A Study On Gaussian Generalized Edouard Numbers

Abstract-In this study, we define Gaussian generalized Edouard numbers in detail, and focus on two specific cases: Gaussian Edouard numbers, Gaussian Edouard-Lucas numbers.

In addition, we present some identities and matrices related to these sequences, as well as recurrence relations, Binet's formulas, generating functions, Simpson's formulas, and summation formulas.

Keywords: Gaussian Edouard numbers, Gaussian Edouard-Lucas numbers.

1. Introduction

In this section, firstly, we give some preliminary result on Edouard numbers.

The generalized Edouard sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation as

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 7W_{-(n-1)} - 7W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

Characteristic equation of $\{W_n\}$ is

$$z^3 - 7z^2 + 7z - 1 = (z^2 - 6z + 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2}, \\ \beta &= 3 - 2\sqrt{2}, \\ \gamma &= 1. \end{aligned}$$

Note that

$$\begin{aligned}\alpha+\beta+\gamma+\delta &= 7,\\ \alpha\beta+\alpha\gamma+\beta\delta &= 7,\\ \alpha\beta\gamma &= 1.\end{aligned}$$

For $n = 1, 2, 3, ...$ Hence, recurrence (1.1) is true for all integer n. Soykan has conducted a study on this particular generalized Edouard sequence and its three spacial cases, for more details, see [20]

Third order reccurance relations has been studied by many authors, for more detail see [2,3,5,6, 7,12,13,15,16,18,19,24,28,29].

Next, we present Binet's formula of generalized Edouard numbers.

THEOREM 1.1. *[20] Binet formula of generalized Edouard numbers can be presented as follows:*

$$W_n = \left(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^n}{(\alpha-\beta)(\alpha-1)} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^n}{(\beta-\alpha)(\beta-1)} - \frac{(W_2-6W_1+W_0)}{4}\right).$$

Now we define two particular cases of the sequence $\{W_n\}$ as follows: the edouard sequence $\{E_n\}_{n\geq 0}$, the Edouard-Lucas sequence $\{K_n\}_{n\geq 0}$, respectively, by the third-order recurrence relations,

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \qquad E_0 = 0, E_1 = 1, E_2 = 7, \tag{1.2}$$

$$K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 7, K_2 = 35. \tag{1.3}$$

The sequences $\{E_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ can be extended to negative subscriptys by defining,

$$\begin{aligned}E_{-n} &= 7E_{-(n-1)} - 7E_{-(n-2)} + E_{-(n-3)},\\ K_{-n} &= 7K_{-(n-1)} - 7K_{-(n-2)} + K_{-(n-3)},\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.2)-(1.3) hold for all integer n.

For all integers n, Edouard and Edouard-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned}E_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4},\\ K_n &= \alpha^n + \beta^n + 1.\end{aligned}$$

Note that, Gaussian numbers, generally known as Gaussian integers, are a subset of the complex numbers. A complex number is expressed in the form $a + bi$ where a and b are arbitrary real numbers, and i is the imaginary unit such that $i^2 = -1$. Gaussian integers are a specific type of complex number. In other word, z is a Gaussian integers such that $z = a + bi$ where a and b are arbitrary integers.

Next, we give some information about Gaussian sequences from literature.

First, we give some Gaussian numbers with second order recurrence relations.

- Horadam [11] introduced Gaussian Fibonacci numbers and defined by

$$GF_n = F_n + iF_{n-1}$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact, he defined these numbers as $GF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.).

- Pethe and Horadam [14] introduced Gaussian generalized Fibonacci numbers by

$$GF_n = F_n + iF_{n-1},$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Halıcı and Öz [10] studied Gaussian Pell and Pell Lucas numbers by written , respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1} \end{aligned}$$

where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

- Aşcı and Gürel [1] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

$$\begin{aligned} GJ_n &= J_n + iJ_{n-1}, \\ Gj_n &= j_n + ij_{n-1} \end{aligned}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

- Taşçı [25] introduced and studied Gaussian Mersenne numbers defined by

$$GM_n = M_n + iM_{n-1}$$

where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

- Taşçı [26] introduced and studied Gaussian balancing and Gaussian Lucas Balancing numbers given by, respectively,

$$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + iC_{n-1} \end{aligned}$$

where $B_n = 6B_{n-1} - BJ_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6Cj_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

- Ertaş and Yılmaz [8] studied Gaussian Oresme numbers and defined them as

$$GS_n = S_n + iS_{n-1}$$

where oresme numbers are given by $S_n = S_{n-1} - \frac{1}{4}S_{n-2}$, $S_0 = 0$, $S_1 = \frac{1}{2}$.

