theorem 3.1. Let $x \in X(A,b)$. If x_k is non-binding for all $j \in J_B$, then the value of x_k can be reduced to $\max[\max_{j \in J_B}\{a_{kj}\}, \max_{j \in (J - J_B)} \{b_j - \sum_{i \in (I - \{k\})} \min(x_i, a_{ij})\}]$.

Proof. Since x_k is non-binding variable for all $j \in J_B$, thus $a_{kj} \leq x_k, \forall j \in J_B$. This implies $\max_{j \in J_B}\{a_{kj}\} \leq x_k$.

Now for $j \in J - J_B$, $\sum_{i \in I} \min(x_i, a_{ij}) \geq b_j$. This implies $\min(x_k, a_{kj}) + \sum_{i \in (I - \{k\})} \min(x_i, a_{ij}) \geq b_j$ i.e. $\min(x_k, a_{kj}) \geq b_j - \sum_{i \in (I - \{k\})} \min(x_i, a_{ij})$. This implies that $x_k \geq b_j - \sum_{i \in (I - \{k\})} \min(x_i, a_{ij}), \forall j \in J - J_B$. Hence $x_k \geq \max_{j \in (J - J_B)} \{b_j - \sum_{i \in (I - \{k\})} \min(x_i, a_{ij})\}$.

From the above two cases, we get $x_k \geq \max[\max_{j \in J_B}\{a_{kj}\}, \max_{j \in (J - J_B)} \{b_j - \sum_{i \in (I - \{k\})} \min(x_i, a_{ij})\}]$.

Theorem 3.2. Let $x \in X(A, b)$. Then

(i) For $j \in J$, $\sum_{i \in I_j} x_i \geq b_j - \sum_{i \in (I - I_j)} a_{ij}$,

(ii) For $j \in J_B$, $\sum_{i \in I_j} x_i = b_j - \sum_{i \in (I - I_j)} a_{ij}$.

Proof. (i) For $x \in X(A, b), \sum_{i \in I} \min(x_i, a_{ij}) \geq b_j, \forall j \in J$. This implies $\sum_{i \in I_j} \min(x_i, a_{ij}) + \sum_{i \in (I - I_j)} \min(x_i, a_{ij}) \geq b_j$. If x_i is a non-binding variable, then $\min(x_i, a_{ij}) \geq a_{ij}, \forall j \in J$ and if x_i is a binding variable, then $\min(x_i, a_{ij}) < a_{ij}$, for some $j \in J$. Thus we get $\sum_{i \in I_j} x_i + \sum_{i \in (I - I_j)} a_{ij} \geq b_j$ i.e. $\sum_{i \in I_j} x_i \geq b_j - \sum_{i \in (I - I_j)} a_{ij}$.

(ii) For $j \in J_B$, $\sum_{i \in I} \min(x_i, a_{ij}) = b_j$. This implies $\sum_{i \in I_j} \min(x_i, a_{ij}) + \sum_{i \in (I - I_j)} \min(x_i, a_{ij}) = b_j$. Using the similar argument above, we get $\sum_{i \in I_j} x_i + \sum_{i \in (I - I_j)} a_{ij} = b_j$ i.e. $\sum_{i \in I_j} x_i = b_j - \sum_{i \in (I - I_j)} a_{ij}$.

Theorem 3.3. The optimal solution of the problem of minimizing the value of $Z = \max(x_1, x_2, x_3, ..., x_n)$ subject to $x_1 + x_2 + x_3 + ... + x_n = m$ is $x_1 = x_2 = x_3 = ... = x_n = \frac{m}{n}$.

Proof. Since the problem involves minimizing the maximum value of $x_1, x_2, x_3, ..., x_n$ under the given constraint $x_1 + x_2 + x_3 + ... + x_n = m$, we can observe that each x_i plays an equal role in minimizing the value of Z. Thus we can conclude that $x_1 = x_2 = x_3 = ... = x_n = x$. So the constraint $x_1 + x_2 + x_3 + ... + x_n = m$ can be reduced to $n \cdot x = m$. Thus, the value of each x_i is $\frac{m}{n}$ and the maximum value of Z is the value of any x_i i.e. $Z = \max(x_1, x_2, x_3, ..., x_n) = \frac{m}{n}$.

