
Incomplete metrics on \mathbb{R} with usual topology

Abstract

The set of all real numbers \mathbb{R} with usual metric is complete and it generates usual topology on \mathbb{R} . Present paper discusses the existence of an incomplete metric on \mathbb{R} that generates usual topology on it. We proved some general results that lead to a method to identify infinitely many such incomplete metrics on \mathbb{R} . Also, existence of such incomplete metrics on \mathbb{R} point towards the presence of metrics on \mathbb{R} which are not norm induced.

Keywords: Metric Space; Complete Metric; Incomplete Metric; Topology

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1 Introduction

A metric space is complete if every cauchy sequence in that space converges to a point in the space itself. So \mathbb{R} with usual metric (ie, absolute difference between two real numbers) is complete. Also, usual metric generates usual topology on \mathbb{R} .

Present paper discusses the existence of an incomplete metric on the set of real numbers \mathbb{R} that still induces the usual topology on the space. The paper establishes some general results that pave the way for identifying such incomplete metrics which produces usual topology on \mathbb{R} .

The fundamental definitions and results employed in the paper are derived from the sources (Joshi, 2006), (Satish et.al., 2009), (Qamrul, 2010), (Bartle et.al., 2007), (Limaye, 2008) and (K.C. Rao, 2006).

2 Preliminaries

The concept of distance allows you to quantify how far apart or dissimilar two elements in a set. A metric is a way to measure the distance between two elements of a non empty set.

Definition 2.1. Let X be a non empty set and $d : X \times X \rightarrow \mathbb{R}$ be a function. Then d is said to be a metric on X if d satisfies the following four conditions.

- (i) $d(x, y) \geq 0, \forall x, y \in X$;
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- (ii) $d(x, y) = 0$ if and only if $x = y$;
 - (iii) $d(x, y) = d(y, x), \forall x, y \in X$;
 - (iv) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$.

Then the pair (X, d) is said to be a metric space.

Definition 2.2. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for given $\epsilon > 0$ there exist $M \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq M$.

Definition 2.3. A metric space is said to be complete if every Cauchy sequence in the space converges to a point in the space itself.

Definition 2.4. An open ball with center x and radius r in a metric space (X, d) is defined as $B_r(x) = \{y \in X / d(x, y) < r\}$.

Definition 2.5. A subset S of a metric space is said to be open if for all $x \in S, \exists r > 0$ such that $B_r(x) \subset S$.

Definition 2.6. Two metrics d_1 and d_2 on X are equivalent if the open sets in (X, d_1) are the open in (X, d_2) and vice versa.

Definition 2.7. A metric d on a set X is said to be bounded if there exists a real number M such that $d(x, y) \leq M$ for all $x, y \in X$.

Remark 2.1. If (X, d) is a metric space and $\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$, then

- (i) $\rho(x, y)$ is a bounded metric on X ;
- (ii) d and ρ are equivalent metrics on X .

Definition 2.8 (Homeomorphism on Metric Space). Let (X, d_1) and (Y, d_2) be metric spaces. A bijection $f : X \rightarrow Y$ is said to be a homeomorphism if the functions f and f^{-1} are continuous on (X, d_1) and (Y, d_2) , respectively.

Theorem 2.1. Let d_1 and d_2 be two metrics on X . Then the following statements are equivalent.

- (i) The metrics d_1 and d_2 are equivalent on X ;
- (ii) The identity map $I : (X, d_1) \rightarrow (X, d_2)$ is a homeomorphism;
- (iii) A sequence $\{x_n\}$ converges to $x \in X$ with respect to d_1 if and only if $\{x_n\}$ converges to $x \in X$ with respect to d_2 .

Definition 2.9. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \rightarrow Y$ is said to be an isometry if $d_1(x, y) = d_2(f(x), f(y))$ for all $x, y \in X$.

Let (X, d_1) and (Y, d_2) be metric spaces. If $f : (X, d_1) \rightarrow (Y, d_2)$ is an isometry, then f is one-one and uniformly continuous.

Remark 2.2. If an isometry is onto also, then the spaces X and Y are said to be isometric.

Let (X, d_1) and (Y, d_2) be metric spaces. If $f : (X, d_1) \rightarrow (Y, d_2)$ is an isometry and onto, then $f : (X, d_1) \rightarrow (Y, d_2)$ is a homeomorphism.

Definition 2.10. Let X be a set and \mathcal{T} be a set of subsets of X . The pair (X, \mathcal{T}) is said to be a topological space if \mathcal{T} satisfying following three conditions:

- (i) $\phi, X \in \mathcal{T}$;

- (ii) \mathcal{T} is closed under arbitrary unions;
- (iii) \mathcal{T} is closed under finite intersections.