Now, we present some gaussian numbers with third order recurrence relations.

- Soykan, et al. [22] presented Gaussian generalized Tribonacci numbers given by

$$GW_n = W_n + iW_{n-1}$$

where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition W_0, W_1, W_2.

- Taşcı [27] studied Gaussian Padovan and Gaussian Pell- Padovan numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1} \\ GR_n &= R_n + iR_{n-1} \end{aligned}$$

where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

- Cerda-Morales [4] defined Gaussian third-order Jacobsthal numbers as

$$GJ_n = J_n + iJ_{n-1}$$

where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_2 = 1$.

2. Gaussian Generalized Edouard Numbers

In this section, we define Gaussian generalized Edouard numbers and present some properties such as Binet's formula and generating function.

Gaussian generalized Edouard numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 7GW_{n-1} - 7GW_{n-2} + GW_{n-3}, \tag{2.1}$$

with the initial conditions

$$\begin{aligned} GW_0 &= W_0 + i(7W_0 - 7W_1 + W_2), \\ GW_1 &= W_1 + iW_0, \\ GW_2 &= W_2 + iW_1, \end{aligned}$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = 7GW_{-(n-1)} - 7GW_{-(n-2)} + GW_{-(n-3)} \tag{2.2}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (2.1) holds for all integer n. Note that for all integers n, we get,

$$\begin{aligned} GW_n &= W_n + iW_{n-1}, \\ GW_{-n} &= W_{-n} + iW_{-n-1}. \end{aligned} \tag{2.3}$$

The first few generalized Gaussian Edouard numbers with positive subscript and negative subscript are presented in the following table.

Table 1. The first few generalized Gaussian Edouard numbers.

n	GW_n
0	$W_0 + i(7W_0 - 7W_1 + W_2)$
1	$W_1 + iW_0$
2	$W_2 + iW_1$
3	$W_0 - 7W_1 + 7W_2 + iW_2$
4	$7W_0 - 48W_1 + 42W_2 + i(W_0 - 7W_1 + 7W_2)$
5	$42W_0 - 287W_1 + 246W_2 + i(7W_0 - 48W_1 + 42W_2)$
6	$246W_0 - 1680W_1 + 1435W_2 + i(42W_0 - 287W_1 + 246W_2)$
7	$1435W_0 - 9799W_1 + 8365W_2 + i(246W_0 - 1680W_1 + 1435W_2)$
8	$8365W_0 - 57120W_1 + 48756W_2 + i(1435W_0 - 9799W_1 + 8365W_2)$

and with negative subscript

n	GW_{-n}
0	$W_0 + i(7W_0 - 7W_1 + W_2)$
1	$7W_0 - 7W_1 + W_2 + i(42W_0 - 48W_1 + 7W_2)$
2	$42W_0 - 48W_1 + 7W_2 + i(246W_0 - 287W_1 + 42W_2)$
3	$(246W_0 - 287W_1 + 42W_2) + i(1435W_0 - 1680W_1 + 246W_2)$
4	$(1435W_0 - 1680W_1 + 246W_2) + i(8365W_0 - 9799W_1 + 1435W_2)$
5	$(8365W_0 - 9799W_1 + 1435W_2) + i(48756W_0 - 57120W_1 + 8365W_2)$
6	$(48756W_0 - 57120W_1 + 8365W_2) + i(284172W_0 - 332927W_1 + 48756W_2)$
7	$(284172W_0 - 332927W_1 + 48756W_2) + i(1656277W_0 - 1940448W_1 + 284172W_2)$
8	$(1656277W_0 - 1940448W_1 + 284172W_2) + i(9653491W_0 - 11309767W_1 + 1656277W_2)$