If we had any inequality among $x_i, i \in I$, then the maximum value would be higher than $\frac{m}{n}$, because the maximum is the largest value in the set. For example, if we were to make some x_i smaller than $\frac{m}{n}$, we would need to make others larger in order to satisfy the constraint

$x_1 + x_2 + x_3 + ... + x_n = m,$ thereby increasing the maximum value of Z. Therefore, the configuration in which each x_i are equal, minimizes the maximum value of Z. Thus, the minimum value of $Z = \max(x_1, x_2, x_3, ..., x_n)$ subject to the constraint $x_1 + x_2 + x_3 + ... + x_n = m$ is $Z = \frac{m}{n}$.

Theorem 3.4. Let $x \in X(A,b)$ and $j \in J_B$. Suppose the following conditions hold:

(i) For each binding variable x_i,

$$x_i = \frac{b_j - \sum_{i \in (I - I_j)} a_{ij}}{|I_j|};$$

(ii) For each non-binding variable x_i,

$$x_i \le \frac{b_j - \sum_{i \in (I - I_j)} a_{ij}}{|I_j|};$$

Then the value

$$Z = \frac{b_j - \sum_{i \in (I - I_j)} a_{ij}}{|I_j|}$$

is the optimal value of the problem (5)-(6).

Proof. Let $x \in X(A,b)$ and $j \in J_B$. Using Theorem 3.2, we get $\sum_{i \in I_j} x_i = b_j - \sum_{i \in (I - I_j)} a_{ij}$. Thus from Theorem 3.3, the optimum value of $Z = \max_{i \in I}\{x_i\} = \max_{i \in I_j}\{x_i\} = \frac{b_j - \sum_{i \in (I - I_j)} a_{ij}}{|I_j|}$.

4. Greedy search method

We will apply greedy approach for solving the problem (5)-(6). Greedy algorithms are a class of algorithms that make locally optimal choices at each step with the hope of finding a global optimum solution. The term greedy reflects the algorithms approach of making the most advantageous choice available at that moment without considering the broader context or future steps. In these algorithms, decisions are made based on the information available at the current moment without considering the consequences of these decisions in the future. The key idea is to select the best possible choice at each step, leading to a solution that may not always be the most optimal but is often good enough for many problems.

The working of greedy algorithm is as follows:

1. Define a goal: The algorithm starts with a clearly defined goal or objective, such as maximizing or minimizing a certain cost or an objective function.

2. Greedy choice: At each step, the algorithm selects the option that appears best to achieve the goal, given the current state.
3. Iterate: The process is repeated, talking the best immediate choice at each step, until an end condition is reached.
4. Final solution: After all steps are completed, the algorithm returns the final result, which may or may not be optimal but is derived from a series of locally optimal choices.

To applying greedy approach we first define the term increment quantity. The Increment quantity corresponding to j^{th} constraint is the change that is required in the value of $\sum_{i\in I}\min(x_i, a_{ij})$ to satisfy the j^{th} constraint. It is given as:

$$\text{Increment quantity} = b_j - \sum_{i\in I}\min(x_i, a_{ij}).$$

Greedy algorithm for finding optimal solution of the problem (5)-(6)

Set iteration count $Iter = 1.$

Step 1. Check consistency of the given system using Theorem 2.1.

Step 2. Construct a vector $\tilde{x}$ by setting all components to their lower bound, i.e. $\tilde{x} = [\tilde{x}_k^L]_{k\in I}$, using Lemma 2.2.

Step 3. Denote the set of all constraints which are not satisfied by the vector $\tilde{x}$ by S.

If $S = \phi$, then, stop. The vector $\tilde{x}$ is the optimal solution. Else go to Step 4.

Step 4. Find increment quantity for each $j \in S$. Let $K = \min_{j\in S}\left\{b_j - \sum_{i\in I}\min(\tilde{x}_i, a_{ij})\right\}$

and $\bar{S} = \left\{ j \in S : b_j - \sum_{i\in I}\min(\tilde{x}_i, a_{ij}) = K \right\}$

where K denotes the minimum of the increment quantity for all $j \in S$ and $\bar{S} \subseteq S$ denotes the set of constraints whose increment quantity equals K.

Step 5. Randomly select a constraint $k \in \bar{S}$.

Step 6. Find the set of binding variables I_k for $k \in \bar{S}$.

Step 7. Let $t = \min\{\tilde{x}_i : i \in I_k\}$ and $P = \{i \in I_k : \tilde{x}_i = t\}$

where t denotes the minimum of the lower bound $\tilde{x}_i$ for all $i \in I_k$ and $P \subseteq I_k$ denotes the set of variables whose lower bound equals t.

