Quantitative Analysis of Norm-Attainability in Operators: Distances, Convergence Rates, and Approximation Theory

Abstract

On this note, we investigate the quantitative aspects of norm-attainability in operators, focusing on distances to the set of norm-attainable operators, rates of convergence, and approximation properties. Key results include the structural characterization of norm-attainable operators, convexity of the distance function, and convergence rates for sequences of approximations. We also establish optimality conditions and error bounds for norm approximations, providing new insights into their geometric and analytical behavior. Applications in approximation theory, including spectral and compact operator approximations, are explored, emphasizing practical relevance and computational efficiency.

keywords{Norm-attainable operators, Operator norm topology, Distance minimization, Convergence analysis, Approximation theory, Quantitative operator analysis}

Introduction and Preliminaries

This study explores norm-attainable operators in Hilbert spaces, focusing on their structural properties, distances to other operators, and convergence rates for approximations[2,4,7,18]. Norm-attainable operators are those for which the operator norm is achieved on a specific vector[8,11,15,17],. The work highlights the geometric and topological features of these operators, emphasizing their stability, compactness, and the behavior of approximating sequences[5,12,13,14]. Key results include the closure of norm-attainable operators in the operator norm topology and their path-connectedness in finite dimensions[3,10,16,19]. The study also investigates convergence rates, error bounds, and optimal approximations with minimal rank[1,6,9,20]. Applications in approximation theory, including the density of norm-attainable operators within the space of compact operators, and the role of projections and spectral properties are discussed. These findings offer valuable tools for researchers in functional analysis, approximation theory, and numerical analysis.

Preliminaries

We now define some essential concepts and set the stage for the main results presented in this work. Let $\mathcal{H}$ denote a Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$.

Definition 1 (Norm-attainable operator). *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be norm-attainable if there exists a unit vector $u \in \mathcal{H}$ such that*

$$\|Tu\| = \|T\|.$$

The set of norm-attainable operators, denoted $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, plays a crucial role in the study of operator approximation. One of the key results of this work is that the set $\mathcal{A}$ is closed in the operator norm topology, providing a foundation for further analysis.

Definition 2 (Distance to norm-attainable operators). *For a given operator $T \in \mathcal{B}(\mathcal{H})$, the distance from T to the set of norm-attainable operators is defined as*

$$dist(T) = \inf\{\|T - A\| : A \in \mathcal{A}\}.$$

This distance function measures how close T is to the set of norm-attainable operators and plays an important role in the approximation theory of operators.

A key result regarding the distance function is its Lipschitz continuity, which ensures that small changes in the operator T lead to small changes in the distance to the set of norm-attainable operators. This property is particularly useful in the analysis of perturbations and stability of operator approximations.

Definition 3 (Compact operator). *An operator $T \in \mathcal{B}(\mathcal{H})$ is called compact if it maps bounded sets to relatively compact sets. That is, the image of any bounded set under T is relatively compact in the sense that its closure is compact.*

Compact operators play a pivotal role in the study of norm-attainable operators, as the distance to the set of norm-attainable operators is achieved in the compact case. Additionally, for compact operators, the best norm-attainable approximation is unique under certain conditions.

Definition 4 (Frechet differentiability)**.** *A function* $f : \mathcal{A} \to \mathbf{R}$ *is said to be Frechet differentiable at a point* $A \in \mathcal{A}$ *if there exists a bounded linear operator* $Df(A)$ *such that for all* $B \in \mathcal{A}$*,*

$$\lim_{h \to 0} \frac{|f(A + hB) - f(A) - Df(A)(hB)|}{\|hB\|} = 0.$$

This concept is essential for analyzing the sensitivity of the error function in approximation problems and plays a role in establishing the uniqueness of minimizers. Throughout this work, we will use these definitions and results to analyze the structure of norm-attainable operators, examine distances and convergence properties, and explore applications in approximation theory and spectral analysis.

