SOLVING QUADRATIC EQUATIONS WITH COMPLEX COEFFICIENTS & DETERMINING THE SECOND ROOT FOR ANY COMPLEX NUMBER

Abstract. This article presents a definitive solution for expressing the second root $z^{1/2}$ in terms of the complex variable z and $\Delta^{1/2}$ as a function of Δ in the context of second-degree equations with complex coefficients. The research yields two direct algorithms and accompanying computer software programs. These algorithms are not only exact but also exceptionally fast, offering a powerful tool for efficiently determining the second root for any complex number and solving quadratic equations with complex coefficients. This innovative approach addresses a longstanding challenge in the field, providing a significant contribution to the accuracy and speed of computations in complex coefficient scenarios.

Math. Subject Classification [2020]: Primary 97F50, 68-04; Secondary 97H30 , 65H04.
Key Words and Phrases: Complex numbers, Equations of the second degree, Algorithm, The second root.

1. Introduction:

1.1. Historical Context:

- Ancient Use of Square Completions: Completing the square was used by ancient mathematicians such as **Diophantus** and **Brahmagupta** [7] [8]. They applied geometric methods to solve quadratic equations.
- **Al-Khwarizmi**'s Influence: **Al-Khwarizmi** [2], a Persian mathematician in the 9th century, wrote a book that introduced the method of completing the square for solving quadratic equations. His work significantly influenced European mathematicians during the Middle Ages.
- European Developments: European mathematicians like **François Viète** [12] and **René Descartes** [6] continued to refine and develop methods for solving quadratic equations, contributing to the eventual formulation of the quadratic formula.

The concept of complex numbers, involving imaginary units, was developed in the 16th and 17th centuries to solve equations that didn't have real solutions. Mathematicians like **Rafael Bombelli** [9] and **John Wallis** [10] made early contributions.

In the late 18th century, **John Wallis** introduced the concept of the *Argand diagram*, a graphical representation of complex numbers on a plane. This laid the groundwork for expressing complex numbers in rectangular form as "$a+bi$", where a and b are the real and imaginary parts, respectively.

Leonhard Euler's formula [11], $e^{i\theta} = \cos\theta + i\sin\theta$, provided a powerful connection between complex numbers and trigonometry. This representation paved the way for expressing complex numbers in polar form, $r(\cos\theta + i\sin\theta)$.

In the 18th century, **Abraham de Moivre** [1] generalized **Euler**'s formula with his theorem, $z^n = r^n [\cos(n\theta) + i\sin(n\theta)]$, providing a method for finding powers of complex numbers. This theorem is fundamental for understanding the roots of complex numbers.

The rectangular form of the square root involves manipulating the polar form and using **De Moivre**'s Theorem. For a complex number $z = a + bi$, the square roots can be expressed in the rectangular form using the quadratic formula and trigonometric functions.

1.2. **Setting up the problem:**

This article aims to give once and for all $z^{\frac{1}{2}}$ in function of z (and so $\Delta^{\frac{1}{2}}$ in function of Δ).

To solve a quadratic equation, which has the form $Ax^2 + Bx + C = 0$, you can use **the quadratic formula**:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

The discriminant is the expression under the square root in the quadratic formula:

$$\Delta = B^2 - 4AC$$

The nature of the roots of equation depends on the value of the discriminant:
If $\Delta \succ 0$, there are two distinct real roots.
If $\Delta = 0$, there is one real root (a repeated root).
If $\Delta \prec 0$, there are two complex conjugate roots.

Remark 1. *This famous discussion remains restricted in the case where all the coefficients A, B and C are real. The problem therefore appears when the coefficients A, B and C are complex numbers.*

- Firstly because we are not allowed to write the square root of a complex number, because the square root don't usualy verify the relationship
$\sqrt{a}\sqrt{b} = \sqrt{ab}$ *for complex number since*
$-1 = i^2 = i.i = \sqrt{(-1)}\sqrt{(-1)} = \sqrt{(-1)^2} = \sqrt{1} = 1$ *(absurd)*

- Secondly, we did not have a direct and simple formula (until the appearance of our publications in 2015 See [3], [4] *and* [5]*) which gives us the second roots of a complex number z as a function of z. That is to say*

$$z^{\frac{1}{2}} = f(z)$$

- Thirdly, if $\Delta \in \mathbb{C} \backslash \mathbb{R}$ *and we cannot write* Δ *in the exponential or the geometric form.*

Hence, $\exists (a, b) \in \mathbb{R} \; / \; \Delta = a + ib$ *(which means* Δ *is a complex not real with* $b \neq 0$*).*

Thereafter, we write $\Delta^{\frac{1}{2}}$ *in the algebraic form* $\Delta^{\frac{1}{2}} = x + iy$ *and we try to find a couple* $(x, y) \in \mathbb{R}^2$ *such that* $(x + iy)^2 = a + ib$.