Step 8. Define average increment quantity I_{avg} for each $i \in P$ by $I_{avg} = \dfrac{K}{|P|}$.

Step 9. Compute $\Delta\tilde{x}_i$ for each $i \in P$ where $\Delta\tilde{x}_i = \min\{\min_{s\in(I_k-P)}\{\tilde{x}_s - t\}, I_{avg}, a_{ik} - \tilde{x}_i\}$ and a_{ik} denotes the upper bound of $\tilde{x}_i$ corresponding to k^{th} constraint.

Step 10. Let $w = \min\{\Delta\tilde{x}_i : i \in P\}$ and $V = \{i \in P : \Delta\tilde{x}_i = w\}$.

Step 11. Update $\tilde{x}$ by increasing the value of each $\tilde{x}_v$, $v \in V$ by w.

Step 12. Set increment quantity by $K = K - w|V|$.

Step 13. If $K = 0$, set $Iter = Iter + 1$. Go to Step 3. Else go to Step 6.

5. Illustrations

Example 5.1. Consider the following optimization problem subject to the system of addition-min FRI:

$\min Z = \max\{x_1, x_2, x_3, x_4\}$

with

$$[x_1 \quad x_2 \quad x_3 \quad x_4] \circ \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.8 \\ 0.8 & 0.3 & 0.3 \\ 0.2 & 0.7 & 0.7 \end{bmatrix} \geq [1.5 \;\; 1.6 \;\; 1.4]$$

where $\circ$ denotes the addition-min composition.

Set iteration count $Iter = 1$.

Step 1. Check the consistency of the given system using Theorem 2.1.

Since $\sum_{i\in I} a_{ij} \geq b_j, \forall\, j \in J$, thus the system is consistent.

Step 2. Construct a vector $\tilde{x}$ by setting all components to their lower bound, i.e. $\tilde{x} = [\tilde{x}_k^L]_{k\in I}$ using Lemma 2.2.

$\tilde{x}_1 = \max\{0, 0.2, 0\} = 0.2$
$\tilde{x}_2 = \max\{0.3, 0.3, 0\} = 0.3$
$\tilde{x}_3 = \max\{0.6, 0.2, 0\} = 0.6$
$\tilde{x}_4 = \max\{0, 0.6, 0\} = 0.6.$

$\tilde{x} = [0.2 \quad 0.3 \quad 0.6 \quad 0.6].$

Step 3. Since $\tilde{x} \circ A = [1.3 \quad 1.4 \quad 1.4]$, thus $S = \{1, 2\}$ is the set of constraints which are not satisfied by the vector $\tilde{x}$. Since $S \neq \phi$, thus we go for Step 4.

Step 4. Find increment quantity for each $j \in S$.

$$K = \min_{j \in S} \left\{ b_j - \sum_{i \in I} \min(\tilde{x}_i, a_{ij}) \right\} = \min\{0.2, 0.2\} = 0.2 \text{ and}$$

$$\bar{S} = \left\{ j \in S : b_j - \sum_{i \in I} \min(\tilde{x}_i, a_{ij}) = 0.2 \right\} = \{1, 2\}.$$

Step 5. Randomly select a constraint $k \in \bar{S}$. Say $k = 1$.

Step 6. The set of binding variables for $k = 1$ is $I_1 = \{2, 3\}$.

Step 7. Find $t = \min\{\tilde{x}_i : i \in I_1\} = \min\{\tilde{x}_2, \tilde{x}_3\} = \min\{0.3, 0.6\} = 0.3$ and $P = \{i \in I_1 : \tilde{x}_i = 0.3\} = \{2\}$.

Step 8. Find average increment quantity $I_{avg} = \dfrac{K}{|\mathrm{P}|} = \dfrac{0.2}{1} = 0.2.$

Step 9. Find $\Delta\tilde{x}_2 = \min\{\min_{s \in (I_1 - P)}\{\tilde{x}_s\} - t, I_{avg}, a_{ik} - \tilde{x}_i\} = \min\{\min_{s \in \{3\}}\{\tilde{x}_s\} - 0.3, 0.2, a_{21} - \tilde{x}_2\}$

$= \min\{0.6 - 0.3, 0.2, 0.5 - 0.3\} = 0.2.$

Step 10. Let $w = \min\{\Delta\tilde{x}_i : i \in P\} = 0.2$ and $V = \{i \in P : \Delta\tilde{x}_i = 0.2\} = \{2\}$.

