1

2 The Numerical Solution for a Maxwell Integral

3 Equation: MRI Brain Scan

4

5

6

7

8

9

101

12 .

13 **ABSTRACT**

14

This paper uses Maxwell’s 4th integral equation model to approximate the magnetic field of an MRI machine given the electric field. MRI scans are a popular method for brain mapping and vital to monitor degenerative diseases of the brain such as Dementia and Alzheimer's. The integral is not analytically solvable, so a numerical approximation is obtained using the Gaussian quadrature method and Romberg’s integral method. The approximations are compared in order to obtain good convergence results. The model considers an electrical current and a time dependent electric field. Additionally, understanding the electrical permittivity and conductivity of the brain is critical to tuning the radio frequency of the scan. The assumption is that the MRI scan is of the patient’s head.

15

16 *Keywords: Maxwell integral equation, Gaussian quadrature method, Romberg’s integral method, MRI*

17

18

# 19 1. INTRODUCTION

20

21 Magnetic resonance imaging, or MRI for short, is a type of procedure doctors use to create images of the body. An MRI

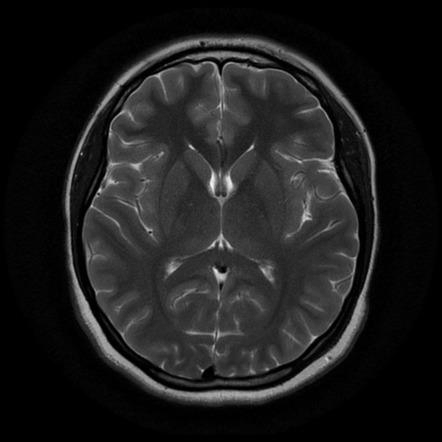
22 scan can produce images of organs, bones, muscles, and blood vessels. To do this, a person is placed within a machine

23 that generates a strong magnetic field (Albani & Bernardi, 1974). The magnetic field causes atoms to align in the same direction, which are then

24 displaced by radio waves. When the atoms return to their original position, radio signals are released allowing for images

25 to be created (Johns Hopkins Medicine, 2019; Ylä-Oijala et al., 2014). This research specifically investigates grey matter as each biological

26 tissue has unique dielectric properties (Abbosh et al., 2024; Borel et al., 2005).

27

28

#### 29 Fig. 1. Image of the human brain from an MRI scan (Radiopaedia, 2024).

30

## 31 1.1 Definitions

32

33 We consider the electromagnetic wave propagation in an adult brain with electric permittivity, magnetic permeability, and

34 electrical conductivity. Refer to table 1 for variable details. The electromagnetic wave with frequency is depicted by the

35 following equations:

36

37 The Maxwell integral equation is given by

38

𝐿 𝐵(𝑙) 𝑑𝑙 = 𝜇𝐼 + 𝜇𝜀

𝛛(𝑎,𝑡) 𝑑𝑎𝑑𝑡, (1)

∮0

39

40 where the electrical field is given by (Colton & Kress, 2013)

41

∫0 ∫0

−1

𝛛𝑡

### (2)

𝑖𝜎

( )

𝐸 𝑎, 𝑡 = (𝜀 + )

|  |  |  |
| --- | --- | --- |
| 42 |  | 𝜔 |
| 43 | and |  |
| 44 |  |  |
|  |  | 𝐸(𝑎) =  4 |
| 45 |  |  |
| 46 | Additionally, the magnetic flux density is given by |  |
| 47 |  | −1 |

### 2 (𝑎)−𝑖𝜔𝑡,

𝑒𝑘|𝑎−𝑡| . (3)

𝜋|𝑎−𝑡|

𝐵(𝑎, 𝑡) = 𝜇

48

### 2(𝑎)

−𝑖𝜔𝑡.