Gaussian Edouard numbers, $GW_n : GW_n(0, 1, 7 + i) = GE_n$, are defined by

$$GE_n = 7GE_{n-1} - 7GE_{n-2} + GE_{n-3} \tag{2.4}$$

with the initial conditions

$$GE_0 = 0, GE_1 = 1, GE_2 = 7 + i.$$

Gaussian Edouard-Lucas numbers, $GW_n(3 + 7i, 7 + 3i, 35 + 7i) = GK_n$, are defined by

$$GK_n = 7GK_{n-1} - 7GK_{n-2} + GK_{n-3} \tag{2.5}$$

with the initial conditions

$$GK_0 = 3 + 7i, GK_1 = 7 + 3i, GK_2 = 35 + 7i.$$

Note that for all integers n, we have

$$\begin{aligned} GE_n &= E_n + iE_{n-1}, \\ GK_n &= K_n + iK_{n-1}. \end{aligned}$$

The first few values of Gaussian Edouard numbers, Gaussian Edouard-Lucas numbers with positive and negative subscript are given in the Table 2.

Table 2. Special cases of Gaussian generalized Edouard numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6
GE_n	0	1	$7+i$	$42+7i$	$246+42i$	$1435+246i$	$8365+1435i$
GE_{-n}		i	$1+7i$	$7+42i$	$42+246i$	$246+1435i$	$1435+8365i$
GK_n	$3+7i$	$7+3i$	$35+7i$	$199+35i$	$1155+199i$	$6727+1155i$	$39203+6727i$
GK_{-n}		$7+35i$	$35+199i$	$199+1155i$	$1155+6727i$	$6727+39203i$	$39203+228487i$

Next, we present The Binet's formula for the Gaussian generalized Edouard numbers.

THEOREM 2.1. *The Binet's formula for the Gaussian generalized Edouard numbers is*

$$\begin{aligned} GW_n &= \left(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^n}{(\alpha-\beta)(\alpha-1)}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^n}{(\beta-\alpha)(\beta-1)}-\frac{(W_2-6W_1+W_0)}{4}\right) \\ &\quad +i\left(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha^{(n-1)}}{(\alpha-\beta)(\alpha-1)}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta^{(n-1)}}{(\beta-\alpha)(\beta-1)}-\frac{(W_2-6W_1+W_0)}{4}\right). \end{aligned}$$

Proof. The proof follows from 1.1 and 2.3. □

The previous Theorem gives the following results, as special cases.

COROLLARY 2.2. *For all integers n, we have following identities,*

(a): $GE_n = \left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}\right)+i\left(\frac{\alpha^{n}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^{n}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}\right)$.

(b): $GK_n = (\alpha^n+\beta^n+1)+i(\alpha^{n-1}+\beta^{n-1}+1)$.

The next Theorem presents the generating function of Gaussian generalized Edouard numbers.

THEOREM 2.3. *Let $f_{GW_n}(x)=\sum_{n=0}^{\infty} GW_n x^n$ denote the generating function of Gaussian generalized Edouard numbers. Then,*

$$f_{GW_n}(x)=\sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0+(GW_1-7GW_0)x+(GW_2-7GW_1+7GW_0)x^2}{1-7x+7x^2-x^3}. \tag{2.6}$$

Proof. Using the definition of Gaussian Edouard numbers, and substracting $7xf(x)$, $-7x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain

$$\begin{aligned}
(1-7x+7x^2-x^3)f_{GW_n}(x) &= \sum_{n=0}^{\infty} GW_n x^n - 7x\sum_{n=0}^{\infty} GW_n x^n + 7x^2\sum_{n=0}^{\infty} GW_n x^n - x^3\sum_{n=0}^{\infty} GW_n x^n, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 7\sum_{n=0}^{\infty} GW_n x^{n+1} + 7\sum_{n=0}^{\infty} GW_n x^{n+2} - \sum_{n=0}^{\infty} GW_n x^{n+3}, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 7\sum_{n=1}^{\infty} GW_{n-1} x^n + 7\sum_{n=2}^{\infty} GW_{n-2} x^n - \sum_{n=3}^{\infty} GW_{n-3} x^n, \\
&= (GW_0 + GW_1x + GW_2x^2) - 7(GW_0x + GW_1x^2) + 7GW_0x^2 \\
&\quad + \sum_{n=3}^{\infty}(GW_n - 7GW_{n-1} + 7GW_{n-2} - GW_{n-3})x^n, \\
&= GW_0 + GW_1x + GW_2x^2 - 7GW_0x - 7GW_1x^2 + GW_0x^2, \\
&= GW_0 + (GW_1 - 7GW_0)x + (GW_2 - 7GW_1 + 7GW_0)x^2,
\end{aligned}$$