Step 11. Updated $\tilde{x} = [0.2 \quad 0.5 \quad 0.6 \quad 0.6]$.

Step 12. Increment quantity $K = K - w|V| = 0$.

Step 13. Since $K = 0$, we repeat the process from Step 3 with updated $\tilde{x} = [0.2 \quad 0.5 \quad 0.6 \quad 0.6]$

Set iteration count $iter = 2$.

Step 3. Since $\tilde{x} \circ A = [1.5 \quad 1.5 \quad 1.6]$, thus $S = \{2\}$ is the set of constraints which are not satisfied by the vector $\tilde{x}$. Since $S \neq \phi$, thus we go for Step 4.

Step 4. Find increment quantity for each $j \in S$.

$$K = \min_{j \in S} \left\{ b_j - \sum_{i \in I} \min(\tilde{x}_i, a_{ij}) \right\} = \min\{0.1\} = 0.1 \text{ and } \bar{S} = \left\{ j \in S : b_j - \sum_{i \in I} \min(\tilde{x}_i, a_{ij}) = 0.1 \right\} = \{2\}.$$

Step 5. Randomly select a constraint $k \in \bar{S}$. Say $k = 2$.

Step 6. The set of binding variables for $k=2$ is $I_2=\{1,4\}$.

Step 7. Find $t=\min\{\tilde{x}_i : i\in I_2\}=\min\{\tilde{x}_1,\tilde{x}_4\}=\min\{0.2,0.6\}=0.2$ and

$P=\{i\in I_2 : \tilde{x}_i=0.2\}=\{1\}$.

Step 8. Find average increment quantity $I_{avg}=\frac{K}{|P|}=\frac{0.1}{1}=0.1$.

Step 9. Find $\Delta\tilde{x}_1=\min\{\min_{i\in(I_2-P)}\{\tilde{x}_s\}-t, I_{avg}, a_{ik}-\tilde{x}_i\}$

$=\min\{\min_{s\in\{4\}}\{\tilde{x}_s\}-0.2, 0.1, a_{12}-\tilde{x}_1\}=\min\{0.6-0.2, 0.1, 0.3-0.2\}=0.1$.

Step 10. Let $w=\min\{\Delta\tilde{x}_i : i\in P\}=0.1$ and $V=\{i\in P:\Delta\tilde{x}_i=0.1\}=\{1\}$.

Step 11. Updated $\tilde{x}=[0.3 \quad 0.5 \quad 0.6 \quad 0.6]$.

Step 12. Increment quantity $K-w|V|=0$.

Step 13. Since $K=0$ we repeat the process from the Step 3 with updated $\tilde{x}=[0.3 \quad 0.5 \quad 0.6 \quad 0.6]$.

Set iteration count $Iter=3$.

Step 3. Since $\tilde{x}\circ A=[1.5 \quad 1.6 \quad 1.7]\geq b$, thus $S=\phi$. Thus $\tilde{x}=[0.3 \quad 0.5 \quad 0.6 \quad 0.6]$ is the optimal solution with $Z=\max\{x_1,x_2,x_3,x_4\}=\max\{0.3,0.5,0.6,0.6\}=0.6$.

Example 5.2. Consider the following optimization problem subject to the system of addition-min FRI:

$\min Z=\max(x_1,x_2,x_3,x_4,x_5,x_6)$

with

$$[x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6]\circ\begin{bmatrix} 0 & 0.5 & 0.8 & 0.7 & 0.8 & 0.6 \\ 0.6 & 0 & 0.7 & 0.5 & 0.6 & 0.7 \\ 0.8 & 0.7 & 0 & 0.6 & 0.9 & 0.7 \\ 0.5 & 0.9 & 0.4 & 0 & 0.7 & 0.9 \\ 0.6 & 0.8 & 0.7 & 0.8 & 0 & 0.5 \\ 0.9 & 0.5 & 0.8 & 0.6 & 0.7 & 0 \end{bmatrix}\geq[2.8 \quad 3.0 \quad 2.0 \quad 2.5 \quad 2.0 \quad 3.0]$$

where, $\circ$ denotes the addition-min composition.

Set iteration count $Iter=1$.

Step 1. Check the consistency of the given system using Theorem 2.1.