This step is a bit long, especially with the distressing calculations that repeats every time when practical work, specially when it comes to teach students how to find $z^{\frac{1}{2}}$ *by hand, mostly when we are solving an equation of second degree with complex coefficients or a second degree differential equations.*

This article applies to establish a definitive expression for the second root, denoted $z^{1/2}$, as a function of the complex variable z and further aims to formulate $\Delta^{1/2}$ as a function of Δ. The focus is on cases where Δ represents the discriminant of second-degree equations with complex coefficients. The outcomes of this research culminate in the development of two direct algorithms and the corresponding computer software programs. These algorithms are exact and exceptionally fast, providing a robust solution for efficiently determining the second root for any complex number and solving quadratic equations with complex coefficients.

1.3. **The second root:** The "second root" of a complex number refers to finding the square root of a complex number.

Let's denote a complex number as $z = a + bi$, where a and b are real numbers and i is the imaginary unit $i^2 = -1$.

1.4. **Formulas for the Square Root of a complex number:**

1.4.1. *If* $z \in \mathbb{R}^*$*:*
$z^{\frac{1}{2}} = \pm i^{\frac{1 - signe(z)}{2}} \sqrt{|z|}$
where
if $z \geq 0$ then $signe(z) = +1$ and $|z| = z$
if $z \leq 0$ then $signe(z) = -1$ and $|z| = -z$
or

$$z^{\frac{1}{2}} = \pm i^{\frac{z - |z|}{2z}} \sqrt{|z|}$$

since $\forall z \in \mathbb{R}^*, signe(z) = \frac{|z|}{z}$

1.4.2. *If* $z \notin \mathbb{R}$*:*

The Polar Form:
If $z = r \left(\cos\theta + i \sin\theta \right)$ is the polar form of a complex number, then the square roots are given by:

$$z^{\frac{1}{2}} = \pm\sqrt{r} \left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right) \right)$$

$$z^{\frac{1}{2}} = \pm\sqrt{r} e^{i\frac{\theta}{2}}$$

Rectangular Form:
If $z = a + ib$ is in rectangular form, the square roots are found by using the quadratic formula:

$$z^{\frac{1}{2}} = \pm\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}} + i.sgn\,(b)\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$$

where sgn(b) is the sign function.

2. Our Contributions

Recently we found a more "beautiful" formula for the second root of any complex number z and which is (in) a function of this complex number z.
Let $z \notin \mathbb{R}$ which means that $z \in \mathbb{C}\backslash\mathbb{R}$

$$z^{\frac{1}{2}} = \pm\frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}}$$

See [3], [4] and [5]
where
$|z| = \sqrt{Re(z)^2 + Im(z)^2}$ (z's module) or $|z| = \sqrt{z\overline{z}}$

$Re(z)$ (the real part of z)

$Im(z)$ (the imaginary part of z).

Proof. First we have $\forall z \notin \mathbb{R}$, $|z| + Re(z) \succ 0$:
$|z|^2 = [Re(z)]^2 + [Im(z)]^2 \succ [Re(z)]^2$
such $z \notin \mathbb{R} \Rightarrow Im(z) \neq 0 \Rightarrow [Im(z)]^2 \succ 0$
$\Rightarrow -|z| \prec Re(z) \prec |z|$
$\Rightarrow 0 \prec Re(z) + |z| \prec 2\,|z|$.

$$\left(\pm\frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}}\right)^2 = \frac{|z|^2 + 2\,|z|\,z + z^2}{2\,(|z| + Re(z))} = \frac{z\overline{z} + 2\,|z|\,z + z^2}{2\,(|z| + Re(z))}$$

$$= \frac{z}{2}\left(\frac{\overline{z} + 2\,|z| + z}{|z| + Re(z)}\right) = \frac{z}{2}\left(\frac{2\,|z| + 2Re(z)}{|z| + Re(z)}\right)$$
$$= z$$

because $\overline{z} + z = 2Re(z)$, $z\overline{z} = |z|^2$. □

Claim 1.