(4)

49 The time dependent Maxwell integral equations follows the following space dependent equations given from (Colton &

50 Kress, 2013)

51

### curl 𝐸 + 𝜇 𝛛𝐸 = 0, (5)

𝛛𝑡

52

53 and

54

### curl 𝐵 − 𝜀 𝛛𝐸 = 𝜎𝐸. (6)

𝛛𝑡

55

56 The space dependent *B* and *E* satisfy the time-harmonic Maxwell equations given by

57

### curl 𝐸 − 𝑖𝑘𝐵 = 0, (7)

58

59 and

60

61

62 The wave number k is given by

63

### curl 𝐵 − 𝑖𝑘𝐸 = 0. (8)

𝑘 = ((𝜀 +

64

𝜎𝑖

𝑤

)𝜇𝜔

2)1/2,

(9)

65 We chose Im 𝑘 = 0. Therefore, the mathematical model of the scattering time harmonic waves from an obstacle, a human

66 brain, leads to a boundary value problem for the reduced Maxwell integral equation.

67

68 In the following table , magnetic permeability, can be considered the absorption coefficient.

69

#### Table 1. Units for electric properties

#### 

Symbol Quantity Units

*A* brain surface area m2

*a* surface area m2

|  |  |  |
| --- | --- | --- |
| *B* | magnetic flux | Tesla |
|  | density |  |
| *I* | electrical current (Schmidt & Webb, | amps |
| *k* | 2016)  electromagnetic | cm2 |
|  | wave number |  |
| *l* | arc length | m |
| *L* | arc length | m |
| T | time patients are | hr |
|  | in MRI scan |  |
| t | time | hr |
| 𝜀 | relative electrical  permittivity | F/m |
|  | (Schmidt & Webb, 2016)  electrical | S/m |
|  | conductivity  (Schmidt & Webb, |  |
|  | 2016)  magnetic | H/m |
|  | permeability |  |
| 𝜔 | angular frequency | rad/s |

## 1.2 Theoretical Framework

72

1. In (1), the double integral was reduced to a single integral with the assumption that time is bounded. The value of 0.75
2. hours was used for t. This new integral equation model was given by

75

∮𝑙 𝐵(𝑙) 𝑑𝑙 = 𝜇𝐼 + 𝜇𝜀 ∫𝐴 𝛛𝐸(𝑎,𝑡) 𝑑𝑎. (10)

0 0 𝛛𝑡

76

1. Referring to (10), partial derivatives were computed for the real and complex parts of the integrand separately. By taking
2. the partial derivative of (3), we obtained 𝑀 = −4𝜋𝑘𝑒𝑘|𝑎−𝑡| + 4𝜋𝑒𝑘|𝑎−𝑡|, the real portion of the integrand becomes

|𝑎−𝑡|

1. 𝐴[ 𝑀 cos(𝜔𝑡) − 𝑒𝑘|𝑎−𝑡| 𝜔𝑠𝑖𝑛(𝜔𝑡)] 𝑑𝑎 and the imaginary portion of the integrand becomes −𝑖

[ sin(𝜔𝑡) −

∫0 (4𝜋|𝑎−𝑡|)2 4𝜋|𝑎−𝑡|

∫0 (4𝜋|𝑎−𝑡|)2

1. 𝑒𝑘|𝑎−𝑡| 𝜔𝑐𝑜𝑠(𝜔𝑡)]𝑑𝑎.

4𝜋|𝑎−𝑡|

1. Together the model is given by

𝐿 ( )

1

𝑖𝜎 −2

### (11)

∮0 𝐵 𝑙

𝑑𝑙 = 𝜇𝐼 + 𝜇𝜀 (𝜀 + )

𝜔

[ cos(𝜔𝑡) − 𝑒𝑘|𝑎−𝑡| 𝜔𝑠𝑖𝑛(𝜔𝑡)] 𝑑𝑎

∫0 (4𝜋|𝑎−𝑡|)2 4𝜋|𝑎−𝑡|

− 𝑖

𝐴 𝑀 sin(𝜔𝑡) 𝑒𝑘|𝑎−𝑡|

### ∫0 [ − 𝜔𝑐(𝜔𝑡)]𝑑𝑎.

2

1. Refer to table 1.

83

# 2. NUMERICAL METHODS

(4𝜋|𝑎−𝑡|)

4𝜋|𝑎−𝑡|

## 2.1 The Framework of the Gaussian quadrature method

86

1. The Gaussian quadrature method was used to modify the integral equation and convert to an integral defined from -1 to 1
2. defined by the following

89

### 𝑏 (𝑥) 𝑑𝑥 = 1

(𝑏−𝑑)𝑡+(𝑏+𝑑)

𝑏−𝑑

### (12)

∫𝑑

90

∫−1 𝑓 (

) ( ) 𝑑𝑡.