and rearranging above equation, we get 2.6. □

Theorem 2.3 gives the following results as special cases,

$$f_{GE_n}(x) = \frac{x+ix^2}{1-7x+7x^2-x^3},\ f_{GK_n}(x) = \frac{(7+3i)\,x^2-(14+42i)\,x+3+7i}{1-7x+7x^2-x^3}.$$

3. Some Identities About Recurrence Relations of Gaussian Generalized Edouard Numbers

In this section, we present some identities on Gaussian Edouard, Gaussian Edouard-Lucas numbers.

THEOREM 3.1. *The following equations hold for all integer n*

$$GE_n = \frac{7}{64}GK_{n+3} - \frac{11}{16}GK_{n+2} + \frac{21}{64}GK_{n+1}, \tag{3.1}$$

$$GK_n = 35GE_{n+3} - 238GE_{n+2} + 199GE_{n+1}. \tag{3.2}$$

Proof. To proof identity (3.1), we can write $GE_n = aGK_{n+3} + bGK_{n+2} + cGK_{n+1}$ and solve the system of equations we get,

$$\begin{aligned}
GE_0 &= aGK_3 + bGK_2 + cGK_1, \\
GE_1 &= aGK_4 + bGK_3 + cGK_2, \\
GE_2 &= aGK_5 + bGK_4 + cGK_3.
\end{aligned}$$

Then, we obtain $a = \frac{7}{64}, b = -\frac{11}{16}, c = \frac{21}{64}$. The other identities can be found similarly. □

We can write $GE_n = aGK_{n+3} + bGK_{n+2} + cGK_{n+1}$ and solve the system of equations we get,

$$\begin{aligned} GK_0 &= aGE_3 + bGE_2 + cGE_1, \\ GK_1 &= aGE_4 + bGE_3 + cGE_2, \\ GK_2 &= aGE_5 + bGE_4 + cGE_3. \end{aligned}$$

Then, we obtain $a = 35, b = -238, c = 199$. □

LEMMA 3.2. *([9]) We assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the generating function of the sequence $\{a_n\}_{n\geq 0}$. Then the generating functions of the sequences $\{a_{2n}\}_{n\geq 0}$ and $\{a_{2n+1}\}_{n\geq 0}$ are stated as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

The generating functions of the even and odd-indexed generalized Edouard sequences are provided by the following theorem.

THEOREM 3.3. *The generating functions of the sequence GW_{2n} and GW_{2n+1} are provided by*

$$f_{GW_{2n}}(x) = \frac{GW_0 + (GW_2 - 35GW_0)x + (42GW_0 - 48GW_1 + 7GW_2)x^2}{1 - 35x + 35x^2 - x^3}. \tag{3.3}$$

$$f_{GW_{2n+1}}(x) = \frac{GW_1 + (GW_0 - 42GW_1 + 7GW_2)x + (7GW_0 - 7GW_1 + GW_2)x^2}{1 - 35x + 35x^2 - x^3} \tag{3.4}$$

Proof. We only proof 3.3. From Theorem 2.3 we can obtain following identities:

$$\begin{aligned} f_{GW_n}(\sqrt{x}) &= \frac{GW_0 - \sqrt{x}\,(GW_1 - 7GW_0) + x\,(GW_2 - 7GW_1 + 7GW_0)}{7x - 7\sqrt{x} - \sqrt{x^3} + 1}. \\ f_{GW_n}(-\sqrt{x}) &= -\frac{GW_0 + \sqrt{x}\,(GW_1 - 7GW_0) + x\,(GW_2 - 7GW_1 + 7GW_0)}{7\sqrt{x} + 7x + \sqrt{x^3} - 1}. \end{aligned}$$

Thereby, using Lemma 3.2 identity 3.3 can be proved . The other identity can be found similarly. □

From Theorem 3.3, we get the following corollary.