$$\boxed{\begin{array}{l} \text{If } z \in \mathbb{R}^* \text{ then } z^{\frac{1}{2}} = \pm i^{\frac{z-|z|}{2z}}\sqrt{|z|} \\ \text{If } z \notin \mathbb{R} \text{ then } z^{\frac{1}{2}} = \pm\frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}} \end{array}}$$

Conclusion 1. *Subsequently we developed an algorithm and finally programed it in the form of an application (**Code of Complexe-Square-Root** see Annex) to be implemented in Androids.*

3. The new way to solve an equation of second degree:

Let $(a,b,c)\in\mathbb{C}^3$, the equation:

$$az^2+bz+c=0$$

$$a\neq 0 \Longrightarrow z^2+\frac{b}{a}z+\frac{c}{a}=0$$

$$\Longleftrightarrow z^2+2\frac{b}{2a}z+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}=0$$

$$\Longleftrightarrow \left(z+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Let $\Delta\in\mathbb{C}$ such that $\Delta=b^2-4ac$.
$\left(z+\frac{b}{2a}\right)^2=\frac{\Delta}{4a^2}=\left(\frac{\Delta^{\frac{1}{2}}}{2a}\right)^2$
with $\Delta^{\frac{1}{2}}$ a second root of Δ (in $\mathbb{C}^*$ we have always two second root $\pm\Delta^{\frac{1}{2}}$).
so

$$z=\frac{-b\pm\Delta^{\frac{1}{2}}}{2a}$$

Then we combine all the possibilities in *just 3 cases*:

(1) First case: If $\Delta\in\mathbb{R}^*$
$\Delta^{\frac{1}{2}}=\pm i^{\frac{1-signe(\Delta)}{2}}\sqrt{|\Delta|}$

So we can generalize:
$z=\frac{1}{2a}\left(-b\pm i^{\frac{1-signe(\Delta)}{2}}\sqrt{|\Delta|}\right)\qquad \text{if } \Delta\in\mathbb{R}^*$
Where
if $\Delta\geq 0$ then $signe(\Delta)=+1$ and $|\Delta|=\Delta$
if $\Delta\leq 0$ then $signe(\Delta)=-1$ and $|\Delta|=-\Delta$
Or just $signe(\Delta)=\frac{|\Delta|}{\Delta}$, so

$$z=\frac{1}{2a}\left(-b\pm i^{\frac{\Delta-|\Delta|}{2\Delta}}\sqrt{|\Delta|}\right)$$

(2) Second case: If $\Delta=0$, it is obvious that we have

$$z=\frac{-b}{2a}$$

(3) Third case: If $\Delta\notin\mathbb{R}$ (means that $\Delta\in\mathbb{C}\backslash\mathbb{R}$)
$\Delta^{\frac{1}{2}}=\pm\frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$
and $z=\frac{-b\pm\Delta^{\frac{1}{2}}}{2a}$
so

$$z=\frac{1}{2a}\left(-b\pm\frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}\right)$$

with

$\|\Delta\| = \sqrt{Re(\Delta)^2 + Im(\Delta)^2}$ (Δ's module) or $\|\Delta\| = \sqrt{\Delta\overline{\Delta}}$

$Re(\Delta)$ (the real part of Δ)

$Im(\Delta)$ (the imaginary part of Δ).

Claim 2.

$$\boxed{\begin{array}{c} \textit{First case: If } \Delta \in \mathbb{R}^* \textit{ then } z = \frac{1}{2a}\left(-b \pm i^{\frac{\Delta-|\Delta|}{2\Delta}}\sqrt{|\Delta|}\right) \\ \textit{Second case: If } \Delta = 0 \textit{ then } z = \frac{-b}{2a} \\ \textit{Third case: If } \Delta \notin \mathbb{R} \textit{ then } z = \frac{1}{2a}\left(-b \pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}\right) \end{array}}$$

