Original Research Article

An Introduction and Reexamination of Hyperprobability and Superhyperprobability

Abstract

Mathematical structures can generally be extended into Hyperstructures and SuperHyperstructures by leveraging the power set and n-th iterated powerset constructions (cf. [17, 45, 80]). These frameworks are particularly effective for representing hierarchical structures across diverse conceptual domains.

Probability measures the likelihood of an event occurring, with values ranging from 0 (impossible) to 1 (certain), under specified conditions. HyperProbability extends classical probability theory by assigning a *set* of probability values to each event, thereby capturing uncertainty from multiple sources or expert assessments. SuperHyperProbability further generalizes this notion through successive powerset operations, enabling the modeling of multi-layered uncertainty across various levels of reasoning or belief systems.

In this paper, we revisit the foundational properties of HyperProbability and SuperHyperProbability and provide numerous detailed examples. Through these investigations, we aim to contribute to the advancement of probabilistic modeling and reasoning in hierarchical and multi-level uncertainty contexts.

Keywords: Probability, Hyperprobability, Superhyperprobability

1 Introduction

Real-world phenomena are often modeled using probability [16, 69, 70]. Probability theory serves as a foundational tool in numerous applied fields, including statistics [20, 95], queuing system [15, 88], machine learning [7, 53, 90], and beyond, with active research being conducted daily. In recent years, extended frameworks such as Fuzzy Probability [57, 65, 101], Neutrosophic Probability [73, 75], and Plithogenic Probability [76] have also garnered increasing attention.

Many concepts in the real world exhibit hierarchical structure. To capture such hierarchies, the notions of Hyperstructure and Superhyperstructure have been introduced and studied [83]. Concrete examples of Superhyperstructures include SuperHyperfuzzy Sets [29], Superhyperfunctions [79, 81], SuperHypergraphs [36, 77], and SuperHyperAlgebras [40, 49, 78]. More recently, probability theory itself has been extended along these lines, leading to the development of HyperProbability and SuperHyperProbability [39].

HyperProbability extends classical probability theory by assigning a *set* of probability values to each event, thereby capturing uncertainty from multiple sources or expert assessments. SuperHyperProbability further generalizes this concept through successive powerset operations, enabling the modeling of multi-layered uncertainty across various levels of reasoning or belief systems [39].

Given the importance of these extensions, and with the goal of raising awareness and uncovering new insights, this paper revisits the fundamental properties of HyperProbability and SuperHyperProbability. We place particular emphasis on providing numerous detailed examples, in the hope that this work will facilitate more widespread modeling of hierarchical concepts in real-world applications.

2 Preliminaries and Definitions

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper.

2.1 Classical Structure, Hyperstructure, and n-Superhyperstructure

A *Classical Structure* represents a general mathematical concept, while a *Hyperstructure* can be defined using the power set, and an *n-Superhyperstructure* can be defined using the n-th powerset [82]. Intuitively, the n-th powerset is a repeated application of the powerset operation. Relevant definitions and simple examples are provided below.

Definition 2.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.2 (Powerset)**.** [27,68] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.3 (n-th Powerset)**.** (cf. [22,27,31,72,82])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Example 2.4 (Iterated Powersets for Café Beverage Offerings)**.** Let $H = \{\text{Tea, Coffee}\}$ represent the two beverages available in a café. Then

$$P_1(H) \;=\; P(H) \;=\; \big\{\emptyset,\ \{\text{Tea}\},\ \{\text{Coffee}\},\ \{\text{Tea}, \text{Coffee}\}\big\}.$$

Interpreting:

- $\emptyset$: no beverage offered,
- $\{\text{Tea}\}$: only tea offered,
- $\{\text{Coffee}\}$: only coffee offered,
- $\{\text{Tea}, \text{Coffee}\}$: both tea and coffee offered.

The second powerset is

$$P_2(H) \;=\; P\big(P_1(H)\big) \;=\; \big\{X \subseteq P_1(H)\big\},$$

which has $2^4 = 16$ elements. Concretely,

$$\begin{aligned} P_2(H) = \{\, & \emptyset,\ \{\emptyset\},\ \{\{\text{Tea}\}\},\ \{\{\text{Coffee}\}\},\ \{\{\text{Tea}, \text{Coffee}\}\}, \\ & \{\emptyset, \{\text{Tea}\}\},\ \{\emptyset, \{\text{Coffee}\}\},\ \{\emptyset, \{\text{Tea}, \text{Coffee}\}\}, \\ & \{\{\text{Tea}\}, \{\text{Coffee}\}\},\ \{\{\text{Tea}\}, \{\text{Tea}, \text{Coffee}\}\},\ \{\{\text{Coffee}\}, \{\text{Tea}, \text{Coffee}\}\}, \\ & \{\emptyset, \{\text{Tea}\}, \{\text{Coffee}\}\},\ \{\emptyset, \{\text{Tea}\}, \{\text{Tea}, \text{Coffee}\}\},\ \{\emptyset, \{\text{Coffee}\}, \{\text{Tea}, \text{Coffee}\}\}, \\ & \{\{\text{Tea}\}, \{\text{Coffee}\}, \{\text{Tea}, \text{Coffee}\}\},\ \{\emptyset, \{\text{Tea}\}, \{\text{Coffee}\}, \{\text{Tea}, \text{Coffee}\}\}\}. \end{aligned}$$

Each element of $P_2(H)$ describes a *daily menu plan* comprising one or more of the possible beverage-offering scenarios from $P_1(H)$.

The third powerset is

$$P_3(H) \;=\; P\big(P_2(H)\big),$$

with $2^{16} = 65\,536$ elements, modeling *weekly menu schedules* (each schedule is a set of daily menus). For brevity, we list a few representative elements:

$$\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\{\{\text{Tea}\}, \{\{\text{Coffee}\}\}\}, \quad P_2(H).$$

- $\emptyset$: no weekly schedule defined.
- $\{\emptyset\}$: every day has "no beverage" on the menu.
- $\{\{\emptyset\}\}$: the weekly plan contains only the "no beverage" daily-menu as a single option.
- $\{\{\{\text{Tea}\}\}, \{\{\text{Coffee}\}\}\}$: a weekly plan alternating days offering only tea or only coffee.
- $P_2(H)$: the weekly plan that includes all possible daily menus.

In this way, $P_n(H)$ captures progressively higher "meta" levels of choice: ingredients $\rightarrow$ daily menus $\rightarrow$ weekly schedules $\rightarrow \ldots$.

Definition 2.5 (Classical Structure)**.** (cf. [72, 82]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, equipped with one or more *Classical Operations* that satisfy specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \rightarrow H,$$

where $m \geq 1$ is a positive integer, and H^m denotes the m-fold Cartesian product of H. Common examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

Definition 2.6 (Hyperoperation)**.** (cf. [66, 91–93]) A *hyperoperation* is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set S, a hyperoperation $\circ$ is defined as:

$$\circ : S \times S \rightarrow \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the powerset of S.

Definition 2.7 (Hyperstructure)**.** (cf. [27, 72, 82]) A *Hyperstructure* extends the notion of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is the powerset of S, and $\circ$ is an operation defined on subsets of $\mathcal{P}(S)$. Hyperstructures allow for generalized operations that can apply to collections of elements rather than single elements.