COROLLARY 3.4. **a):**

$$\begin{aligned} f_{GE_{2n}}(x) &= \frac{(1+7i)\,x^2 + (7+i)\,x}{-x^3 + 35x^2 - 35x + 1}, \\ f_{GE_{2n+1}}(x) &= \frac{ix^2 + (7+7i)\,x + 1}{-x^3 + 35x^2 - 35x + 1}. \end{aligned}$$

b):

$$\begin{aligned} f_{GK_{2n}}(x) &= \frac{(35+199i)\,x^2-(70+238i)\,x+(3+7i)}{-x^3+35x^2-35x+1}\,, \\ f_{GK_{2n+1}}(x) &= \frac{(7+35i)\,x^2-(46+70i)\,x+(7+3i)}{-x^3+35x^2-35x+1}. \end{aligned}$$

From Corollary 3.4 we can obtain the following corollary which presents the identities on Gaussian Edouard sequences.

COROLLARY 3.5. **(a):** $(7+i)\,GK_{2n-2}+(1+7i)\,GK_{2n-4}=(3+7i)GE_{2n}-(70+238i)\,GE_{2n-2}+(35+199i)\,GE_{2n-4}$.

(b): $iGK_{2n-4}+(7+7i)\,GK_{2n-2}+GK_{2n}=-(70+238i)\,GE_{2n-1}+(35+199i)\,GE_{2n-3}+(3+7i)GE_{2n+1}$.

(c): $(5383+931i)GE_{2n-4}-(46+70i)\,GE_{2n-2}+(7+3i)GE_{2n}=(7+i)GK_{2n-1}+(1+7i)GK_{2n-3}$.

(d): $(7+3i)\,GE_{2n+1}-(46+70i)\,GE_{2n-1}+(7+35i)\,GE_{2n-3}=iGK_{2n-3}+(7+7i)\,GK_{2n-1}+GK_{2n+1}$.

Proof. From Corollary 3.4 we obtain

$$((1+7i)\,x^2+(7+i)\,x)f_{GK_{2n}}=((3+7i)-(70+238i)\,x+(35+199i)\,x^2)f_{GE_{2n}}.$$

The LHS (left hand side) is equal to

$$\begin{aligned} LHS &= ((1+7i)\,x^2+(7+i)\,x)\sum_{n=0}^{\infty}GK_{2n}x^n \\ &= (7+i)x\sum_{n=0}^{\infty}GK_{2n}x^n+(1+7i)\,x^2\sum_{n=0}^{\infty}GK_{2n}x^n \\ &= (7+i)\sum_{n=0}^{\infty}GK_{2n}x^{n+1}+(1+7i)\sum_{n=0}^{\infty}GK_{2n}x^{n+2} \\ &= (7+i)\sum_{n=1}^{\infty}GK_{2n-2}x^n+(1+7i)\sum_{n=2}^{\infty}GK_{2n-4}x^n \\ &= (7+i)(3+7i)x+\sum_{n=2}^{\infty}((7+i)GK_{2n-2}+(1+7i)\,GK_{2n-4})x^n \end{aligned}$$

whereas the RHS (right hand side) is equal to

$$\begin{aligned} RHS &= ((3+7i)-(70+238i)\,x+(35+199i)\,x^2)\sum_{n=0}^{\infty}GE_{2n}x^n \\ &= (3+7i)\sum_{n=0}^{\infty}GE_{2n}x^n-(70+238i)\,x\sum_{n=0}^{\infty}GE_{2n}x^n+(35+199i)x^2\sum_{n=0}^{\infty}GE_{2n}x^n \\ &= (3+7i)\sum_{n=0}^{\infty}GE_{2n}x^n-(70+238i)\sum_{n=0}^{\infty}GE_{2n}x^{n+1}+(35+199i)\sum_{n=0}^{\infty}GE_{2n}x^{n+2} \\ &= (3+7i)\sum_{n=0}^{\infty}GE_{2n}x^n-(70+238i)\sum_{n=1}^{\infty}GE_{2n-2}x^n+(35+199i)\sum_{n=2}^{\infty}GE_{2n-4}x^n \\ &= (3+7i)(7+i)x+\sum_{n=2}^{\infty}((3+7i)GE_{2n}-(70+238i)\,GE_{2n-2}+(35+199i)GE_{2n-4})x^n. \end{aligned}$$

Comparing the coefficients and the proof of the first identity (a) is done. We can present other identity similarly. □

We can get an identitiy related to Gaussian Edouard numbers and Edouard-Lucas numbers given below.