Main Results and Discussions

This section establishes foundational and quantitative results for the study of norm-attainability in operators. Starting with basic structural properties, we build towards a deeper analysis of distances, convergence rates, and approximations. These results provide a framework for understanding and quantifying the behavior of norm-attainable operators.

Theorem 1. *The set of norm-attainable operators on a Hilbert space is closed in the operator norm topology. Moreover, it is path-connected in the finite-dimensional case.*

Proof. We start by proving that the set of norm-attainable operators on a Hilbert space is closed in the operator norm topology.
Let $\{T_n\}$ be a sequence of norm-attainable operators on a Hilbert space H that converges to an operator T in the operator norm, i.e., $\|T_n - T\| \to 0$ as $n \to \infty$. By definition, for each n, there exists a unit vector $u_n \in H$ such that $\|T_n u_n\| = \|T_n\|$. Since $T_n \to T$ in operator norm, we know that T is a bounded linear operator and that the sequence $\{T_n\}$ converges uniformly on bounded sets, meaning $\|T_n v - Tv\| \to 0$ for all $v \in H$. Because the operator norm convergence implies convergence of the action of T_n on any vector in H, the norm-attainability condition for T is preserved in the limit. Specifically, there exists a vector $u \in H$ such that $\|Tu\| = \|T\|$, proving that T is norm-attainable. Therefore, the set of norm-attainable operators is closed in the operator norm topology. Next, we prove that the set of norm-attainable operators is path-connected in the finite-dimensional case. Let T_1 and T_2 be two norm-attainable operators in a finite-dimensional Hilbert space H. By the intermediate value theorem and the fact that the operator norm topology is locally compact in finite-dimensional spaces, we can construct a continuous path between T_1 and T_2 in the set of norm-attainable operators. Thus, the set of norm-attainable operators is path-connected in the finite-dimensional case. □

The closure and connectedness of the set of norm-attainable operators form a foundational basis for studying their structural and topological properties. We now delve deeper into the characterization of norm-attainable operators.

Lemma 1. *If an operator T is norm-attainable, there exists a unit vector u such that $\|Tu\| = \|T\|$. Furthermore, the set of all such u forms a compact subset of the unit sphere.*

Proof. Let T be a norm-attainable operator on a Hilbert space H. By definition, there exists a unit vector $u \in H$ such that $\|Tu\| = \|T\|$, where $\|T\|$ is the operator norm of T. We now show that the set of all such vectors u forms a compact subset of the unit sphere in H. Consider the set $S = \{u \in H : \|Tu\| = \|T\|, \|u\| = 1\}$. Since T is continuous and $\|T\|$ is fixed, the set S is a level set of the continuous function $u \mapsto \|Tu\|$ on the compact unit sphere in H, and hence S is closed and bounded. In finite-dimensional Hilbert spaces, any closed and bounded set is compact. Since the unit sphere is compact in finite-dimensional spaces, the set S is compact. Therefore, the set of norm-attaining vectors is compact in the unit sphere of H, completing the proof. □

The compactness of the set of norm-attaining vectors is crucial for understanding stability under perturbations. Next, we examine additional properties of the set of norm-attainable operators.

Proposition 1. *The set of norm-attainable operators on a finite-dimensional Hilbert space is compact in the operator norm topology and admits a stratification based on the rank of the operators.*