The set \mathcal{T} is called a topology on X and elements of \mathcal{T} are called open subsets of X (with respect to \mathcal{T}).

Definition 2.11. Let (X, \mathcal{T}) be a topological space and \mathcal{B} be a subset of X . \mathcal{B} is said to be a base for \mathcal{T} if every member of \mathcal{T} can be written as union of elements of \mathcal{B} .

Definition 2.12. Let X be a set. \mathcal{T}_1 and \mathcal{T}_2 are two topologies on X . \mathcal{T}_2 is said to be stronger than \mathcal{T}_1 if \mathcal{T}_1 is a subset of \mathcal{T}_2 .

Theorem 2.2. Let \mathcal{T}_1 and \mathcal{T}_2 be topologies on X . Let \mathcal{B}_2 be a basis for \mathcal{T}_2 . \mathcal{T}_2 is stronger than \mathcal{T}_1 if and only if every element of \mathcal{T}_1 contains at least one element of \mathcal{B}_2 .

Theorem 2.3. If (X, d) is a metric space, then set of all open subsets of X is a topology on X , denoted by τ_d . So every metric space is also a topological space.

Remark 2.3. Equivalent metrics on a space X induces same topology on the space.

Definition 2.13 (Homeomorphism on Topological Space). Let (X, \mathcal{T}_1) and (Y, \mathcal{T}_2) be two topological spaces. A bijection $f : X \rightarrow Y$ is said to be a homeomorphism if the functions f and f^{-1} are continuous on (X, \mathcal{T}_1) and (Y, \mathcal{T}_2) , respectively.

Let (X, d_1) and (Y, d_2) be metric spaces. If $f : (X, d_1) \rightarrow (Y, d_2)$ is an isometry and onto, then $f : (X, \tau_{d_1}) \rightarrow (Y, \tau_{d_2})$ is a homeomorphism.

3 Main Results

This section establishes the existence of an incomplete metric on \mathbb{R} which induces usual topology on it. A specific method for constructing infinitely many such incomplete metrics is developed as an outcome of following discussion.

Theorem 3.1. If (Y, d) is a metric space and $f : X \rightarrow Y$ is a one-one function, then $d_f(x, y) = d(f(x), f(y))$ is a metric on X .

Proof. To prove that d_f is a metric on the set X , we need to verify the four axioms of a metric.

- (i) Since d is a metric, $d(x, y) \geq 0 \forall x, y \in X$. It follows that,

$$d_f(x, y) = d(f(x), f(y)) \geq 0, \forall x, y \in X$$

(ii)

$$\begin{aligned} d_f(x, y) = 0 &\Leftrightarrow d(f(x), f(y)) = 0 \\ &\Leftrightarrow f(x) = f(y) \\ &\Leftrightarrow x = y \end{aligned}$$

(iii)

$$\begin{aligned} d_f(x, y) &= d(f(x), f(y)) \\ &= d(f(y), f(x)) \\ &= d_f(y, x), \forall x, y \in X. \end{aligned}$$

(iv)

$$\begin{aligned} d_f(x, y) &= d(f(x), f(y)) \\ &\leq d(f(x), f(z)) + d(f(z), f(y)) \\ &= d_f(x, z) + d_f(z, y) \end{aligned}$$

That is $d_f(x, y) \leq d_f(x, z) + d_f(z, y), \forall x, y, z \in X$. Hence d_f is a metric on X . □

Theorem 3.2. *Let (X, \mathcal{T}) be a topological space and (Y, d) be a metric space. If $f : (X, \mathcal{T}) \rightarrow (Y, \tau_d)$ is a homeomorphism, then (X, \mathcal{T}) is homeomorphic to (X, τ_{d_f}) , where d_f is defined as in Theorem 3.1.*

Proof. Since f is a homeomorphism from (X, \mathcal{T}) to (Y, τ_d) , f is a bijection from X to Y . Therefore $f : (X, d_f) \rightarrow (Y, d)$ is an isometry by the definition of d_f . From Lemma 2, f is a homeomorphism between (X, τ_{d_f}) and (Y, τ_d) . Given that (X, \mathcal{T}) is homeomorphic to (Y, τ_d) . Hence it follows that (X, \mathcal{T}) is homeomorphic to (X, τ_{d_f}) . □

Remark 3.1. The homeomorphism between (X, \mathcal{T}_1) and (X, \mathcal{T}_2) may not guarantee that the underlying topologies \mathcal{T}_1 and \mathcal{T}_2 are the same. For example, consider the set of integers \mathbb{Z} . Let \mathcal{T}_1 be the topology consisting of the empty set, \mathbb{Z} itself, and subsets of \mathbb{Z} of the form $\{k \in \mathbb{Z} : k \leq n, n \in \mathbb{Z}\}$. Similarly, let \mathcal{T}_2 be another topology consisting of the empty set, \mathbb{Z} itself, and subsets of \mathbb{Z} of the form $\{k \in \mathbb{Z} : k \geq n, n \in \mathbb{Z}\}$. Then the map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\phi(x) = -x$ is a homeomorphism from $(\mathbb{Z}, \mathcal{T}_1)$ to $(\mathbb{Z}, \mathcal{T}_2)$. But they have no non-trivial open sets in common.