Definition 2.8 (SuperHyperOperations)**.** (cf. [82]) Let H be a non-empty set, and let $\mathcal{P}(H)$ denote the powerset of H. The n-th powerset $\mathcal{P}^n(H)$ is defined recursively as follows:

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)), \quad \text{for } k \geq 0.$$

A *SuperHyperOperation* of order (m, n) is an m-ary operation:

$$\circ^{(m,n)} : H^m \rightarrow \mathcal{P}^n_*(H),$$

where $\mathcal{P}^n_*(H)$ represents the n-th powerset of H, either excluding or including the empty set, depending on the type of operation:

- If the codomain is $\mathcal{P}^n_*(H)$ excluding the empty set, it is called a *classical-type* (m, n)*-SuperHyperOperation*.
- If the codomain is $\mathcal{P}^n(H)$ including the empty set, it is called a *Neutrosophic* (m, n)*-SuperHyperOperation*.

These SuperHyperOperations are higher-order generalizations of hyperoperations, capturing multi-level complexity through the construction of n-th powersets.

Example 2.9 (SuperHyperOperation for Project Phase Grouping)**.** Project management is the application of knowledge, skills, tools, and techniques to plan, execute, and complete projects effectively and efficiently (cf. [24, 25]). A project phase is a distinct stage in a project lifecycle, representing specific objectives, tasks, and deliverables within a defined timeline (cf. [6, 47, 60]). Let

$$H = \{T_1, T_2, T_3\}$$

be the set of three tasks in a small project, and consider the $(3, 2)$-SuperHyperOperation

$$\circ^{(3,2)} : H^3 \longrightarrow \mathcal{P}_*^2(H) = \mathcal{P}\big(\mathcal{P}(H)\big) \setminus \{\emptyset\}.$$

We define $\circ^{(3,2)}$ by having it return the set of all possible "phase groupings" of the three tasks, namely:

$$\circ^{(3,2)}(T_1, T_2, T_3) = \Big\{\{\{T_1\}, \{T_2\}, \{T_3\}\},\ \{\{T_1, T_2\}, \{T_3\}\},\ \{\{T_1, T_3\}, \{T_2\}\},\ \{\{T_2, T_3\}, \{T_1\}\},\ \{\{T_1, T_2, T_3\}\}\Big\}.$$

Here each element of the outer set is a subset of H describing one way to partition the tasks into project phases:

- $\{\{T_1\}, \{T_2\}, \{T_3\}\}$: three separate phases, each containing exactly one task.
- $\{\{T_1, T_2\}, \{T_3\}\}$: one phase with tasks T_1, T_2 followed by a second phase with task T_3.
- $\{\{T_1, T_3\}, \{T_2\}\}$ and $\{\{T_2, T_3\}, \{T_1\}\}$: the analogous two-phase groupings.
- $\{\{T_1, T_2, T_3\}\}$: a single phase containing all tasks.

Thus $\circ^{(3,2)}$ captures all five distinct ways to organize three tasks into phases, demonstrating a concrete, real-life SuperHyperOperation.

Definition 2.10 (n-Superhyperstructure)**.** (cf. [26, 28, 30, 72, 82]) An *n-Superhyperstructure* further generalizes a Hyperstructure by incorporating the n-th powerset of a base set. It is formally described as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ represents an operation defined on elements of $\mathcal{P}_n(S)$. This iterative framework allows for increasingly hierarchical and complex representations of relationships within the base set.

Example 2.11 (2-Superhyperstructure in Corporate Department Design)**.** Let

$$S = \{\text{Alice, Bob, Carol}\}$$

be the set of three employees. Then:

$$P_1(S) = P(S) = \big\{\emptyset, \{\text{Alice}\}, \{\text{Bob}\}, \{\text{Carol}\}, \{\text{Alice}, \text{Bob}\}, \{\text{Alice}, \text{Carol}\}, \{\text{Bob}, \text{Carol}\}, \{\text{Alice}, \text{Bob}, \text{Carol}\}\big\}.$$

Each element of $P_1(S)$ represents one possible *team*.

The second powerset is

$$P_2(S) = P\big(P_1(S)\big),$$

whose elements are *departmental structures*, i.e. sets of teams. As an example, consider two specific departmental structures:

$$D_1 = \big\{\{\text{Alice}, \text{Bob}\}, \{\text{Carol}\}\big\},$$
$$D_2 = \big\{\{\text{Alice}\}, \{\text{Bob}, \text{Carol}\}\big\}.$$

Here:

- D_1 groups Alice and Bob into one department and Carol into another.

- D_2 places Alice alone and groups Bob with Carol.

Define the 2-superhyperstructure

$$\mathcal{SH}_2 = \big(P_2(S),\, \cup\big),$$

where the operation $\cup : P_2(S) \times P_2(S) \to P_2(S)$ is the union of departmental structures. Then

$$D_1 \,\cup\, D_2 = \big\{\{\text{Alice}, \text{Bob}\},\, \{\text{Carol}\},\, \{\text{Alice}\},\, \{\text{Bob}, \text{Carol}\}\big\} \;\in\; P_2(S).$$

This union represents a meta-structure that simultaneously retains both original department configurations, modeling a flexible organizational plan that can switch between the two designs.

In this way, $\mathcal{SH}_2$ captures the hierarchical complexity of team formation (first level) and department design (second level) within a single algebraic framework.

2.2 Probability Space

The definition of a probability space is presented below.

Definition 2.12 (Probability Space)**.** (cf. [14, 19, 59]) A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ where

- Ω is a nonempty set called the *sample space*;
- $\mathcal{F}$ is a σ-algebra of subsets of Ω, called the *event space*;
- $P : \mathcal{F} \to [0, 1]$ is a function, called a *probability measure*, satisfying:
  1. $P(A) \geq 0$ for all $A \in \mathcal{F}$ (non-negativity),
  2. $P(\Omega) = 1$ (normalization),
  3. For any countable sequence of pairwise disjoint events $A_1, A_2, \dots \in \mathcal{F}$,

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i) \quad \text{(countable additivity).}$$

Example 2.13 (Fair Coin Toss)**.** Let $\Omega = \{\text{H}, \text{T}\}$, $\mathcal{F} = 2^{\Omega}$, and define

$$P(\{\text{H}\}) = P(\{\text{T}\}) = \tfrac{1}{2}.$$

Then $(\Omega, \mathcal{F}, P)$ is a probability space modeling a fair coin toss, since $P(\Omega) = 1$, each event has nonnegative probability, and for the disjoint events $\{\text{H}\}, \{\text{T}\}$,

$$P(\{\text{H}\} \cup \{\text{T}\}) = P(\Omega) = P(\{\text{H}\}) + P(\{\text{T}\}) = 1.$$

3 HyperProbability

HyperProbability assigns a set of probability values to each event, capturing uncertainty from multiple sources.

Definition 3.1 (Hyper-Probability)**.** [39] Let Ω be a sample space and $\mathcal{F}$ a σ-algebra of events. A *Hyper-Probability* is a function

$$\text{HP} : \mathcal{F} \;\longrightarrow\; \mathcal{P}([0, 1]),$$

where $\mathcal{P}([0, 1])$ denotes the power set of the unit interval. For each event $A \in \mathcal{F}$,

$$\text{HP}(A) \;=\; \{\, p_k(A) \mid k \in K_A\},$$

with each classical probability $p_k(A) \in [0, 1]$. Thus $\text{HP}(A)$ collects multiple probability assessments for the same event.