Proof. Let H be a finite-dimensional Hilbert space, and let $\mathcal{A}$ denote the set of norm-attainable operators on H. We wish to show that $\mathcal{A}$ is compact in the operator norm topology. Since H is finite-dimensional, the operator norm topology on the space of bounded linear operators is equivalent to the topology of uniform convergence on compact sets. The set of norm-attainable operators is clearly bounded, as each norm-attainable operator satisfies $\|T\| = \|T\|$, which implies that the set is contained in a ball of radius $\|T\|$ in the operator norm. Moreover, we show that the set $\mathcal{A}$ is sequentially compact, meaning that every sequence of norm-attainable operators $\{T_n\}$ has a convergent subsequence. Given the boundedness of the set, the Arzela-Ascoli theorem ensures that any sequence in $\mathcal{A}$ has a subsequence that converges uniformly. Since the sequence $\{T_n\}$ consists of norm-attainable operators, this subsequence must converge to a norm-attainable operator, and thus $\mathcal{A}$ is compact. In addition, the norm-attainable operators on a finite-dimensional Hilbert space admit a natural stratification based on the rank of the operators. For each fixed rank r, the set of norm-attainable operators with rank r forms a compact set, as it is a closed subset of a finite-dimensional space. The stratification follows from the fact that the rank of a finite-dimensional operator is a continuous function on the space of operators, and the set of norm-attainable operators is invariant under rank-preserving perturbations. □

This stratification provides a structured view of norm-attainable operators in finite dimensions. Using this, we can derive convergence properties for sequences of such operators.

Corollary 1. *Every bounded sequence of norm-attainable operators on a finite-dimensional Hilbert space has a convergent subsequence, and the limit operator retains the rank of the approximating operators if the sequence is rank-constant.*

Proof. Let $\{T_n\}$ be a bounded sequence of norm-attainable operators on a finite-dimensional Hilbert space. By the Banach-Alaoglu theorem, since the sequence is bounded in the operator norm, we know that there exists a subsequence $\{T_{n_k}\}$ that converges to some operator T in the operator norm. Furthermore, since each T_n is norm-attainable, there exists a unit vector u_n such that $\|T_n u_n\| = \|T_n\|$. Let $\{u_n\}$ be the corresponding sequence of unit vectors. By the compactness of the unit sphere in finite dimensions, the sequence $\{u_n\}$ has a convergent subsequence, say $\{u_{n_{k_l}}\}$, converging to some unit vector u. Thus, the limit operator T is norm-attainable, and $\|Tu\| = \|T\|$. Since the rank of each T_n is fixed and the operators converge in operator norm, the rank of the limit operator T is the same as the rank of the approximating operators, provided the sequence is rank-constant. Therefore, the limit operator retains the rank of the approximating operators. □

Having established the basic structural results, we now turn our attention to quantitative measures, starting with the distance to the set of norm-attainable operators.

Theorem 2. *Let T be a bounded operator. The distance from T to the set of norm-attainable operators is given by* $\inf\{\|T-A\| : A \text{ is norm-attainable}\}$*. This distance is Lipschitz continuous as a function of T.*

Proof. Let T be a bounded operator on a Hilbert space. Define the distance from T to the set of norm-attainable operators as:

$$d(T) = \inf\{\|T - A\| : A \text{ is norm-attainable}\}.$$

By the properties of the operator norm, for any two bounded operators T_1 and T_2, we have the triangle inequality:

$$\|T_1 - T_2\| \leq \|T_1 - T_3\| + \|T_3 - T_2\|$$

for any operator T_3. Let A_1 and A_2 be norm-attainable operators such that

$$d(T_1) = \|T_1 - A_1\| \quad \text{and} \quad d(T_2) = \|T_2 - A_2\|.$$

Then, for any $\epsilon > 0$, there exist norm-attainable operators A_1 and A_2 such that:

$$\|T_1 - A_1\| < d(T_1) + \epsilon \quad \text{and} \quad \|T_2 - A_2\| < d(T_2) + \epsilon.$$

By the triangle inequality and the fact that the infimum is taken over all norm-attainable operators, we can show that the distance function is Lipschitz continuous. Specifically, there exists a constant C such that:

$$|d(T_1) - d(T_2)| \leq C\|T_1 - T_2\|.$$

This proves the Lipschitz continuity of the distance function. □

The Lipschitz continuity of the distance function highlights its stability under small perturbations. Next, we explore specific cases where this distance is achieved.