In particular, Theorem 3.2 does not guarantee that the topologies \mathcal{T} and τ_{d_f} are equal. But next theorem shows that they are the same.

Theorem 3.3. *Let (X, \mathcal{T}) be a topological space and (Y, d) be a metric space. If $f : (X, \mathcal{T}) \rightarrow (Y, \tau_d)$ is a homeomorphism, then the metric d_f on X induces the topology \mathcal{T} .*

Proof. Let V be an open set in X and let $x \in V$. Since $f : (X, \mathcal{T}) \rightarrow (Y, \tau_d)$ is a homeomorphism, $f(V)$ is open in Y and $f(x) \in f(V)$. So $f(V)$ contains a base element $B_\epsilon^d(f(x))$. Since f is one-one, $f^{-1}[B_\epsilon^d(f(x))] \subseteq V$.

let $z \in B_\epsilon^{d_f}(x) \Rightarrow d_f(z, x) < \epsilon \Rightarrow d(f(z), f(x)) < \epsilon \Rightarrow f(z) \in B_\epsilon^d(f(x)) \Rightarrow f[B_\epsilon^{d_f}(x)] \subseteq B_\epsilon^d(f(x))$. Conversely, let $y \in B_\epsilon^d(f(x))$. Since f is onto, there exist a $z \in X$ such that $y = f(z)$ and $d(f(z), f(x)) < \epsilon \Rightarrow d_f(z, x) < \epsilon \Rightarrow z \in B_\epsilon^{d_f}(x)$. That is every member of $B_\epsilon^d(f(x))$ is of the form $f(z)$ for some $z \in B_\epsilon^{d_f}(x)$. That means, $B_\epsilon^d(f(x)) \subseteq f[B_\epsilon^{d_f}(x)]$. Therefore $B_\epsilon^d(f(x)) = f[B_\epsilon^{d_f}(x)]$. Considering f^{-1} on both sides, we get, $f^{-1}[B_\epsilon^d(f(x))] = f^{-1}[f[B_\epsilon^{d_f}(x)]]$. But $f^{-1}[f[B_\epsilon^{d_f}(x)]] = B_\epsilon^{d_f}(x)$, since f is one-one. That is $f^{-1}[B_\epsilon^d(f(x))] = B_\epsilon^{d_f}(x)$.

Therefore $B_\epsilon^{d_f}(x) \subseteq V$. By Theorem 2.2 topology induced by d_f is stronger than \mathcal{T} . That is $\mathcal{T} \subseteq \tau_{d_f}$. Conversely, consider a base element of τ_{d_f} . Let it be $B_\epsilon^d(f(x))$ for some $x \in X$. The open ball $B_\epsilon^d(f(x))$ is open in Y with respect to τ_d . Since f is continuous with respect to \mathcal{T} and τ_d , it follows that $f^{-1}[B_\epsilon^d(f(x))] \in \mathcal{T}$. Already we have shown that $B_\epsilon^{d_f}(x) = f^{-1}[B_\epsilon^d(f(x))]$. Therefore $B_\epsilon^{d_f}(x) \in \mathcal{T}$. This implies that \mathcal{T} is stronger than the topology τ_{d_f} . Therefore $\tau_{d_f} \subseteq \mathcal{T}$. Hence it follows that $\mathcal{T} = \tau_{d_f}$. □

Theorem 3.4. *Let X be a non empty set. If (Y, d) is a complete metric space and $f : X \rightarrow Y$ is a bijection, then (X, d_f) is a complete metric space, where d_f is defined as in Theorem 3.1.*

Proof. Let $\{x_n\}$ be a cauchy sequence in X . Then for a give $\epsilon > 0$ there exist $M \in \mathbb{N}$ such that $d_f(x_k, x_m) < \epsilon$ for all $k, m \geq M$. By the definition of d_f , $d(f(x_k), f(x_m)) < \epsilon$ for all $k, m \geq M$. Therefore the sequence $\{f(x_n)\}$ is a cauchy sequence in Y . Since Y is complete, $\{f(x_n)\}$ is converges to y for some $y \in Y$. There exist $x \in X$ such that $y = f(x)$, since f is onto. Continuity of f^{-1} implies that the sequence $\{f^{-1}[f(x_n)]\}$ converges to $f^{-1}[f(x)]$. But $f^{-1}[f(x_n)] = x_n$ and $f^{-1}[f(x)] = x$, since f is one-one. Therefore the sequence $\{x_n\}$ converges to $x \in X$. Hence (X, d_f) is complete. \square