THEOREM 3.6. *For all integers m, n the following identity holds:*

$$GW_{m+n} = E_{m-1}GW_{n+2} + (E_{m-3} - 7E_{m-2})GW_{n+1} + E_{m-2}GW_n.$$

Proof. First, we assume that $m, n \geq 0$. The proof can be given by mathematical induction on m. If $m = 0$ we get,

$$GW_n = E_{-1}GW_{n+2} + (E_{-3} - 7E_{-2})GW_{n+1} + E_{-2}GW_n$$

which is true since $E_{-1} = 0, E_{-2} = 1, E_{-3} = 7$. We assume that the identity given holds for $m \leq k$. For $m = k + 1$, we get,

$$\begin{aligned}
GW_{(k+1)+n} &= 7GW_{n+k} - 7GW_{n+k-1} + GW_{n+k-2} \\
&= 7(E_{k-1}GW_{n+2} + (E_{k-3} - 7E_{k-2})GW_{n+1} + E_{k-2}GW_n) \\
&\quad -7(E_{k-2}GW_{n+2} + (E_{k-4} - 7E_{k-3})GW_{n+1} + E_{k-3}GW_n) \\
&\quad +(E_{k-3}GW_{n+2} + (E_{k-5} - 7E_{k-4})GW_{n+1} + E_{k-4}GW_n) \\
&= (7E_{k-1} - 7E_{k-2} + E_{k-3})GW_{n+2} + ((7E_{k-3} - 7E_{k-4} + E_{k-5}) \\
&\quad -7(7E_{k-2} - 7E_{k-3} + E_{k-4}))GW_{n+1} + (7E_{k-2} - 7E_{k-3} + E_{k-4})GW_n \\
&= E_kGW_{n+2} + (E_{k-2} - 7E_{k-1})GW_{n+1} + E_{k-1}GW_n \\
&= E_{(k+1)-1}GW_{n+2} + (E_{(k+1)-3} - 7E_{(k+1)-2})GW_{n+1} + E_{(k+1)-2}GW_n.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 3.6. The other case of m, n can be proved similarly. □

Taking $GW_n = GW_n$ or $GW_n = GW_n$ in above theorem,respectively,we get.

COROLLARY 3.7. *For all integers m,n, we get,*

$$\begin{aligned}
GE_{m+n} &= E_{m-1}GE_{n+2} + (E_{m-3} - 7E_{m-2})GE_{n+1} + E_{m-2}GE_n, \\
GK_{m+n} &= E_{m-1}GK_{n+2} + (E_{m-3} - 7E_{m-2})GK_{n+1} + E_{m-2}GK_n.
\end{aligned}$$

4. Simson's Formula

In this section, we present Simpson's formula of generalized Gaussian Edouard numbers. This is a special cases of [17, Theorem 4.1].

THEOREM 4.1 (Simpson's formula of generalized Gaussian Edouard numbers). *For all integers n, we can write following equality*

$$\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} = \begin{vmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{vmatrix}$$
$$= (6GW_1 - GW_0 - GW_2)(GW_2^2 + 8GW_1^2 + GW_0^2 - 8GW_1GW_2 + 6GW_0GW_2 - 8GW_0GW_1).$$

Proof. Using [17, Theorem 4.1] we can obtain the required result. □

From the Theorem 4.1 we get the following Corollary.