Since, $\sum_{i\in I} a_{ij} \geq b_j, \forall j \in J,$ thus the system is consistent.

Step 2. Construct a vector $\tilde{x}$ by setting all components to their lower bound, i.e. $\tilde{x} = [\tilde{x}_k^L]_{k\in I}$ using Lemma 2.2.

$$\tilde{x}_1 = \max\{0, 0.1, 0, 0, 0, 0.2\} = 0.2$$
$$\tilde{x}_2 = \max\{0, 0, 0, 0, 0, 0.3\} = 0.3$$
$$\tilde{x}_3 = \max\{0.2, 0.3, 0, 0, 0, 0.3\} = 0.3$$
$$\tilde{x}_4 = \max\{0, 0.5, 0, 0, 0, 0.5\} = 0.5$$
$$\tilde{x}_5 = \max\{0, 0.4, 0, 0.1, 0, 0.1\} = 0.4$$
$$\tilde{x}_6 = \max\{0.3, 0.1, 0, 0, 0, 0\} = 0.3$$

$$\tilde{x} = [0.2 \quad 0.3 \quad 0.3 \quad 0.5 \quad 0.4 \quad 0.3]$$

Step 3. Since $\tilde{x} \circ A = [1.8 \quad 1.7 \quad 1.6 \quad 1.5 \quad 1.6 \quad 1.7]$, thus $S = \{1, 2, 3, 4, 5, 6\}$ is the set of constraints which are not satisfied by the vector $\tilde{x}$. Since $S \neq \phi$, thus we go for the Step 4.

Step 4. Find increment quantity for each $j \in S$.

$$K = \min_{j\in S}\left\{b_j - \sum_{i\in I}\min(\tilde{x}_i, a_{ij})\right\} = \min\{1.0, 1.3, 0.4, 1.0, 0.4, 1.3\} = 0.4 \text{ and}$$
$$\bar{S} = \left\{j \in S : b_j - \sum_{i\in I}\min(\tilde{x}_i, a_{ij}) = 0.4\right\} = \{3, 5\}.$$

Step 5. Randomly select a constraint $k \in \bar{S}$. say $k = 3$.

Step 6. The set of binding variables for $k = 3$ is $I_3 = \{1, 2, 5, 6\}$.

Step 7. Find $t = \min\{\tilde{x}_i : i \in I_3\} = \min\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_5, \tilde{x}_6\} = \min\{0.2, 0.3, 0.4, 0.3\} = 0.2$ and $P = \{i \in I_3 : \tilde{x}_i = 0.2\} = \{1\}$.

Step 8. Find average increment quantity $I_{avg} = \dfrac{K}{|\mathrm{P}|} = \dfrac{0.4}{1} = 0.4$.

Step 9. Find $\Delta\tilde{x}_1 = \min\{\min_{s\in(I_3-P)}\{\tilde{x}_s\} - t, I_{avg}, a_{ik} - \tilde{x}_i\}$

$$= \min\{\min_{s\in I_3-\{1\}}\{\tilde{x}_s\} - 0.2, 0.4, 0.8 - 0.2\} = \min\{0.3 - 0.2, 0.4, 0.8 - 0.2\} = \min\{0.1, 0.4, 0.6\} = 0.1.$$

Step 10. Let $w = \min\{\Delta\tilde{x}_i : i \in P\} = 0.1$ and $V = \{i \in P : \Delta\tilde{x}_i = w\} = \{i \in P : \Delta\tilde{x}_i = 0.1\} = \{1\}$

Step 11. Updated $\tilde{x} = [0.3 \quad 0.3 \quad 0.3 \quad 0.5 \quad 0.4 \quad 0.3]$.

Step 12. Increment quantity $K - w|V| = 0.4 - 0.1 = 0.3.$

Step 13. Since $K \neq 0,$ we repeat the process from Step 6 with $K = 0.3$, $k = 3$ and $\tilde{x} = [0.3 \quad 0.3 \quad 0.3 \quad 0.5 \quad 0.4 \quad 0.3].$

Step 6. The set of binding variables for $k = 3$ is $I_3 = \{1,2,5,6\}.$

Step 7. Find $t = \min\{\tilde{x}_i : i \in I_3\} = \min\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_5, \tilde{x}_6\} = \min\{0.3, 0.3, 0.4, 0.3\} = 0.3$ and $P = \{i \in I_k : \tilde{x}_i = 0.3\} = \{1,2,6\}.$