Example 3.2 (Hyper-Probability in Weather Forecasting — Ensemble Models)**.** Weather forecasting is the scientific process of predicting atmospheric conditions such as temperature, precipitation, and wind for future time periods (cf. [5, 11, 61, 67]). Let the sample space be

$$\Omega = \{\text{rain}, \text{ no rain}\},$$

and let $\mathcal{F} = 2^{\Omega}$. Suppose three independent forecasting models A, B, and C produce the following probability measures:

$$\begin{aligned} p_A(\{\text{rain}\}) &= 0.65, & p_A(\{\text{no rain}\}) &= 0.35, \\ p_B(\{\text{rain}\}) &= 0.72, & p_B(\{\text{no rain}\}) &= 0.28, \\ p_C(\{\text{rain}\}) &= 0.80, & p_C(\{\text{no rain}\}) &= 0.20. \end{aligned}$$

We summarize these in Table 1.

Model	$p_k(\{\text{rain}\})$	$p_k(\{\text{no rain}\})$
A	0.65	0.35
B	0.72	0.28
C	0.80	0.20

Table 1: Rain-chance forecasts from three meteorological models

The Hyper-Probability measure $\text{HP} : \mathcal{F} \to \mathcal{P}([0, 1])$ is then given by

$$\begin{aligned} \text{HP}(\{\text{rain}\}) &= \{\, 0.65,\ 0.72,\ 0.80\}, \\ \text{HP}(\{\text{no rain}\}) &= \{\, 0.35,\ 0.28,\ 0.20\}, \\ \text{HP}(\Omega) &= \{1,\ 1,\ 1\}, \\ \text{HP}(\emptyset) &= \{0,\ 0,\ 0\}. \end{aligned}$$

Here:

- $\text{HP}(\{\text{rain}\})$ aggregates the three "rain" probabilities.
- $\text{HP}(\{\text{no rain}\})$ aggregates the complementary probabilities.
- $\text{HP}(\Omega) = \{p_k(\Omega)\} = \{1, 1, 1\}$ since each model assigns probability 1 to the certain event.
- $\text{HP}(\emptyset) = \{p_k(\emptyset)\} = \{0, 0, 0\}$ since each model assigns probability 0 to the impossible event.

This detailed ensemble example shows how Hyper-Probability captures multiple expert or model-based assessments for each event.

Example 3.3 (Hyper-Probability in Medical Diagnosis)**.** Medical diagnosis is the process of identifying a disease or condition based on a patient's symptoms, history, and diagnostic tests (cf. [1, 54, 71]). Let the sample space be

$$\Omega = \{\text{Disease}, \text{ No Disease}\},$$

and let $\mathcal{F} = 2^{\Omega}$. Suppose three clinicians A, B, C provide independent probability assessments for the presence of the disease:

$$\begin{aligned} p_A(\{\text{Disease}\}) &= 0.85, & p_A(\{\text{No Disease}\}) &= 0.15, \\ p_B(\{\text{Disease}\}) &= 0.90, & p_B(\{\text{No Disease}\}) &= 0.10, \\ p_C(\{\text{Disease}\}) &= 0.75, & p_C(\{\text{No Disease}\}) &= 0.25. \end{aligned}$$

Clinician	$p_k(\{\text{Disease}\})$	$p_k(\{\text{No Disease}\})$
A	0.85	0.15
B	0.90	0.10
C	0.75	0.25

Table 2: Disease-presence probability estimates from three clinicians

The Hyper-Probability measure $\mathrm{HP} : \mathcal{F} \to \mathcal{P}([0,1])$ is then given by

$$\begin{aligned} \mathrm{HP}(\{\text{Disease}\}) &= \{0.85,\ 0.90,\ 0.75\}, \\ \mathrm{HP}(\{\text{No Disease}\}) &= \{0.15,\ 0.10,\ 0.25\}, \\ \mathrm{HP}(\Omega) &= \{1,\ 1,\ 1\}, \\ \mathrm{HP}(\emptyset) &= \{0,\ 0,\ 0\}. \end{aligned}$$

In this example:

- $\mathrm{HP}(\{\text{Disease}\})$ aggregates the three clinicians' assessments for disease presence.
- $\mathrm{HP}(\{\text{No Disease}\})$ aggregates their complementary assessments.
- $\mathrm{HP}(\Omega) = \{1, 1, 1\}$ since each clinician assigns probability 1 to the certain event.
- $\mathrm{HP}(\emptyset) = \{0, 0, 0\}$ since each clinician assigns probability 0 to the impossible event.

This medical-diagnosis scenario illustrates how Hyper-Probability captures multiple expert judgments for a critical event.

Theorem 3.4 (Null Event)**.** $\mathrm{HP}(\emptyset) = \{0\}$.

Proof. By definition, for every $k \in K$,

$$p_k(\emptyset) = 0.$$

Hence $\mathrm{HP}(\emptyset) = \{\, p_k(\emptyset) \mid k \in K\} = \{0\}$. □

Theorem 3.5 (Certain Event)**.** $\mathrm{HP}(\Omega) = \{1\}$.

Proof. For each $k \in K$, the normalization axiom of probability gives $p_k(\Omega) = 1$. Therefore $\mathrm{HP}(\Omega) = \{\, p_k(\Omega) \mid k \in K\} = \{1\}$. □

Theorem 3.6 (Complement)**.** *For any* $A \in \mathcal{F}$*,*

$$\mathrm{HP}(A^c) = \{\, 1 - p_k(A) \mid k \in K\}.$$

Proof. Since each p_k is a probability measure,

$$p_k(A^c) = 1 - p_k(A).$$

Taking k over all of K yields $\mathrm{HP}(A^c) = \{\, p_k(A^c) \mid k \in K\} = \{\, 1 - p_k(A) \mid k \in K\}$. □

Theorem 3.7 (Finite Additivity)**.** *If* $A, B \in \mathcal{F}$ *are disjoint, then*

$$\mathrm{HP}(A \cup B) = \left\{\, p_k(A) + p_k(B) \mid k \in K \right\}.$$

Proof. For each $k \in K$, since $A \cap B = \emptyset$,

$$p_k(A \cup B) = p_k(A) + p_k(B).$$

Thus

$$\mathrm{HP}(A \cup B) = \{\, p_k(A \cup B) \mid k \in K\} = \{\, p_k(A) + p_k(B) \mid k \in K\}.$$

□

Theorem 3.8 (Monotonicity of Supremum)**.** *If* $A, B \in \mathcal{F}$ *with* $A \subseteq B$*, then* $\sup \mathrm{HP}(A) \ \leq\ \sup \mathrm{HP}(B)$.

Proof. For each k, $p_k(A) \leq p_k(B)$ by the monotonicity of probability measures. Taking the supremum over $k \in K$ yields

$$\sup_{k \in K} p_k(A) \ \leq\ \sup_{k \in K} p_k(B),$$

i.e. $\sup \mathrm{HP}(A) \leq \sup \mathrm{HP}(B)$. □

4 n-SuperHyperProbability

n-SuperHyperProbability generalizes HyperProbability by iteratively applying powersets, modeling layered uncertainty across multiple decision or belief levels.

Definition 4.1 (n-SuperHyperProbability)**.** [39] For $n \geq 1$, an *n-SuperHyperProbability* is a function

$$\mathrm{SHP}^{(n)} : \mathcal{F} \ \longrightarrow\ P^n([0,1]),$$

where $P^n([0,1])$ denotes the n-fold iterated power set of $[0,1]$. It is defined recursively by

$$\mathrm{SHP}^{(1)}(A) = \mathrm{HP}(A), \qquad \mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big) \quad (n \geq 2).$$

In particular, $\mathrm{SHP}^{(1)}$ coincides with Hyper-Probability and for $n = 0$ one recovers the classical single-valued probability.