2 2

91 Using Gaussian Quadrature nodes, the integral was approximated by a numerical sum given by

92

𝑛

∑

𝑖=1

93

𝑐𝑖𝑃(𝑥𝑖). (13)

1. where 𝑃(𝑥𝑖) are the Legendre polynomials for n, and 𝑐𝑖 are the Gaussian Quadrature weights. We used n = 5 nodes to
2. obtain good convergence results.
3. If *P*(*x*) is any polynomial of degree less than 2*n*, then we can convert the integral by a sum given by

97

1 𝑛

### (14)

∫ 𝑃(𝑥)𝑑𝑥 = ∑ 𝑐𝑖𝑃(𝑥𝑖)

98 where

99

−1

### 𝑐 = ∫1

∏𝑛

𝑖=1

𝑥−𝑥𝑗 𝑑𝑥. (15)

100

𝑖 −1

𝑗=1 𝑥𝑖−𝑥𝑗

𝑗≠1

1. The method was chosen due to the strategically selected nodes from the Lagrange polynomials. This method allows for
2. good convergence results for ellipsoidal regions.

## 2.2 The Framework of Romberg’s Integration method

104

1. To validate our results from the Gaussian Quadrature method, we approximated our integral equation using Romberg’s
2. integral method. This method is based on an advanced form of composite trapezoidal rule given by

107

### ℎ [(𝑎) + 2 ∑𝑛−1 (𝑥 ) + 𝑓(𝑏)]. (16)

2 𝑗=1 𝑗

108

109

110

111

112

113

114

115

116

#### A diagram of a circle AI-generated content may be incorrect.Fig. 2. Sample image of how trapezoidal rule is used to approximate a curve.

Romberg’s approximation is given by

117

𝑅𝑖,𝑘

= 4𝑖𝑅𝑖−1,−𝑅𝑖−1,𝑘−1. (17)

4𝑖−1

118

119

120

121

122

123

124

125

126

127

128

129

130

131

132

133

134

135

136

137

138

139

140

141

142

𝑅𝑖, represents the approximation of the integral at the i-th row and k-th column. Romberg’s integration is an extrapolation technique which takes a sequence of solutions to an integral and calculates a better approximation.

𝑅1,1

𝑅2,1 𝑅2,2

𝑅3,1 𝑅3,2 𝑅3,3

⋮ ⋮ ⋮

𝑅𝑛,1 𝑅𝑛,2 𝑅𝑛,3 ⋯ 𝑅𝑛,𝑛

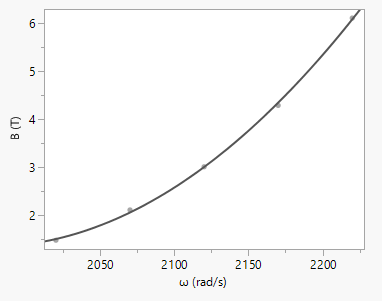
#### Fig. 3. The Romberg’s integral diagram

Romberg’s method was chosen in attempt to see how well an advanced form of the trapezoidal method could approximate an ellipsoidal region.

# NUMERICAL RESULTS

## Gaussian Quadrature Approximations

The following figures show the relationships between each variable and its impact on the magnetic flux density, *B*.



#### Fig. 4. The effects of variation of angular frequency.

In the above figure, the interval 2020 to 2220 cm-1 with the Gaussian Quadrature approximation. The model was given by

143

144

145

146

147

148

149

150

151

152

153

154

155

156

157

158

159

160

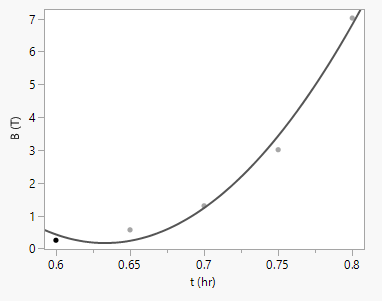
161

162

163

164

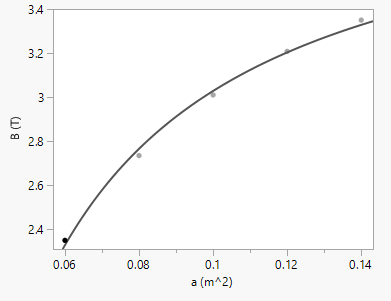
(𝜔) = −45.49231 + 0.0228768𝜔 + 0.000078823(𝜔 − 2120)2. The rate of change of the magnetic flux density in terms of angular frequency at 2120 rad/s is given by 0.0228768. This is a reasonable rate of change.