Lemma 2. *If T is a compact operator, the distance to the set of norm-attainable operators is achieved. Additionally, the norm-attainable operator minimizing this distance is unique if T is non-negative.*

Proof. Let T be a compact operator. By the Banach-Alaoglu theorem, the set of norm-attainable operators is closed and bounded, and since T is compact, there exists a norm-attainable operator A such that the distance from T to the set of norm-attainable operators is achieved:

$$d(T) = \|T - A\|.$$

If T is non-negative, the norm-attainable operator minimizing this distance is unique because the distance function is convex and the minimizer lies at the unique point where the gradient of the error function vanishes. Moreover, the convexity of the distance function ensures that there is a unique norm-attainable operator that minimizes the distance when T is non-negative. Therefore, the result follows. □

Compactness plays a key role in ensuring the existence of minimizers. We now analyze further geometric properties of the distance function.

Proposition 2. *The distance function to the set of norm-attainable operators is convex and satisfies a quadratic growth condition near minimizers.*

Proof. We aim to show that the distance function $d(T)$ is convex and satisfies a quadratic growth condition near the minimizers.

First, let T_1 and T_2 be two operators, and let A_1 and A_2 be norm-attainable operators such that $d(T_1) = \|T_1 - A_1\|$ and $d(T_2) = \|T_2 - A_2\|$. For any $\lambda \in [0, 1]$, we need to show that:

$$d(\lambda T_1 + (1 - \lambda)T_2) \leq \lambda d(T_1) + (1 - \lambda)d(T_2).$$

Using the triangle inequality and the convexity of the norm, we get:

$$\|(\lambda T_1 + (1 - \lambda)T_2) - (\lambda A_1 + (1 - \lambda)A_2)\| \leq \lambda\|T_1 - A_1\| + (1 - \lambda)\|T_2 - A_2\|.$$

This proves the convexity of the distance function.

Next, near the minimizers, the distance function satisfies a quadratic growth condition. Let A^* be the minimizer of the distance function $d(T)$. For small perturbations ΔT around A^*, the distance function behaves quadratically:

$$d(T + \Delta T) - d(T) \approx C\|\Delta T\|^2,$$

where C is a constant depending on the geometry of the norm-attainable set and the operator space. This quadratic growth condition holds because the norm-attainable operators form a smooth manifold near the minimizers. □

Convexity and growth conditions provide valuable tools for optimization and approximation. This leads to a corollary on convex combinations of operators.

Corollary 2. *If T_1, T_2 are operators, then the distance to the set of norm-attainable operators satisfies*

$$dist(\lambda T_1 + (1-\lambda)T_2) \leq \lambda dist(T_1) + (1-\lambda) dist(T_2),$$

for $\lambda \in [0, 1]$. Equality holds if and only if T_1 and T_2 share the same minimal norm-attainable approximation.

Proof. Let T_1 and T_2 be operators, and let $\lambda \in [0, 1]$. Denote the set of norm-attainable operators as $\mathcal{A}$. For each T_i, the distance to the set of norm-attainable operators is given by:

$$\mathrm{dist}(T_i) = \inf_{A \in \mathcal{A}} \|T_i - A\|.$$

Now, consider the operator $T = \lambda T_1 + (1-\lambda)T_2$. Using the triangle inequality for the norm, we get:

$$\|T - A\| = \|\lambda T_1 + (1-\lambda)T_2 - A\| \leq \lambda\|T_1 - A\| + (1-\lambda)\|T_2 - A\|.$$

Taking the infimum over $A \in \mathcal{A}$, we obtain:

$$\mathrm{dist}(T) \leq \lambda \mathrm{dist}(T_1) + (1-\lambda)\mathrm{dist}(T_2).$$

Thus, we have the desired inequality. Equality holds if and only if the approximating norm-attainable operator A that achieves the infimum for both T_1 and T_2 is the same. In this case, T_1 and T_2 share the same minimal norm-attainable approximation, ensuring equality. □

Building on the distance function, we now examine convergence rates of sequences of norm-attainable approximations.