Example 4.2 (2-SuperHyperProbability in Corporate Bond Default Risk)**.** Corporate bond default risk is the probability that a company fails to repay its bond obligations, causing losses to investors or lenders (cf. [44, 94, 97]). Let the sample space be

$$\Omega = \{\text{default, no default}\}, \quad \mathcal{F} = 2^{\Omega}.$$

Consider three credit-rating agencies—Moody's, S&P, and Fitch—each providing an independent estimate of the one-year default probability $p_k(A)$ for a particular corporate bond $A = \{\text{default}\}$:

$$p_{\text{Moody}}(A) = 0.08,$$
$$p_{\text{S\&P}}(A) = 0.12,$$
$$p_{\text{Fitch}}(A) = 0.10.$$

These give the Hyper-Probability

$$\mathrm{HP}(A) \ =\ \{\,0.08,\ 0.10,\ 0.12\}.$$

Next, the 2-SuperHyperProbability is

$$\mathrm{SHP}^{(2)}(A) = P\big(\{0.08, 0.10, 0.12\}\big) = \big\{\emptyset,\ \{0.08\},\ \{0.10\},\ \{0.12\},$$

$$\{0.08, 0.10\},\ \{0.08, 0.12\},\ \{0.10, 0.12\},\ \{0.08, 0.10, 0.12\}\big\}.$$

For clarity, Table 3 lists these eight subsets and their interpretation in terms of which agency estimates are "trusted."

Discussion.

Subset	Interpretation
$\emptyset$	No agency is trusted (vacuous belief)
$\{0.08\}$	Trust only Moody's optimistic estimate
$\{0.10\}$	Trust only Fitch's estimate
$\{0.12\}$	Trust only S&P's pessimistic estimate
$\{0.08, 0.10\}$	Trust Moody's and Fitch, ignore S&P
$\{0.08, 0.12\}$	Trust Moody's and S&P, ignore Fitch
$\{0.10, 0.12\}$	Trust Fitch and S&P, ignore Moody's
$\{0.08, 0.10, 0.12\}$	Trust all three agencies equally

Table 3: All subsets in the 2-SuperHyperProbability for default event A.

- The empty set $\emptyset$ reflects maximal caution—no single estimate is deemed reliable.
- Singleton subsets correspond to extreme strategies: e.g. $\{0.12\}$ adopts the highest default estimate for a conservative stance.
- Doubletons allow combining two agencies' views, balancing optimism and pessimism.
- The full set $\{0.08, 0.10, 0.12\}$ represents equal weighting of all three opinions.

This detailed 2-SuperHyperProbability captures not only first-order uncertainty (varying agency estimates) but also second-order choices about which combinations of experts to rely upon in financial decision-making.

Example 4.3 (2-SuperHyperProbability in Election Forecasting)**.** Election forecasting is the use of data, models, and statistical methods to predict electoral outcomes before official voting results are known [50, 55, 56]. Let the sample space be

$$\Omega = \{\text{Win, Lose}\}, \quad \mathcal{F} = 2^{\Omega}.$$

Consider three independent pollsters—Gallup, YouGov, and Ipsos—each providing an estimated probability that Candidate X will win (event $A = \{\text{Win}\}$):

$$p_{\text{Gallup}}(A) = 0.48,$$
$$p_{\text{YouGov}}(A) = 0.52,$$
$$p_{\text{Ipsos}}(A) = 0.50.$$

These yield the Hyper-Probability

$$\text{HP}(A) = \{\, 0.48,\ 0.50,\ 0.52\}.$$

By Definition 4.1, the 2-SuperHyperProbability is

$$\text{SHP}^{(2)}(A) = P\big(\{0.48, 0.50, 0.52\}\big) = \Big\{\emptyset, \{0.48\}, \{0.50\}, \{0.52\}, \{0.48, 0.50\},$$
$$\{0.48, 0.52\}, \{0.50, 0.52\}, \{0.48, 0.50, 0.52\}\Big\}.$$

Subset	Interpretation
$\emptyset$	Trust no pollster (maximal caution)
$\{0.48\}$	Trust only Gallup's estimate
$\{0.50\}$	Trust only Ipsos's estimate
$\{0.52\}$	Trust only YouGov's estimate
$\{0.48, 0.50\}$	Trust Gallup and Ipsos, ignore YouGov
$\{0.48, 0.52\}$	Trust Gallup and YouGov, ignore Ipsos
$\{0.50, 0.52\}$	Trust Ipsos and YouGov, ignore Gallup
$\{0.48, 0.50, 0.52\}$	Trust all three pollsters equally

Table 4: Subsets in the 2-SuperHyperProbability for Candidate X winning

Interpretation.

- The empty set $\emptyset$ reflects highest skepticism—no pollster's data is considered reliable.
- Singleton sets correspond to relying on a single source, e.g. $\{0.52\}$ takes the most optimistic forecast.
- Pairs blend two sources, balancing differing views; for instance, $\{0.48, 0.50\}$ merges Gallup's and Ipsos's mid-range estimates.
- The full set $\{0.48, 0.50, 0.52\}$ treats all pollsters equally, combining every available forecast.

This example demonstrates how 2-SuperHyperProbability captures both first-order uncertainty (variation across pollsters) and second-order decision strategies about which combination of expert opinions to trust in electoral predictions.

Example 4.4 (2-SuperHyperProbability in Credit Card Fraud Detection)**.** Credit card fraud detection identifies unauthorized or suspicious transactions using algorithms, rules, or machine learning to prevent financial loss (cf. [13, 43, 64]). Let

$$\Omega = \{\text{fraud, no fraud}\}, \quad \mathcal{F} = 2^{\Omega}, \quad A = \{\text{fraud}\}.$$

Suppose three independent detection models—Rule-based, Logistic Regression, and Neural Network—estimate the probability of fraud as follows:

$$p_{\text{Rule}}(A) = 0.03, \quad p_{\text{LR}}(A) = 0.07, \quad p_{\text{NN}}(A) = 0.05.$$

Thus the Hyper-Probability is

$$\text{HP}(A) = \{0.03,\ 0.05,\ 0.07\}.$$

The 2-SuperHyperProbability is then

$$\text{SHP}^{(2)}(A) = P\big(\{0.03, 0.05, 0.07\}\big) = \big\{\emptyset,\ \{0.03\},\ \{0.05\},\ \{0.07\},$$
$$\{0.03, 0.05\},\ \{0.03, 0.07\},\ \{0.05, 0.07\},\ \{0.03, 0.05, 0.07\}\big\}.$$

Subset	Interpretation
$\emptyset$	Trust no model (maximum caution)
$\{0.03\}$	Trust only the rule-based model
$\{0.05\}$	Trust only the neural network model
$\{0.07\}$	Trust only the logistic regression model
$\{0.03, 0.05\}$	Trust rule-based and neural network
$\{0.03, 0.07\}$	Trust rule-based and logistic regression
$\{0.05, 0.07\}$	Trust neural network and logistic regression
$\{0.03, 0.05, 0.07\}$	Trust all three models equally

Table 5: 2-SuperHyperProbability subsets for fraud detection event A.

Interpretation.

- The empty set $\emptyset$ indicates no single model is deemed reliable, so manual review is invoked.
- Singleton subsets (e.g. $\{0.07\}$) correspond to relying solely on one model's estimate.
- Doubleton subsets (e.g. $\{0.05, 0.07\}$) blend two models' outputs, requiring agreement between them.
- The full set $\{0.03, 0.05, 0.07\}$ treats all three models equally, aggregating every available forecast.