4. Algorithm

$\rightarrow a, b, c$

$\rightarrow \Delta = b^2 - 4ac$

$\rightarrow$ if $\Delta = 0$ so $z = \frac{1}{2a}$ end

$\rightarrow$ else

- $\rightarrow$ if $\Delta \in \mathbb{R}^*$ so $z = \frac{1}{2a}\left(-b \pm i^{\frac{\Delta-|\Delta|}{2\Delta}}\sqrt{|\Delta|}\right)$ end
- $\rightarrow$ else
  - $\rightarrow \delta = \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$
  - $\rightarrow z = \frac{-b\pm\delta}{2a}$ end

Conclusion 2. *Subsequently we developed a computer program in the form of an application (**Solving a Quadratic Equation with Complex coefficients Code:** see Appendix) to be implemented in androids.*

4.1. Example 1:

Calculate all second root of

$$z = 3 - 4i$$

It is impossible to write it explicitly as an algebraic expression in the exponential or the geometric form:

because $tg\theta = -\frac{4}{3} \Rightarrow \arg(z) = \theta = arctg\left(-\frac{4}{3}\right)$ or $\arg(z) = \theta = arctg\left(-\frac{4}{3}\right) + \pi$ and $arctg\left(-\frac{4}{3}\right)$ can not be writing as an algebraic expression like $arctg\,(1) = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ ...

By applying our formula:

$$z^{\frac{1}{2}} = \pm\frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}}$$

$|z| = \sqrt{9+16}$

$|z| = 5$

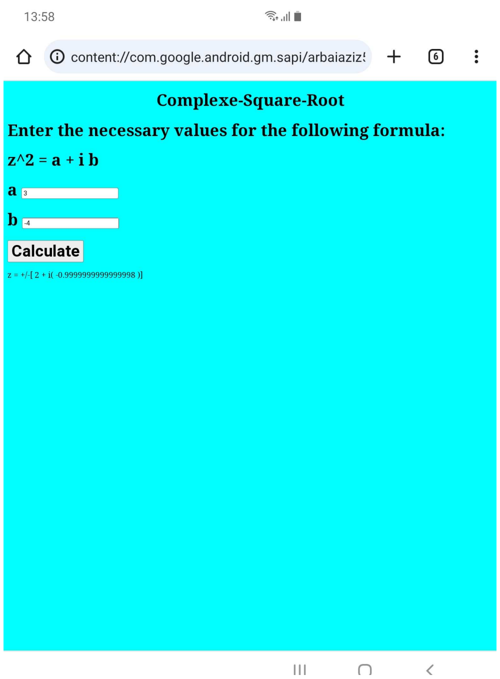
so

$$z^{\frac{1}{2}} = \pm\frac{\frac{5+3-4i}{2}}{\sqrt{\frac{5+3}{2}}}$$

$\Longrightarrow$

$$z^{\frac{1}{2}} = \pm\frac{4-2i}{2} = \pm(2-i)$$

FIGURE 1. Example 1: Results by applying the code.



4.2. **Example 2:**
To solve in $\mathbb{C}$ the equation:

$$z^2 + (2-2i)z - (7+26i) = 0$$

We have $\Delta = (2-2i)^2 + 4.\,(7+26i) = -8i + 28 + 104i = 28 + 96i$
$\Longrightarrow$

$$\Delta = 28 + 96i$$

It is impossible to write Δ explicitly as an algebraic expression in the exponential or the geometric form.

$\Delta \in \mathbb{C}/\mathbb{R} \Longrightarrow \Delta^{\frac{1}{2}} = \pm\frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$ and $z = \frac{-b\pm\Delta^{\frac{1}{2}}}{2a}$

$\|\Delta\| = \sqrt{(28)^2 + (96)^2}$

$\|\Delta\| = 100$

so

$$\Delta^{\frac{1}{2}} = \pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}} = \pm \frac{\frac{100+28+96i}{2}}{\sqrt{\frac{100+28}{2}}}$$