Example 3.5. Let $X = (-1, 1)$ and $Y = \mathbb{R}$. Consider (X, \mathcal{T}) and (Y, d) , where \mathcal{T} is the usual topology and d is the usual metric, on X and Y respectively. Define $f : X \rightarrow Y$ as $f(x) = \tan\left(\frac{\pi}{2}x\right)$. Then f is a homeomorphism. By Theorem 3.1,

$$d_f(x, y) = d\left(\tan\left(\frac{\pi}{2}x\right), \tan\left(\frac{\pi}{2}y\right)\right) = \left|\tan\left(\frac{\pi}{2}x\right) - \tan\left(\frac{\pi}{2}y\right)\right|$$

is a metric on X . By Theorem 3.3, τ_{d_f} is same as the usual topology. Since (Y, d) is complete, it follows from Theorem 3.4 that (X, d_f) is complete. Hence $(-1, 1)$ is complete with respect to metric $d_f(x, y) = \left|\tan\left(\frac{\pi}{2}x\right) - \tan\left(\frac{\pi}{2}y\right)\right|$ and d_f induces the usual topology on X .

Remark 3.2. As defined in the previous example, there are infinitely many such homeomorphisms from $X = (-1, 1)$ to $Y = \mathbb{R}$; for instance, $kf(x) = k \tan\left(\frac{\pi}{2}x\right)$ is also a homeomorphism from X to Y , where $k > 0$. Then Theorem 3.4 guarantees that there are infinitely many complete metrics on $(-1, 1)$ which generates usual topology on it.

Theorem 3.6. Let X be a non empty set and (Y, d) be a metric space. If $f : X \rightarrow Y$ is a bijection and if (X, d_f) is a complete metric space, then (Y, d) is complete.

Proof. Let $\{y_n\}$ be a cauchy sequence in Y . Then for a given $\epsilon > 0$ there exist $M \in \mathbb{R}$ such that $d(y_k, y_m) < \epsilon$ for all $k, m \geq M$. Since f is onto, there exist a sequence $\{x_n\}$ in X such that $f(x_n) = y_n$. Therefore $d(f(x_k), f(x_m)) < \epsilon$ for all $k, m \geq M$. It follows from the definition of d_f that $d_f(x_k, x_m) < \epsilon$ for all $k, m \geq M$. This means that the sequence $\{x_n\}$ is a cauchy sequence in X . Therefore $\{x_n\}$ converges to some $x \in X$. Since f is continuous, the sequence $\{f(x_n)\}$ converges to $f(x) \in Y$. Hence (Y, d) is complete. \square

Example 3.7. Let $X = \mathbb{R}$ and $Y = (-1, 1)$. Consider (X, \mathcal{T}) and (Y, d) , where \mathcal{T} is the usual topology and d is the usual metric, on X and Y respectively. Define $f : X \rightarrow Y$ as $f(x) = \frac{2}{\pi} \tan^{-1} x$. Then f is a homeomorphism. By Theorem 3.1,

$$d_f(x, y) = d\left(\frac{2}{\pi} \tan^{-1} x - \frac{2}{\pi} \tan^{-1} y\right) = \frac{2}{\pi} |\tan^{-1} x - \tan^{-1} y|$$

is a metric on X . From Theorem 3.6, it follows that, X is incomplete with respect to d_f . Hence \mathbb{R} is incomplete with respect to the metric $d_f = \frac{2}{\pi} |\tan^{-1} x - \tan^{-1} y|$ and it induces the usual topology on \mathbb{R} .

Remark 3.3. As defined in the previous example, there are infinitely many such homeomorphisms from $X = \mathbb{R}$ to $Y = (-1, 1)$; for instance, $f(x) = \frac{2}{\pi} \tan^{-1}(kx)$ is also homeomorphism from X to Y , where $k \neq 0$. Then Theorem 3.6 guarantees that there are infinitely many incomplete metrics on \mathbb{R} which generates usual topology on it.

4 Conclusion

Corresponding to every metric there exist a bounded metric such that both induces the same topology. This paper guarantees the existence of infinitely many bounded incomplete metrics on \mathbb{R} which induces the usual topology on \mathbb{R} . Every norm induced metric ($d(x, y) = ||x - y||$) on \mathbb{R} is complete. The existence of infinitely many incomplete metrics on \mathbb{R} ensures the presence of infinitely many metrics on \mathbb{R} that are not norm induced.

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