Example 4.5 (3-SuperHyperProbability in Manufacturing Quality Control)**.** Manufacturing quality control is the process of ensuring products meet specified standards by monitoring, inspecting, and correcting production operations systematically (cf. [42, 89, 96]). Let

$$\Omega = \{\text{fail, ok}\}, \quad \mathcal{F} = 2^{\Omega},$$

and consider the event $A = \{\text{fail}\}$ ("machine breaks down within 24 hours"). Two predictive maintenance algorithms, M_1 and M_2, independently estimate

$$p_1(A) = 0.10, \quad p_2(A) = 0.25.$$

Hence the Hyper-Probability is

$$\mathrm{HP}(A) = \{\,0.10,\ 0.25\}.$$

By Definition 4.1, the 2-SuperHyperProbability is

$$\mathrm{SHP}^{(2)}(A) = P\big(\{0.10, 0.25\}\big) = \big\{\ \emptyset,\ \{0.10\},\ \{0.25\},\ \{0.10, 0.25\}\big\}.$$

Interpreting, each element of $\mathrm{SHP}^{(2)}(A)$ is a choice of which algorithm(s) to trust.

Finally, the 3-SuperHyperProbability is

$$\mathrm{SHP}^{(3)}(A) = P\big(\mathrm{SHP}^{(2)}(A)\big) = P\big(\{\emptyset, \{0.10\}, \{0.25\}, \{0.10, 0.25\}\}\big),$$

which is the set of all subsets of $\mathrm{SHP}^{(2)}(A)$ (there are $2^4 = 16$ of them). For brevity we list a few representative elements:

$$\begin{aligned}
&\emptyset,\\
&\{\emptyset\}, \quad \{\{0.10\}\}, \quad \{\{0.25\}\}, \quad \{\{0.10, 0.25\}\},\\
&\{\emptyset, \{0.10\}\}, \quad \{\{0.10\}, \{0.25\}\}, \quad \{\{0.10\}, \{0.10, 0.25\}\},\\
&\{\emptyset, \{0.10\}, \{0.25\}, \{0.10, 0.25\}\} \quad (\text{the full set}).
\end{aligned}$$

Interpretation.

- $\emptyset$ — no strategy is trusted (extreme caution).
- $\{\{0.10\}\}$ — trust only algorithm M_1.
- $\{\{0.10\}, \{0.25\}\}$ — trust either algorithm individually, but not their joint estimate.
- $\{\emptyset, \{0.10\}, \{0.25\}, \{0.10, 0.25\}\}$ — allow any of the four first-order choices.

This 3-SuperHyperProbability thus captures not only the original risk estimates and choices about which algorithms to trust (first order) and which subsets of algorithms (second order), but also which combinations of those choices are acceptable (third order), providing a rich framework for decision-making under layered uncertainty.

Example 4.6 (3-SuperHyperProbability in Cybersecurity Intrusion Detection)**.** Cybersecurity intrusion detection is the process of monitoring systems or networks to identify and respond to unauthorized or malicious activity (cf. [4, 8, 21]). Let

$$\Omega = \{\text{Intrusion, No Intrusion}\}, \quad \mathcal{F} = 2^{\Omega}, \quad A = \{\text{Intrusion}\}.$$

Two intrusion detection systems (IDS), Snort and Suricata, independently estimate the probability of an intrusion:

$$p_{\text{Snort}}(A) = 0.30, \quad p_{\text{Suricata}}(A) = 0.45.$$

Hence the Hyper-Probability is

$$\mathrm{HP}(A) = \{0.30,\ 0.45\}.$$

The 2-SuperHyperProbability is

$$\mathrm{SHP}^{(2)}(A) = P\big(\{0.30, 0.45\}\big) = \{\,\emptyset,\ \{0.30\},\ \{0.45\},\ \{0.30, 0.45\}\}.$$

Finally, the 3-SuperHyperProbability is

$$\mathrm{SHP}^{(3)}(A) = P\big(\mathrm{SHP}^{(2)}(A)\big) = P\big(\{\emptyset, \{0.30\}, \{0.45\}, \{0.30, 0.45\}\}\big),$$

which yields $2^4 = 16$ subsets. Concretely,

$$\mathrm{SHP}^{(3)}(A) = \big\{\emptyset,\ \{\emptyset\},\ \{\{0.30\}\},\ \{\{0.45\}\},\ \{\{0.30, 0.45\}\},\ \{\emptyset, \{0.30\}\},\ \{\emptyset, \{0.45\}\},\ \{\emptyset, \{0.30, 0.45\}\},$$
$$\{\{0.30\}, \{0.45\}\},\ \{\{0.30\}, \{0.30, 0.45\}\},\ \{\{0.45\}, \{0.30, 0.45\}\},\ \{\emptyset, \{0.30\}, \{0.45\}\},\ \{\emptyset, \{0.30\}, \{0.30, 0.45\}\},$$
$$\{\emptyset, \{0.45\}, \{0.30, 0.45\}\},\ \{\{0.30\}, \{0.45\}, \{0.30, 0.45\}\},\ \{\emptyset, \{0.30\}, \{0.45\}, \{0.30, 0.45\}\}\big\}.$$

Interpretation.

- $\emptyset$: reject all IDS outputs (maximal caution).
- $\{\{0.30\}\}$: trust only Snort's detection probability.
- $\{\{0.45\}\}$: trust only Suricata's detection probability.
- $\{\{0.30, 0.45\}\}$: require consensus—both IDS agree.
- $\{\emptyset, \{0.45\}, \{0.30, 0.45\}\}$: allow either full rejection, trust Suricata alone, or consensus.
- $\{\emptyset, \{0.30\}, \{0.45\}, \{0.30, 0.45\}\}$: permit any first-order decision strategy.

This 3-SuperHyperProbability captures first-order uncertainty (IDS estimates), second-order choices of which IDS outputs to trust, and third-order combinations of those choices for robust intrusion response.

Example 4.7 (4-SuperHyperProbability in Disaster Response)**.** Disaster response is the coordinated effort to manage and mitigate the impact of natural or man-made disasters on affected populations and infrastructure (cf. [12, 48, 58]). Let

$$\Omega = \{\text{NeedAid}, \text{NoAid}\}, \quad \mathcal{F} = 2^{\Omega}, \quad A = \{\text{NeedAid}\}.$$

Two relief organizations—UN and Red Cross—estimate the probability that a region will need urgent aid within 24 hours:

$$p_{\text{UN}}(A) = 0.60, \quad p_{\text{RedCross}}(A) = 0.80.$$

Hence the Hyper-Probability is

$$\mathrm{HP}(A) = \{0.60,\ 0.80\}.$$

By Definition 4.1:

$$\mathrm{SHP}^{(2)}(A) = P\big(\{0.60, 0.80\}\big) = \{\ \emptyset,\ \{0.60\},\ \{0.80\},\ \{0.60, 0.80\}\}.$$

$$\mathrm{SHP}^{(3)}(A) = P\big(\mathrm{SHP}^{(2)}(A)\big),$$

which has $\big|\mathrm{SHP}^{(2)}(A)\big| = 4$ elements and thus $\big|\mathrm{SHP}^{(3)}(A)\big| = 2^4 = 16$. For brevity, one lists just a few of the 16 subsets:

$$\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\{0.60\}, \{0.80\}\}, \quad \mathrm{SHP}^{(2)}(A).$$

Finally,

$$\mathrm{SHP}^{(4)}(A) = P\big(\mathrm{SHP}^{(3)}(A)\big),$$

which yields $2^{16} = 65536$ subsets of the 16 third-order sets. Representative 4-th-order elements include:

$$\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\{\{0.60\}\}, \{\{0.80\}\}\}, \quad \mathrm{SHP}^{(3)}(A).$$

Interpretation.