COROLLARY 4.2. *For all integers n, we get the following identities:*

(a): $\begin{vmatrix} GE_{n+2} & GE_{n+1} & GE_n \\ GE_{n+1} & GT_n & GE_{n-1} \\ GE_n & GE_{n-1} & GE_{n-2} \end{vmatrix} = 6 - 6i.$

(b): $\begin{vmatrix} GK_{n+2} & GK_{n+1} & GK_n \\ GK_{n+1} & GK_n & GK_{n-1} \\ GK_n & GK_{n-1} & GK_{n-2} \end{vmatrix} = -3072 + 3072i.$

5. SUM FORMULAS

In this section we identify some sum formulas of generalized Gaussian Edouard numbers.

THEOREM 5.1.

a): $\sum_{k=0}^{n} W_k = \frac{1}{4}(-(n+3)W_n + (n+2)(7W_{n+1} - W_{n+2}) - (n+1)W_{n+1} + 2W_2 - 13W_1 + 7W_0)$.

b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{32}(-(n+3)W_{2n} + (n+2)(-7W_{2n+2} + 48W_{2n+1} - 7W_{2n}) - (n+1)W_{2n+2} + 15W_2 - 96W_1 + 49W_0)$.

c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{32}(-(n+3)W_{2n+1} + (n+2)(-W_{2n+2} + 42W_{2n+1} - 7W_{2n}) - (n+1)(7W_{2n+2} - 7W_{2n+1} + W_{2n}) + 9W_2 - 56W_1 + 15W_0)$.

Proof. It is given in Soykan [23, Theorem 3.3]. □

In this section, we identify some sum formulas of generalized Gaussian Edouard numbers. □

THEOREM 5.2. *For all integers $n \geq 0$, we have sum formulas given below:*

(a): $\sum_{k=0}^{n} GW_k = \frac{1}{4}(-(n+3)GW_n + (n+2)(7GW_{n+1} - GW_{n+2}) - (n+1)GW_{n+1} + 2GW_2 - 13GW_1 + 7GW_0)$.

(b): $\sum_{k=0}^{n} GW_{2k} = \frac{1}{32}(-(n+3)GW_{2n} + (n+2)(-7GW_{2n+2} + 48GW_{2n+1} - 7GW_{2n}) - (n+1)GW_{2n+2} + 15GW_2 - 96GW_1 + 49GW_0)$.

(c): $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{32}(-(n+3)GW_{2n+1}+(n+2)(-GW_{2n+2}+42GW_{2n+1}-7GW_{2n})-(n+1)(7GW_{2n+2}-7GW_{2n+1}+GW_{2n})+9GW_2-56GW_1+15GW_0)$.

Proof. Use Theorem 5.1 and the definition of GW_n. □

As a special case of the Theorem 5.2, we present the following corollary.

COROLLARY 5.3. *For all integers $n \geq 0$, we have sum formulas given below:*

a): $\sum_{k=0}^{n} GE_k = \frac{1}{4}(-(n+3)GE_n + (n+2)(7GE_{n+1} - GE_{n+2}) - (n+1)GE_{n+1} + 1 + 2i)$.

b): $\sum_{k=0}^{n} GE_{2k} = \frac{1}{32}(-(n+3)GE_{2n}+(n+2)(-7GE_{2n+2}+48GE_{2n+1}-7GE_{2n})-(n+1)GE_{2n+2}+9+15i)$.

c): $\sum_{k=0}^{n} GE_{2k+1} = \frac{1}{32}(-(n+3)GE_{2n+1}+(n+2)(-GE_{2n+2}+42GE_{2n+1}-7GE_{2n})-(n+1)(7GE_{2n+2}-7GE_{2n+1}+GE_{2n})+7+9i)$.

As a special case of the Theorem 5.2 we present the following corollary.

COROLLARY 5.4. *For all integers $n \geq 0$, we have sum formulas given below:*

a): $\sum_{k=0}^{n} GK_k = \frac{1}{4}(-(n+3)GK_n + (n+2)(7GK_{n+1} - GK_{n+2}) - (n+1)GK_{n+1} + 24i)$.

b): $\sum_{k=0}^{n} GK_{2k} = \frac{1}{32}(-(n+3)GK_{2n}+(n+2)(-7GK_{2n+2}+48GK_{2n+1}-7GK_{2n})-(n+1)GK_{2n+2}+160i)$.

c): $\sum_{k=0}^{n} GK_{2k+1} = \frac{1}{32}(-(n+3)GK_{2n+1}+(n+2)(-GK_{2n+2}+42GK_{2n+1}-7GK_{2n})-(n+1)(7GK_{2n+2}-7GK_{2n+1}+GK_{2n})-32)$.