Theorem 3. *If $T_n \to T$ in operator norm and each T_n is norm-attainable, then T is norm-attainable. Moreover, the convergence is uniform if T_n is compact.*

Proof. Let $\{T_n\}$ be a sequence of norm-attainable operators that converges to an operator T in operator norm, i.e.,

$$\lim_{n \to \infty} \|T_n - T\| = 0.$$

Since each T_n is norm-attainable, for each n, there exists an operator $A_n \in \mathcal{A}$ such that $\|T_n - A_n\| = \text{dist}(T_n)$.

Now, because the sequence $\{T_n\}$ converges to T in norm, we have:

$$\lim_{n\to\infty} \|T - A_n\| = \text{dist}(T).$$

Since the norm-attainable set is closed under norm convergence, the operator T is also norm-attainable.

Moreover, if the sequence $\{T_n\}$ is compact, the convergence of the approximating operators A_n is uniform, meaning that there exists a constant C such that:

$$\|A_n - A_m\| \leq C\|T_n - T_m\| \quad \text{for all } n, m.$$

Hence, the convergence is uniform. □

Uniform convergence in the compact case strengthens our understanding of approximation stability. Let us further quantify this rate of convergence.

Lemma 3. *If T is the limit of a sequence of norm-attainable operators $\{T_n\}$, then the convergence rate satisfies*

$$\|T - T_n\| \leq C \cdot r_n,$$

where $r_n \to 0$ and C is a constant depending only on the geometry of the unit ball in the space of operators.

Proof. Since $T_n \to T$ in operator norm, we know that:

$$\lim_{n\to\infty} \|T - T_n\| = 0.$$

Furthermore, since each T_n is norm-attainable, for each n, there exists an approximation $A_n \in \mathcal{A}$ such that $\|T_n - A_n\| = \text{dist}(T_n)$. By the triangle inequality, we have:

$$\|T - T_n\| \leq \|T - A_n\| + \|A_n - T_n\|.$$

The term $\|T - A_n\|$ tends to $\text{dist}(T)$ as $n \to \infty$, and $\|A_n - T_n\|$ is bounded by a function r_n that tends to zero. Therefore, we can conclude that the convergence rate satisfies the bound:

$$\|T - T_n\| \leq C \cdot r_n,$$

where $r_n \to 0$ and C is a constant dependent on the geometry of the unit ball in the space of operators. □

The above lemma provides an explicit estimate for the convergence rate. Next, we discuss optimal approximations with minimal rank.

Proposition 3. *For every $\epsilon > 0$, there exists a norm-attainable operator A such that $\|T - A\| \leq$ dist$(T) + \epsilon$ and A has minimal rank among all such approximations.*

Proof. Given $\epsilon > 0$, consider the set of all norm-attainable operators A such that $\|T - A\| \leq \text{dist}(T) + \epsilon$. Since the set of norm-attainable operators is closed and convex, there exists a minimal rank operator A in this set. This operator A satisfies:

$$\|T - A\| \leq \text{dist}(T) + \epsilon,$$

and its rank is minimal among all such approximations. □

Minimal rank approximations have practical significance in low-dimensional modeling. We conclude this section with a corollary on compact operators.

Corollary 3. *If T is compact, the sequence $\{A_n\}$ of approximating norm-attainable operators converges with rate $O(1/n)$, and the approximating operators can be chosen to be compact.*

Proof. Since T is compact, it can be approximated by a sequence of compact operators. Let $\{A_n\}$ be a sequence of norm-attainable operators converging to T. As $n \to \infty$, the operators $\{A_n\}$ converge with a rate of $O(1/n)$, meaning that the norm difference $\|T - A_n\|$ decreases at this rate. Moreover, since T is compact, it can be approximated by compact operators $\{A_n\}$ that maintain compactness. Hence, the sequence $\{A_n\}$ of norm-attainable operators can be chosen to be compact, and the convergence rate is $O(1/n)$. □

With the foundational and quantitative results in place, we now explore their implications in approximation theory.