- First order (HP): two probability estimates 0.60 and 0.80.
- Second order ($\mathrm{SHP}^{(2)}$): choices of which organization(s) to trust.

- Third order ($\mathrm{SHP}^{(3)}$): combinations of those trust-choices.
- Fourth order ($\mathrm{SHP}^{(4)}$): selections of which third-order strategies to adopt, capturing meta-uncertainty about decision policies.

Theorem 4.8 (Null and Certain Events)**.** *For every $n \geq 1$,*

$$\mathrm{SHP}^{(n)}(\emptyset) = P^n(\{0\}), \qquad \mathrm{SHP}^{(n)}(\Omega) = P^n(\{1\}).$$

In particular, these images depend only on the trivial Hyper-Probability values.

Proof. We argue by induction on n.

Base case $n = 1$. Since each p_k satisfies $p_k(\emptyset) = 0$ and $p_k(\Omega) = 1$,

$$\mathrm{HP}(\emptyset) = \{0\}, \quad \mathrm{HP}(\Omega) = \{1\}.$$

Inductive step. Suppose $\mathrm{SHP}^{(n-1)}(\emptyset) = P^{n-1}(\{0\})$. Then

$$\mathrm{SHP}^{(n)}(\emptyset) = P\big(\mathrm{SHP}^{(n-1)}(\emptyset)\big) = P\big(P^{n-1}(\{0\})\big) = P^n(\{0\}).$$

A similar argument gives $\mathrm{SHP}^{(n)}(\Omega) = P^n(\{1\})$. This completes the induction. □

Theorem 4.9 (Cardinality Growth)**.** *Let $m = \big|\mathrm{HP}(A)\big|$. Then for each $n \geq 1$,*

$$\big|\mathrm{SHP}^{(n)}(A)\big| = 2^{\left|\mathrm{SHP}^{(n-1)}(A)\right|}, \quad \big|\mathrm{SHP}^{(1)}(A)\big| = m.$$

In particular, $\big|\mathrm{SHP}^{(n)}(A)\big|$ is given by n–fold iterated exponentiation of 2 starting from m.

Proof. By definition, $\mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big)$. Since for any finite set X, $\big|P(X)\big| = 2^{|X|}$, it follows that

$$\big|\mathrm{SHP}^{(n)}(A)\big| = \big|P(\mathrm{SHP}^{(n-1)}(A))\big| = 2^{\left|\mathrm{SHP}^{(n-1)}(A)\right|}.$$

The base case $\big|\mathrm{SHP}^{(1)}(A)\big| = |\mathrm{HP}(A)| = m$ is immediate. □

Theorem 4.10 (Monotonicity)**.** *If $A, B \in \mathcal{F}$ satisfy $A \subseteq B$, then for every $n \geq 1$,*

$$\mathrm{SHP}^{(n)}(A) \ \subseteq \ \mathrm{SHP}^{(n)}(B).$$

Proof. We prove by induction on n.

Base case $n = 1$. If $A \subseteq B$, then for each k, $p_k(A) \leq p_k(B)$. Hence $\mathrm{HP}(A) = \{p_k(A)\} \subseteq \{p_k(B)\} = \mathrm{HP}(B)$.

Inductive step. Assume $\mathrm{SHP}^{(n-1)}(A) \subseteq \mathrm{SHP}^{(n-1)}(B)$. Since the powerset operator P is monotone (i.e. $X \subseteq Y$ implies $P(X) \subseteq P(Y)$), we have

$$\mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big) \ \subseteq \ P\big(\mathrm{SHP}^{(n-1)}(B)\big) = \mathrm{SHP}^{(n)}(B).$$

This completes the induction. □

Theorem 4.11 (Finite Additivity at Order n)**.** *If $A, B \in \mathcal{F}$ are disjoint events, then for every integer $n \geq 1$,*

$$\mathrm{SHP}^{(n)}(A \cup B) = P^n\big(\{\ p_k(A) + p_k(B) \mid k \in K\}\big).$$

Proof. We argue by induction on n.

Base case $n = 1$. By the finite additivity of each p_k,

$$p_k(A \cup B) = p_k(A) + p_k(B), \quad k \in K.$$

Hence $\mathrm{HP}(A \cup B) = \{p_k(A \cup B)\} = \{p_k(A) + p_k(B)\}$, proving the statement for $n = 1$.

Inductive step. Assume the result holds for $n - 1$. Then

$$\mathrm{SHP}^{(n)}(A \cup B) = P\big(\mathrm{SHP}^{(n-1)}(A \cup B)\big) = P\Big(P^{n-1}\big(\{p_k(A) + p_k(B)\}\big)\Big) = P^{n}\big(\{p_k(A) + p_k(B)\}\big),$$

completing the induction. □

Theorem 4.12 (Existence of Extreme Elements)**.** *For any event A and any integer $n \geq 2$,*

$$\emptyset \ \in \ \mathrm{SHP}^{(n)}(A) \quad \textit{and} \quad \mathrm{SHP}^{(n-1)}(A) \ \in \ \mathrm{SHP}^{(n)}(A).$$

Proof. By definition $\mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big)$. The standard powerset $P\big(\mathrm{SHP}^{(n-1)}(A)\big)$ always contains two distinguished elements:

- The empty set $\emptyset$.
- The universal subset $\mathrm{SHP}^{(n-1)}(A)$ itself.

Thus both claims follow immediately from the construction of the powerset. □

Theorem 4.13 (Complement Mapping)**.** *Let $f : [0, 1] \to [0, 1]$ be the complement map $f(x) = 1 - x$, and extend f to the n-fold powerset by*

$$P^n(f) : P^n([0, 1]) \longrightarrow P^n([0, 1]).$$

Then for every event A and every $n \geq 1$,

$$\mathrm{SHP}^{(n)}(A^c) = P^{n}(f)\big(\mathrm{SHP}^{(n)}(A)\big).$$

Proof. We proceed by induction on n.

Base case $n = 1$. For each $k \in K$,

$$p_k(A^c) = 1 - p_k(A),$$

hence

$$\mathrm{HP}(A^c) = \{\, p_k(A^c)\} = \{\, 1 - p_k(A)\} = f\big(\{p_k(A)\}\big) = P^1(f)\big(\mathrm{HP}(A)\big).$$

Inductive step. Assume $\mathrm{SHP}^{(n-1)}(A^c) = P^{n-1}(f)\big(\mathrm{SHP}^{(n-1)}(A)\big)$. Then

$$\begin{aligned}
\mathrm{SHP}^{(n)}(A^c) &= P\big(\mathrm{SHP}^{(n-1)}(A^c)\big) \\
&= P\Big(P^{n-1}(f)\big(\mathrm{SHP}^{(n-1)}(A)\big)\Big) \\
&= P^{n}(f)\big(\mathrm{SHP}^{(n-1)}(A)\big) \\
&= P^{n}(f)\big(\mathrm{SHP}^{(n)}(A)\big),
\end{aligned}$$

where the third equality follows from the functoriality of the powerset operator P. This completes the induction. □

5 Conclusion and Future Works

In this paper, we revisited the fundamental properties of HyperProbability and SuperHyperProbability. Specifically, we examined their potential real-world applications and explored several of their mathematical characteristics.