$\Longrightarrow$

$$\Delta^{\frac{1}{2}} = \pm \frac{\frac{128+96i}{2}}{8} = \pm(8+6i)$$

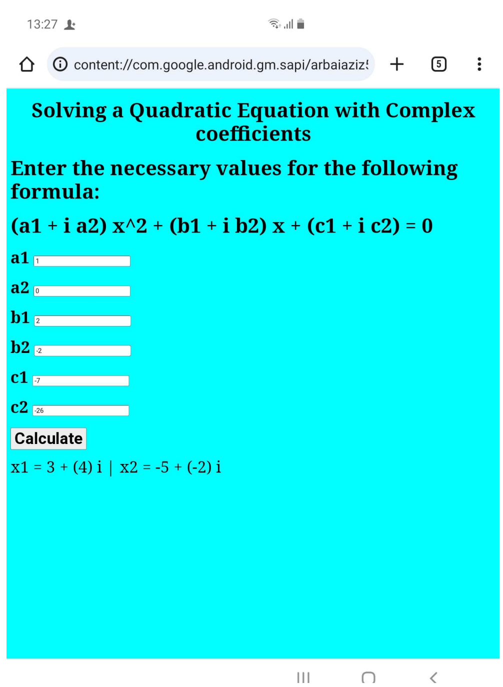
$\Longrightarrow$

$$z = \frac{-(2-2i) \pm (8+6i)}{2}$$

$\Longrightarrow z = 3+4i$ or $z = -5-2i$

$$S = \{3+4i, -5-2i\}$$

FIGURE 2. Example 2: Results by applying the code.



5. Conclusion

In the realm of mathematical discoveries, this research marks a paradigm shift in our understanding of complex numbers and their roots. The study not only introduces a groundbreaking formula for the square root of any complex number but also revolutionizes the methodology employed in solving quadratic equations with complex coefficients.

Key Contributions:

1. Novel Square Root Formula:

- The research introduces a revolutionary formula for the square root of complex numbers, transcending conventional methods. The formula is characterized by its elegance, efficiency, and applicability to a broad spectrum of complex numerical scenarios.

We found the general formula that gives us the second roots (square root) of a complex number.

So, let $\alpha \in \mathbb{C}\backslash\mathbb{R}$, then

$$X^2 = \alpha \Longleftrightarrow X = \pm \frac{\frac{\|\alpha\|+\alpha}{2}}{\sqrt{\frac{\|\alpha\|+Re(\alpha)}{2}}}$$

2. Transformation of Quadratic Equation Solving:

- A novel approach to solving quadratic equations involving complex coefficients is presented. The newfound square root formula serves as the cornerstone for this transformative methodology, offering a fresh perspective on quadratic equation solutions in the complex number domain.

We have a new method and new way to resolve an equation of second degree with complex coefficients.There are two cases:

$\Delta \in \mathbb{R}$ or $\Delta \notin \mathbb{R}$ (with $\Delta = b^2 - 4ac$).

3. Scientific Significance:

- The research not only expands the theoretical framework of complex analysis but also holds profound implications for diverse scientific fields where complex numbers play a pivotal role. Applications range from physics to engineering, where the comprehension and manipulation of complex roots are integral.

4. Pedagogical Impact:

- The pedagogical implications of this research are substantial. The innovative formula and solving approach provide a pedagogically rich platform for educators to engage students in a deeper exploration of complex numbers and quadratic equations. This not only enhances mathematical comprehension but also nurtures a sense of curiosity and innovation in learners.

5. Ultimately, the results of this research culminate in the development of two direct algorithms and the corresponding computer programs see Annex "Code".

This research stands as a testament to the perpetual evolution of mathematical thought. The unveiling of a novel square root formula for complex numbers and its subsequent influence on quadratic equations open avenues for further exploration and applications in the ever-expanding landscape of mathematical sciences. The collaborative effort of theoretical innovation and pedagogical insight underscores the transformative potential of this breakthrough.

6. Annex "Code"

6.1. Complexe-Square-Root Code:

```
<!DOCTYPE html>
<html lang="es">
<head>
<meta name="viewport" comtent="width=device-width, initial-scale=1.0">
<meta charset="UTF-8">
<meta name="author" content="Aziz Arbai">
<title>Complexe-Square-Root</title>
<script>
function calculate() {
var a = Number(document.getElementById("a").value);
var b = Number(document.getElementById("b").value);
var c = ((a**2 + b**2)**(1/2) + a)**(1/2);
var z;
if (a < 0 && b == 0) {
document.getElementById("response").innerHTML = "z = +/- i " + (-a)**(1/2);
} else {
document.getElementById("response").innerHTML = "z = +/- " + c/(2**(1/2))
+ " + i " + b/(c*(2**(1/2)));
}
return 0;
}
</script>
</head>
<body style="background-color:cyan">
<header>
<h1 style="text-align: center"> Complexe-Square-Root</h1>
</header>
<div>
<h1>Enter the necessary values for the following formula:</h1>
<h1>z^2 = a + i b </h1>
</div>
<div>
<h1>a
<input id="a" type="text" value=""></h1>
<h1>b
<input id="b" type="text" value=""></h1>
<button onclick="calculate()" style="font-size: 200%"><b>Calculate</b></button>
</div>
<div>
<p id="response"></p>
</div>
</body>
</html>
```