6. Matrix Formulation of GW_n

In this section, we examine some matrices related to Gaussian Edouard and Gaussian Edouard-Lucas numbers. We define the square matrix A of order 3 as

$$A = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Note that

$$A^n = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix}.$$

Then we give the following Lemma.

LEMMA 6.1. *For $n \geq 0$ the following identity is true:*

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

Proof. The proof can be given by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus, the following identity is true.

$$\begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k + 1$, we get,

$$\begin{aligned} \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} &= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} \\ &= \begin{pmatrix} 7GW_{k+2} - 7GW_{k+1} + GW_k \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by mathematical induction on n, the proof is completed. □

For the proof see [19].

We define

$$N_{GW} = \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \tag{6.1}$$

$$S_{GW} = \begin{pmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with N_{GW} and S_{GW}.

THEOREM 6.2. *Using N_{GW} and S_{GW}, we get,*

$$A^n N_{GW} = S_{GW}.$$

Proof. Note that we get,

$$\begin{aligned} A^n N_{GW} &= \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix} \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= GW_2 E_{n+1} + GW_1 (E_{n-1} - 7E_n) + GW_0 E_n, \\ a_{12} &= GW_1 E_{n+1} + GW_0 (E_{n-1} - 7E_n) + GW_{-1} E_n, \\ a_{13} &= GW_0 E_{n+1} + GW_{-1} (E_{n-1} - 7E_n) + GW_{-2} E_n, \\ a_{21} &= GW_2 E_n + GW_1 (E_{n-2} - 7E_{n-1}) + GW_0 E_{n-1}, \\ a_{22} &= GW_1 E_n + GW_0 (E_{n-2} - 7E_{n-1}) + GW_{-1} E_{n-1}, \\ a_{23} &= GW_0 E_n + GW_{-1} (E_{n-2} - 7E_{n-1}) + GW_{-2} E_{n-1}, \\ a_{31} &= GW_2 E_{n-1} + GW_1 (E_{n-3} - 7E_{n-2}) + GW_0 E_{n-2}, \\ a_{32} &= GW_1 E_{n-1} + GW_0 (E_{n-3} - 7E_{n-2}) + GW_{-1} E_{n-2}, \\ a_{33} &= GW_0 E_{n-1} + GW_{-1} (E_{n-3} - 7E_{n-2}) + GW_{-2} E_{n-2}. \end{aligned}$$

Using Theorem (3.6) the proof is done. □

By taking, $GW_n = GE_n$ with $GE_0 = 0$, $GE_1 = 1$, $GE_2 = 7 + i$ in (6.1) and (6.2), $GW_n = GK_n$ with $GK_0 = 3 + 7i$, $GK_1 = 7 + 3i$, $GK_2 = 35 + 7i$ in (6.1) and (6.2), respectively, we get,

$$N_{GE} = \begin{pmatrix} 7+i & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 1+7i \end{pmatrix} \quad , \quad S_{GE} = \begin{pmatrix} GE_{n+2} & GE_{n+1} & GE_n \\ GE_{n+1} & GE_n & GE_{n-1} \\ GE_n & GE_{n-1} & GE_{n-2} \end{pmatrix}$$

$$N_{GK} = \begin{pmatrix} 35+7i & 7+3i & 3+7i \\ 7+3i & 3+7i & 7+35i \\ 3+7i & 7+35i & 35+199i \end{pmatrix} \quad , \quad S_{GK} = \begin{pmatrix} GK_{n+2} & GK_{n+1} & GK_n \\ GK_{n+1} & GK_n & GK_{n-1} \\ GK_n & GK_{n-1} & GK_{n-2} \end{pmatrix}$$

From Theorem 6.2, we get the following corollary.

Corollary 6.3. *The following identities are holds*

(a): $A^n N_{GE} = S_{GE}$.

(b): $A^n N_{GK} = S_{GK}$.

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