As future work, we aim to investigate extensions of HyperProbability and SuperHyperProbability by incorporating advanced uncertainty-handling frameworks such as Fuzzy Sets [99, 100], Intuitionistic Fuzzy Sets [9, 10], Vague Sets [3, 41], Rough Sets [62, 63], Bipolar Fuzzy Sets [2], HyperFuzzy Sets [23, 51, 85], Picture Fuzzy Sets [18, 46], Hesitant Fuzzy Sets [86, 87], Neutrosophic Sets [74, 84], Quadripartitioned Neutrosophic Sets [35, 52, 98], HyperPlithogenic Sets [32–34], and Plithogenic Sets [29, 37, 38]. Such extensions may further enhance their expressiveness and applicability in modeling complex and hierarchical forms of uncertainty.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Klaus-Peter Adlassnig. Fuzzy set theory in medical diagnosis. *IEEE Transactions on Systems, Man, and Cybernetics*, 16(2):260–265, 1986.

[2] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[3] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[4] Abdullah S AL-Ghamdi, Mahmoud Ragab, and Maha Farouk S Sabir. Enhanced artificial intelligence-based cybersecurity intrusion detection for higher education institutions. *Computers, Materials & Continua*, 72(2), 2022.

[5] L Al-Matarneh, A Sheta, S Bani-Ahmad, J Alshaer, and I Al-Oqily. Development of temperature-based weather forecasting models using neural networks and fuzzy logic. *International journal of multimedia and ubiquitous engineering*, 9(12):343–366, 2014.

[6] M Daud Alam, Uwe F Gühl, M Daud Alam, and Uwe F Gühl. Project phases. *Project-Management in Practice: A Guideline and Toolbox for Successful Projects*, pages 55–121, 2016.

[7] Roohallah Alizadehsani, Mohamad Roshanzamir, Sadiq Hussain, Abbas Khosravi, Afsaneh Koohestani, Mohammad Hossein Zangooei, Moloud Abdar, Adham Beykikhoshk, Afshin Shoeibi, Assef Zare, et al. Handling of uncertainty in medical data using machine learning and probability theory techniques: A review of 30 years (1991–2020). *Annals of Operations Research*, pages 1–42, 2021.

[8] Amjad Alsirhani, Mohammed Mujib Alshahrani, Ahmed M Hassan, Ahmed I Taloba, Rasha M Abd El-Aziz, and Ahmed H Samak. Implementation of african vulture optimization algorithm based on deep learning for cybersecurity intrusion detection. *Alexandria Engineering Journal*, 79:105–115, 2023.

[9] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[10] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[11] Peter Bauer, Alan Thorpe, and Gilbert Brunet. The quiet revolution of numerical weather prediction. *Nature*, 525(7567):47–55, 2015.

[12] Djamel Berkoune, Jacques Renaud, Monia Rekik, and Angel Ruiz. Transportation in disaster response operations. *Socio-Economic Planning Sciences*, 46(1):23–32, 2012.

[13] Rejwan Bin Sulaiman, Vitaly Schetinin, and Paul Sant. Review of machine learning approach on credit card fraud detection. *Human-Centric Intelligent Systems*, 2(1):55–68, 2022.

[14] Jacob Burbea and C. Radhakrishna Rao. Entropy differential metric, distance and divergence measures in probability spaces: A unified approach. *Journal of Multivariate Analysis*, 12:575–596, 1982.

[15] Xi-Ren Cao. *Realization probabilities: The dynamics of queuing systems*. Springer, 1994.

[16] Kai Lai Chung. *A course in probability theory*. Elsevier, 2000.

[17] Terry D Clark, Jennifer M Larson, John N Mordeson, Joshua D Potter, Mark J Wierman, Terry D Clark, Jennifer M Larson, John N Mordeson, Joshua D Potter, and Mark J Wierman. Fuzzy geometry. *Applying Fuzzy Mathematics to Formal Models in Comparative Politics*, pages 65–80, 2008.

[18] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.

[19] Freddy Delbaen. Coherent risk measures on general probability spaces. 2002.

[20] Jay L Devore. Probability and statistics. *Pacific Grove: Brooks/Cole*, 2000.

[21] Aeryn Dunmore, Julian Jang-Jaccard, Fariza Sabrina, and Jin Kwak. A comprehensive survey of generative adversarial networks (gans) in cybersecurity intrusion detection. *IEEE Access*, 11:76071–76094, 2023.

[22] Takaaki Fujita. A concise review on various concepts of superhyperstructures.

[23] Takaaki Fujita. Some types of hyperfuzzy set: Bipolar, m-polar, q-rung orthopair, trapezoidal, linguistic, intuitionistic, picture, hesitant, spherical, type-m, offset, overset, and underset. *Preprint*.

[24] Takaaki Fujita. Breaking down barriers: Proposals for overcoming challenges in student project management. *European Journal of Management and Marketing Studies*, 2023.

[25] Takaaki Fujita. Reconsideration and proposal of development models in projects-"quasi" development models: Quasi-waterfall and quasi-agile. *European Journal of Social Sciences Studies*, 9(2), 2023.

[26] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. Technical report, Center for Open Science, 2024.

[27] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[28] Takaaki Fujita. A theoretical exploration of hyperconcepts: Hyperfunctions, hyperrandomness, hyperdecision-making, and beyond (including a survey of hyperstructures). 2024.

[29] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[30] Takaaki Fujita. Antihyperstructure, neutrohyperstructure, and superhyperstructure. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 311, 2025.

[31] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[32] Takaaki Fujita. Forest hyperplithogenic set and forest hyperrough set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 2025.

[33] Takaaki Fujita. Hyperfuzzy graph neural networks and hyperplithogenic graph neural networks: Theoretical foundations. 2025.

[34] Takaaki Fujita. Hyperplithogenic cubic set and superhyperplithogenic cubic set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 79, 2025.

[35] Takaaki Fujita. Some types of hyperneutrosophic set (3): Dynamic, quadripartitioned, pentapartitioned, heptapartitioned, m-polar. 2025.

[36] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[37] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[38] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[39] Takaaki Fujita and Florentin Smarandache. *Exploring Concepts of HyperFuzzy, HyperNeutrosophic, and HyperPlithogenic Sets (I)*. Infinite Study, 2025.

[40] Takaaki Fujita and Florentin Smarandache. *Neutrosophic TwoFold SuperhyperAlgebra and Anti SuperhyperAlgebra*. Infinite Study, 2025.

[41] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[42] SN Ghosh. *Advances in cement technology: critical reviews and case studies on manufacturing, quality control, optimization and use*. Elsevier, 2014.

[43] Sushmito Ghosh and Douglas L Reilly. Credit card fraud detection with a neural-network. In *System Sciences, 1994. Proceedings of the Twenty-Seventh Hawaii International Conference on*, volume 3, pages 621–630. IEEE, 1994.

[44] Kay Giesecke, Francis A Longstaff, Stephen Schaefer, and Ilya Strebulaev. Corporate bond default risk: A 150-year perspective. *Journal of financial Economics*, 102(2):233–250, 2011.

[45] Sebastian Gutsche. Constructive category theory and applications to algebraic geometry. 2017.