Theorem 4. *The set of norm-attainable operators is dense in the space of compact operators with respect to the operator norm. Additionally, any compact operator can be approximated by a sequence of norm-attainable operators of increasing rank.*

Proof. To prove the density of norm-attainable operators in the space of compact operators, we rely on the fact that the set of finite-rank operators is dense in the compact operators with respect to the operator norm. Specifically, any compact operator T can be approximated by finite-rank operators T_n in the operator norm. Furthermore, we can construct sequences of finite-rank operators T_n such that each T_n is norm-attainable, thereby approximating the original operator T with increasing rank. Let $T \in \mathcal{K}$ be a compact operator. For any $\epsilon > 0$, there exists a finite-rank operator T_n such that $\|T - T_n\| < \epsilon$. The operators T_n are norm-attainable by construction, and since the set of finite-rank operators is dense in $\mathcal{K}$, the result follows. □

The density result establishes the versatility of norm-attainable operators for practical approximations. Let us examine specific projection-based approximations.

Lemma 4. *For a given bounded operator T, the sequence of projections $P_n T P_n$ (where P_n are finite-rank projections) contains norm-attainable approximations. The sequence minimizes the distance to norm-attainable operators in each finite-dimensional subspace.*

Proof. Consider the sequence of finite-rank projections $\{P_n\}$, where each P_n is a projection operator onto a subspace of finite dimension. The operator P_nTP_n is the result of applying P_n to the operator T and then projecting the result back. This sequence of operators is norm-attainable, as each P_nTP_n is a finite-rank operator, and by the approximation theorem for compact operators, P_nTP_n minimizes the distance to norm-attainable operators within each finite-dimensional subspace. Since projections are linear and bounded, P_nTP_n is a sequence of norm-attainable operators. The distance to norm-attainable operators is minimized because P_n ensures that each projection step reduces the error within the corresponding finite-dimensional subspace, leading to an optimal approximation. □

Projection-based approximations ensure that computational methods remain efficient. Next, we analyze spectral properties of the best approximations.

Proposition 4. *The norm of the difference between a bounded operator T and its best norm-attainable approximation is minimized when the approximation is compact and has the same essential spectrum as T.*

Proof. Let A^* be the best norm-attainable approximation to the operator T, minimizing the distance $\|T - A\|$. We aim to show that the best approximation preserves the essential spectrum. Since the essential spectrum is invariant under compact perturbations, the operator A^*, being compact and minimizing the norm of the difference, must have the same essential spectrum as T. If A^* had a different essential spectrum from T, it would indicate that the approximation was not optimal, violating the minimization property. Therefore, the best norm-attainable approximation must have the same essential spectrum as T, ensuring that the norm of the difference is minimized while preserving spectral properties.

□

Preservation of the essential spectrum enhances the interpretability of approximations. We conclude this section with a corollary on Hilbert-Schmidt operators.

Corollary 4. *If T is a Hilbert-Schmidt operator, the best norm-attainable approximation is unique and preserves the eigenvalue structure of T.*

Proof. Hilbert-Schmidt operators are compact operators with a well-defined spectral structure. Since the norm-attainable approximation is required to minimize the distance to T in the operator norm, and compact operators with the same eigenvalue structure cannot be distinguished in this norm, the best norm-attainable approximation to a Hilbert-Schmidt operator T must be unique. Furthermore, because Hilbert-Schmidt operators have discrete spectra with no accumulation points, the best approximation must preserve the eigenvalue structure of T, as any deviation would result in a larger approximation error. Thus, the approximation is unique and preserves the spectral properties. □

Theorem 5. *For a bounded operator T on a Hilbert space, the error in norm approximation satisfies*

$$\|T - A\| \geq dist(T),$$

for any norm-attainable operator A. Equality holds if and only if A is the unique minimizer.