6.2. Solving a Quadratic Equation with Complex coefficients Code:

```
<!DOCTYPE html>
```

```
<html lang="es">
  <head>
    <meta name="viewport" comtent="width=device-width, initial-scale=1.0">
      <meta charset="UFT-8">
      <meta name="author" content="Aziz Arbai">
      <title>Solving a Quadratic Equation with Complex coefficients</title>
      <script>
        function calculate() {
          var ar = Number(document.getElementById("a1").value);
          var ai = Number(document.getElementById("a2").value);
          var br = Number(document.getElementById("b1").value);
          var bi = Number(document.getElementById("b2").value);
          var cr = Number(document.getElementById("c1").value);
          var ci = Number(document.getElementById("c2").value);
          var d1 = (br**2) - (bi**2) - (4 * ar * cr) + (4 * ai * ci);
          var d2 = (2 * br * bi) - (4 * ar * ci) - (4 * ai * cr);
          var d3 = (((d1)**2) + ((d2)**2))**(0.5);
          var f = 2 * (ar**2 + ai**2);
          var x11, x12, x21, x22;
          if (d2 == 0) {
            if (d1 > 0) {
              x11 = (((-1) * ar * br) - (ai * bi) + (ar * (d1**(1/2)))) / f;
              x12 = (((-1) * ar * bi) + (ai * br) - (ai * (d1**(1/2)))) / f;
              x21 = (((-1) * ar * br) - (ai * bi) - (ar * (d1**(1/2)))) / f;
              x22 = (((-1) * ar * bi) + (ai * br) + (ai * (d1**(1/2)))) / f;
            } else if (d1 <= 0) {
              x11 = (((-1) * ar * br) - (ai * bi) + (ai * (((-1) * d1)**(1/2)))) / f;
              x12 = (((-1) * ar * bi) + (ai * br) + (ar * (((-1) * d1)**(1/2)))) / f;
              x21 = (((-1) * ar * br) - (ai * bi) - (ai * (((-1) * d1)**(1/2)))) / f;
              x22 = (((-1) * ar * bi) + (ai * br) - (ar * (((-1) * d1)**(1/2)))) / f;
            } else {
              alert("Indeterminated definition of the algorithm.");
              return 0;
            }
          } else {
            var S = ((d3 + d1) / 2)**(1/2);
            var I = d2 / (2 * S);
            x11 = (((-1) * ar * br) + (ar * S) - (ai * bi) + (ai * I)) / f;
            x12 = (((-1) * ar * bi) + (ar * I) + (ai * br) - (ai * S)) / f;
            x21 = (((-1) * ar * br) - (ar * S) - (ai * bi) - (ai * I)) / f;
            x22 = (((-1) * ar * bi) - (ar * I) + (ai * br) + (ai * S)) / f;
          }
          document.getElementById("response").innerHTML = "x1 = " + x11
+ " + (" + x12 + ") i | x2 = " + x21 + " + (" + x22 + ") i";
          return 0;
  }
  </script>
  </head>
```

```
<body style=”background-color:cyan”>
    <header>
      <h1 style=”text-align: center”>Solving a Quadratic Equation with Complex
coefficients</h1>
    </header>
    <div>
        <h1>Enter the necessary values for the following formula:</h1>
        <h1>(a1 + i a2) xˆ2 + (b1 + i b2) x + (c1 + i c2) = 0</h1>
    </div>
    <div>
        <h1>a1
        <input id=”a1” type=”text” value=””></h1>
        <h1>a2
        <input id=”a2” type=”text” value=””></h1>
        <h1>b1
        <input id=”b1” type=”text”></h1>
        <h1>b2
        <input id=”b2” type=”text”></h1>
        <h1>c1
        <input id=”c1” type=”text”></h1>
        <h1>c2
        <input id=”c2” type=”text”></h1>
      <button onclick=”calculate()” style=”font-size: 200%”><b>Calculate</b></button>
    </div>
    <div>
        <p id=”response”></p>
    </div>
  </body>
</html>
```

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