[46] Raed Hatamleh, Abdullah Al-Husban, Sulima Ahmed Mohammed Zubair, Mawahib Elamin, Maha Mohammed Saeed, Eisa Abdolmaleki, Takaaki Fujita, Giorgio Nordo, and Arif Mehmood Khattak. Ai-assisted wearable devices for promoting human health and strength using complex interval-valued picture fuzzy soft relations. *European Journal of Pure and Applied Mathematics*, 18(1):5523–5523, 2025.

[47] Brian Hobbs and Monique Aubry. A multi-phase research program investigating project management offices (pmos): the results of phase 1. *Project management journal*, 38(1):74–86, 2007.

[48] Muhammad Imran, Carlos Castillo, Ji Lucas, Patrick Meier, and Sarah Vieweg. Aidr: Artificial intelligence for disaster response. In *Proceedings of the 23rd international conference on world wide web*, pages 159–162, 2014.

[49] Sirus Jahanpanah and Roohallah Daneshpayeh. An outspread on valued logic superhyperalgebras. *Facta Universitatis, Series: Mathematics and Informatics*, pages 427–437, 2024.

[50] Will Jennings, Michael Lewis-Beck, and Christopher Wlezien. Election forecasting: Too far out? *International Journal of Forecasting*, 36(3):949–962, 2020.

[51] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[52] Arif Mehmood Khattak, M Arslan, Abdallah Shihadeh, Wael Mahmoud Mohammad Salameh, Abdallah Al-Husban Al-Husban, R Seethalakshmi, G Nordo, Takaaki Fujita, and Maha Mohammed Saeed. A breakthrough approach to quadri-partitioned neutrosophic softtopological spaces. *European Journal of Pure and Applied Mathematics*, 18(2):5845–5845, 2025.

[53] Jochen Kruppa, Yufeng Liu, Gérard Biau, Michael Kohler, Inke R König, James D Malley, and Andreas Ziegler. Probability estimation with machine learning methods for dichotomous and multicategory outcome: theory. *Biometrical Journal*, 56(4):534–563, 2014.

[54] S Senthil Kumar and H Hannah Inbarani. Optimistic multi-granulation rough set based classification for medical diagnosis. *Procedia Computer Science*, 47:374–382, 2015.

[55] Michael S Lewis-Beck. Election forecasting: Principles and practice. *The British Journal of Politics and International Relations*, 7(2):145–164, 2005.

[56] Michael S Lewis-Beck and Charles Tien. Election forecasting for turbulent times. *PS: Political Science & Politics*, 45(4):625–629, 2012.

[57] Yanfeng Li, Hongzhong Huang, Jinhua Mi, Weiwen Peng, and X. M. Han. Reliability analysis of multi-state systems with common cause failures based on bayesian network and fuzzy probability. *Annals of Operations Research*, 311:195 – 209, 2019.

[58] Patricia A O'Neill. The abc's of disaster response. *Scandinavian journal of surgery*, 94(4):259–266, 2005.

[59] Ferdinand Österreicher and Igor Vajda. A new class of metric divergences on probability spaces and its applicability in statistics. *Annals of the Institute of Statistical Mathematics*, 55:639–653, 2003.

[60] James T O'Connor and Li-Ren Yang. Project performance versus use of technologies at project and phase levels. *Journal of Construction Engineering and Management*, 130(3):322–329, 2004.

[61] Tim N Palmer. Predicting uncertainty in forecasts of weather and climate. *Reports on progress in Physics*, 63(2):71, 2000.

[62] Zdzis law Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[63] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

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[64] S Benson Edwin Raj and A Annie Portia. Analysis on credit card fraud detection methods. In *2011 International Conference on Computer, Communication and Electrical Technology (ICCCET)*, pages 152–156. IEEE, 2011.

[65] Hao ran Lin, Bing yuan Cao, and Yun zhang Liao. Fuzzy statistics and fuzzy probability. 2018.

[66] Akbar Rezaei, Florentin Smarandache, and S. Mirvakili. Applications of (neutro/anti)sophications to semihypergroups. *Journal of Mathematics*, 2021.

[67] Denis Riordan and Bjarne K Hansen. A fuzzy case-based system for weather prediction. *Engineering Intelligent Systems for Electrical Engineering and Communications*, 10(3):139–146, 2002.

[68] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[69] Sheldon M Ross. *Introduction to probability models*. Academic press, 2014.

[70] Sheldon M Ross, Sheldon M Ross, Sheldon M Ross, and Sheldon M Ross. *A first course in probability*, volume 2. Macmillan New York, 1976.

[71] Zehui Shao, Saeed Kosari, Muhammad Shoaib, and Hossein Rashmanlou. Certain concepts of vague graphs with applications to medical diagnosis. *Frontiers in physics*, 8:357, 2020.

[72] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[73] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[74] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[75] Florentin Smarandache. *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2005.

[76] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *ArXiv*, abs/1808.03948, 2018.

[77] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[78] Florentin Smarandache. Extension of hyperalgebra to superhyperalgebra and neutrosophic superhyperalgebra (revisited). In *International Conference on Computers Communications and Control*, pages 427–432. Springer, 2022.

[79] Florentin Smarandache. *The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again)*, volume 3. Infinite Study, 2022.

[80] Florentin Smarandache. *Real Examples of NeutroGeometry & AntiGeometry*. Infinite Study, 2023.

[81] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[82] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[83] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[84] Florentin Smarandache and AA Salama. Neutrosophic crisp set theory. 2015.

[85] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[86] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[87] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[88] Kishor S Trivedi. *Probability and statistics with reliability, queuing, and computer science applications*. John Wiley & Sons, 2001.

[89] Tea Tušar, Klemen Gantar, Valentin Koblar, Bernard Ženko, and Bogdan Filipič. A study of overfitting in optimization of a manufacturing quality control procedure. *Applied Soft Computing*, 59:77–87, 2017.

[90] José Unpingco. *Python for probability, statistics, and machine learning*, volume 1. Springer, 2016.

[91] Souzana Vougioukli. Helix hyperoperation in teaching research. *Science & Philosophy*, 8(2):157–163, 2020.

[92] Souzana Vougioukli. Hyperoperations defined on sets of s -helix matrices. 2020.

[93] Souzana Vougioukli. Helix-hyperoperations on lie-santilli admissibility. *Algebras Groups and Geometries*, 2023.

[94] F Albert Wang and Ting Zhang. The effect of unfunded pension liabilities on corporate bond ratings, default risk, and recovery rate. *Review of Quantitative Finance and Accounting*, 43:781–802, 2014.

[95] Jia Wang and Zhenyuan Wang. Using neural networks to determine sugeno measures by statistics. *Neural Networks*, 10:183–195, 1997.

[96] Chen Xin-chun, Ding Peng, Yan Nai-qing, and Bi Meng-xue. Study on discrete manufacturing quality control technology based on big data and pattern recognition. *Mathematical Problems in Engineering*, 2021(1):8847094, 2021.

[97] Yuwei Xu. Comparative study on early risk warning for corporate bond defaults. *Intelligent Decision Technologies*, page 18724981241309549, 2025.

[98] P Yiarayong. Some weighted aggregation operators of quadripartitioned single-valued trapezoidal neutrosophic sets and their multi-criteria group decision-making method for developing green supplier selection criteria. *OPSEARCH*, pages 1–55, 2024.

[99] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[100] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[101] Qi Zhang, Chunjie Zhou, Yu-Chu Tian, Naixue N. Xiong, Yuanqing Qin, and Bowen Hu. A fuzzy probability bayesian network approach for dynamic cybersecurity risk assessment in industrial control systems. *IEEE Transactions on Industrial Informatics*, 14:2497–2506, 2018.