#### Fig. 5. The effects of variation of time.

In the above figure, the interval 0.6 to .8 hours with the Gaussian Quadrature approximation. The model is given by

(𝑡) = −21.14066 + 31.96712𝑡 + 239.02146(𝑡 − 0.7)2. The rate of change of the magnetic flux density in terms of time has a rapid change at approximately 39 minutes.



#### Fig. 6. The effects of variation of area.

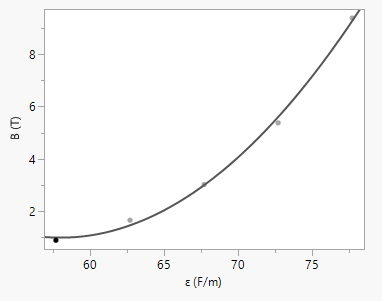
In the above figure, the interval 0.06 to .14 meters2 with the Gaussian Quadrature approximation. The model is given by

1

(𝑎) = 4.0789938 − 0.1050665(). The rate of change of the magnetic flux density in terms of surface area ranges from

𝑎

7.29628472 to 16.4166406, representing a child brain to an adult brain. This is only considering the change of area, and all other variables are kept constant.

165

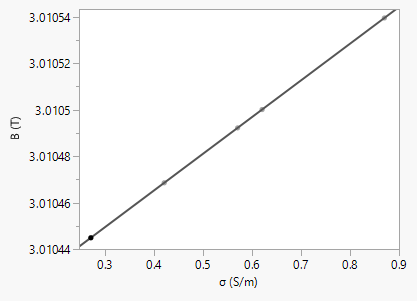
166

#### 167 Fig. 7. The effects of variation of relative electrical permittivity.

168

1. In the above figure, the interval 57.7 to 77.7 Farads/meter with the Gaussian Quadrature approximation. The model is
2. given by 𝐵(𝜀) = −17.19093 + 0.2849845𝜀 + 0.0144537(𝜀 − 67.7)2. The rate of change of the magnetic flux density in
3. terms of relative electrical permittivity is increasing. Thus, the higher the frequency, the higher the rate of change.

172

173

174

175

#### 176 Fig. 8. The effects of variation of electrical conductivity.

177

1. In the above figure, the interval 0.27 to 0.87 Siemens/meter with the Gaussian Quadrature approximation. The model is
2. given by 𝐵(σ) = 3.0104023 + 0.0001576σ. The rate of change of the magnetic flux density in terms of electrical
3. conductivity is given by 0.0001576. Thus, the magnetic flux density and electrical conductivity are directly proportional.

181

182

183

184

185

186

187

188

189

190

191

192

193

194

195

196

197

198

199

200

201

202

203

204

205

206

207

208

209

210

211

212

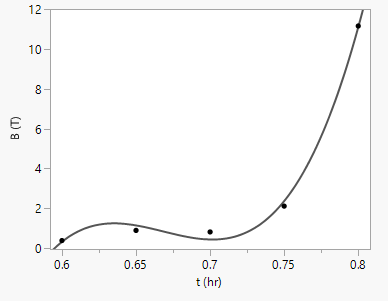
213

#### A graph with numbers and a line AI-generated content may be incorrect.Fig. 9. The effects of variation of electrical current

In the above figure, the interval 34.589 to 42.589 amps with the Gaussian Quadrature approximation. The model is given by (𝐼) = 3.0104437 + 0.0000012565𝐼. The rate of change of the magnetic flux density and electrical current are directly proportional. The higher the current, the higher the magnetic flux density in an MRI brain scan.

## Romberg Approximations

The Romberg figures for angular frequency, surface area, relative electrical permitivity, electrical conductivity, and electrical current were similar to the Gaussian Qudrature figures, except for the magnetic flux density in terms of time, given by figure 3.