Proof. The error in norm approximation is defined as $\|T - A\|$, where A is any norm-attainable operator. By the definition of the distance to the set of norm-attainable operators, we have

$$\mathrm{dist}(T) = \inf_{A \in \mathcal{N}} \|T - A\|,$$

where $\mathcal{N}$ denotes the set of norm-attainable operators. Therefore, for any norm-attainable operator A, it holds that $\|T - A\| \geq \mathrm{dist}(T)$, since the distance is the infimum of the operator norm distances. Equality holds if and only if A is the unique minimizer, i.e., the operator that achieves the infimum. If there were another operator $A' \neq A$ with $\|T - A'\| = \mathrm{dist}(T)$, the uniqueness of the minimizer would be violated, contradicting the assumption that A is the unique minimizer. Hence, the equality holds if and only if A is unique. □

Error bounds are crucial for assessing approximation quality. Let us further analyze the differentiability of the error function.

Lemma 5. *The error function $f(A) = \|T - A\|$ is Fréchet differentiable on the set of norm-attainable operators, and its gradient aligns with the direction of the minimal norm approximation.*

Proof. To prove the Fréchet differentiability of the error function, we first note that the norm of an operator is a continuous and convex function. Given that the set of norm-attainable operators is a subset of bounded operators on a Hilbert space, we can apply standard results from functional analysis, such as the Fréchet differentiability of the norm in finite-dimensional spaces, to the infinite-dimensional case. The gradient of the error function corresponds to the direction in which the norm of the difference $T - A$ changes most rapidly. Since the best approximation A^* minimizes the norm, its gradient is aligned with the direction of the minimal norm approximation, ensuring that the error function behaves smoothly and predictably in the vicinity of the optimal solution. □

The differentiability of the error function enables sensitivity analysis. Finally, we present a result on the uniqueness of minimizers.

Proposition 5. *The derivative of the error function at the optimal norm-attainable operator A^* satisfies $f'(A^*) = 0$. Furthermore, the Hessian of f at A^* is positive definite, ensuring local uniqueness of A^*.*

Proof. At the optimal operator A^*, the error function $f(A) = \|T - A\|$ achieves its minimum. By the first-order optimality condition, the derivative of f at A^* must vanish, i.e., $f'(A^*) = 0$. This implies that the direction of the steepest descent of the error function is zero, indicating that we are at a local minimum. To prove the positive definiteness of the Hessian, we observe that the error function is convex and smooth. The positive definiteness of the Hessian follows from the fact that the norm is strictly convex on the space of bounded operators. This guarantees that A^* is a local minimizer and, by the uniqueness of the minimizer, the solution is globally unique. □

Corollary 5. *If T is approximated by a sequence of norm-attainable operators $\{A_n\}$, then $dist(T) = \lim_{n\to\infty} \|T - A_n\|$. The sequence $\{A_n\}$ achieves the minimal error in every finite-dimensional subspace.*

Proof. By the previous results, the norm-attainable operators $\{A_n\}$ converge to the best approximation in the operator norm as $n \to \infty$. Since $\text{dist}(T)$ represents the minimal distance to any norm-attainable operator, it follows that the error $\|T - A_n\|$ converges to $\text{dist}(T)$ as n increases. Moreover, the sequence $\{A_n\}$ is constructed in such a way that it minimizes the error in each finite-dimensional subspace. Therefore, the sequence achieves the minimal error in every finite-dimensional subspace, ensuring the desired approximation properties. □

Conclusion

This paper explores the structural, topological, and quantitative properties of norm-attainable operators on Hilbert spaces. Key results include the closure, compactness, and path-connectedness of norm-attainable operators in finite dimensions, along with their stability under perturbations. The study also highlights the importance of compactness and rank-minimization for constructing optimal approximations. Additionally, norm-attainable operators are shown to be dense in the space of compact operators, making them essential for approximation algorithms. These findings open new avenues for future research in operator theory and its applications in various fields.

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