Step 8. Find average increment quantity $I_{avg} = \dfrac{K}{|\mathrm{P}|} = \dfrac{0.3}{3} = 0.1.$

Step 9. Find $\Delta\tilde{x}_1 = \min\{\min\limits_{s \in (I_3 - P)} \{\tilde{x}_s\} - t, \text{Average increment}, a_{ik} - \tilde{x}_i\}$

$$\Delta\tilde{x}_1 = \min\{\min_{s \in (I_3 - \{1,2,6\})} \{\tilde{x}_s\} - t, 0.1, a_{13} - \tilde{x}_1\} = \min\{0.4 - 0.3, 0.1, 0.8 - 0.3\} = \min\{0.1, 0.1, 0.5\} = 0.1$$

$$\Delta\tilde{x}_2 = \min\{\min_{s \in (I_3 - P)} \{\tilde{x}_s\} - t, \text{Average increment}, a_{ik} - \tilde{x}_i\}$$

$$\Delta\tilde{x}_2 = \min\{\min_{s \in (I_3 - \{1,2,6\})} \{\tilde{x}_s\} - t, 0.1, a_{23} - \tilde{x}_2\} = \min\{0.4 - 0.3, 0.1, 0.7 - 0.3\} = \min\{0.1, 0.1, 0.4\} = 0.1$$

$$\Delta\tilde{x}_6 = \min(\min_{s \in (I_3 - P)} \{\tilde{x}_s - t\}, \text{Average increment}, a_{ik} - \tilde{x}_i)$$

$$\Delta\tilde{x}_6 = \min\{\min_{s \in (I_3 - \{1,2,6\})} \{\tilde{x}_s\} - t, 0.1, a_{63} - \tilde{x}_6\} = \min\{0.4 - 0.3, 0.1, 0.8 - 0.3\} = \min\{0.1, 0.1, 0.5\} = 0.1$$

Step 10. Let $w = \min\{\Delta\tilde{x}_i : i \in P\} = 0.1$ and $V = \{i \in P : \Delta\tilde{x}_i = 0.1\} = \{1,2,6\}.$

Step 11. Updated $\tilde{x} = [0.4 \quad 0.4 \quad 0.3 \quad 0.5 \quad 0.4 \quad 0.4].$

Step 12. Increment quantity $K - w|V| = 0.$

Step 13. Since $K = 0,$ we repeat the process from Step 3 with updated $\tilde{x} = [0.4 \quad 0.4 \quad 0.3 \quad 0.5 \quad 0.4 \quad 0.4].$

After repeating the steps of the algorithm, the solution obtained for the given problem is $\tilde{x} = [0.5833 \quad 0.5833 \quad 0.6667 \quad 0.6667 \quad 0.6667 \quad 0.575].$

For the above $\tilde{x},$ we compute

$$\tilde{x} \circ A = [2.925 \quad 3.000 \quad 2.808 \quad 2.925 \quad 3.075 \quad 3.000]$$

Using theorem 3.1, $\tilde{x}_6$ is the only non-binding variable for $J_B = \{2,6\},$ thus we can reduced it to

$$\max\{\max_{j \in \{2,6\}} \{a_{6j}\}, \max_{j \in \{1,3,4,5\}} \{b_j - \sum_{i \in \{1,2,3,4,5\}} \min(x_i, a_{ij})\}\}$$

$$= \max\{\max\{0.5, 0\}, \max\{0.45, -0.2333, 0.15, -0.5\}\}$$

$$= \max\{0.5, 0.45\} = 0.5.$$

Thus updated $\tilde{x} = [0.5833 \quad 0.5833 \quad 0.6667 \quad 0.6667 \quad 0.6667 \quad 0.5]$. Now, using theorem 3.4, corresponding to constrain 3, we conclude that the current solution vector is an optimal solution for the given problem 5.2.

6. Conclusion

In this paper, addition-min FRI are considered. Some properties of solution existence and solvability for finding solutions of addition-min FRI are recalled. A greedy approach based algorithm, using the concept of binding variables, is introduced to find the optimal solution of min-max programming problem subject to addition-min FRI. Greedy approach does not assure the optimality of the solution. Theorem 3.4 provides the sufficient condition for the solution to be optimal corresponding to problem (5)-(6). A step-by-step explanation with illustrations using examples is given to make the algorithm clear.

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