#### Fig. 10. The effects of variation of time.

In the above figure, the interval 0.6 to .8 hours with the Romberg’s approximation. The model is given by (𝑡) = 1.6377515 + 1.7068899𝑡 + 526.62232(𝑡 − 0.7)2 + 5554.5386(𝑡 − 0.7)3. Let 𝑑𝐵 = (𝑡) and 𝑃′(𝑡) is changing from a

𝑑𝑡

negative growth rate to a positive growth rate between time of 0.65 and 0.7.

## Approximation Tables

Table 2 to table 7 compares the Gaussian Quadrature approximations to the Romberg approximations. Each highlighted approximation falls within approximately 3 and 7 Tesla. We obtained the following values for the given variables see table 1 where the citations are included.

214

215

216

217

218

219

#### Table 2. The magnetic flux density versus angular frequency.

|  |  |  |
| --- | --- | --- |
| 𝜔  (rad/s) | Gaussian Quadrature Approximation (n=5) | Romberg Approximation R15,15 |
| 2020 | 1.48032196 | 0.08357981 |
| 2070 | 2.11159406 | 0.82315871 |
| 2120 | 3.01049217 | 2.11631864 |
| 2170 | 4.28994208 | 4.22204993 |
| 2220 | 6.11033835 | 7.47317820 |

The variables were 𝜀 = 67.7 F , 𝜎 = 0.57 S , 𝜇 = 4𝜋10−7 H , 𝐼 = 38.589 amps, 𝑡 = 0.75 hr, and 𝑎 = 0.1 m2.

m m m

220

221

222

223

224

#### Table 3. The magnetic flux density versus time.

|  |  |  |
| --- | --- | --- |
| Time (hr) | Gaussian Quadrature  Approximation (n=5) | Romberg Approximation R15,15 |
| .6 | 0.25194142 | 0.38833686 |
| .65 | 0.56930675 | 0.89837299 |
| .7 | 1.30229513 | 0.82113636 |
| .75 | 3.01049217 | 2.11631864 |
| .8 | 7.02312865 | 11.15603600 |

The variables were 𝜀 = 67.7 F , 𝜎 = 0.57 S , 𝜇 = 4𝜋10−7 H , 𝐼 = 38.589 amps, 𝜔 = 2120 rad , and 𝑎 = 0.1 m2.

m m m s

#### 225 Table 4. The magnetic flux density versus area

226

|  |  |  |
| --- | --- | --- |
| Area (m2) | Gaussian Quadrature  Approximation (n=5) | Romberg Approximation R15,15 |
| 0.06 | 2.34996622 | 1.89666554 |
| 0.08 | 2.73485727 | 2.04114519 |
| 0.1 | 3.01049217 | 2.11631864 |
| 0.12 | 3.20821544 | 2.15577796 |
| 0.14 | 3.35030169 | 2.17670071 |

227

228 The variables were 𝜀 = 67.7 F , 𝜎 = 0.57 S , 𝜇 = 4𝜋10−7 H , 𝐼 = 38.589 amps, 𝑡 = 0.75 hr, and 𝜔 = 2120 rad. The best

m m m s

229

230

231

232

233

234

results for adult brains were with an area between 0.1 m2 and 0.14 m2.

#### Table 5. The magnetic flux density versus electrical permittivity.

|  |  |  |
| --- | --- | --- |
| 𝜀 (F/m) | Gaussian Quadrature Approximation (n=5) | Romberg Approximation R15,15 |
| 57.7 | 0.87394413 | 0.63012826 |
| 62.7 | 1.64467706 | 1.17016128 |
| 67.7 | 3.01049217 | 2.11631864 |
| 72.7 | 5.37734408 | 3.73915099 |
| 77.7 | 9.39793994 | 6.47024540 |

The variables were 𝜎 = 0.57 S , 𝜇 = 4𝜋10−7 H , 𝐼 = 38.589 amps, 𝑡 = 0.75 hr, 𝜔 = 2120 rad , and 𝑎 = 0.1 m2. The best

m m s

235 approximation, as previously suggested, was 67.7 F/m for electrical permittivity (Schmidt & Webb, 2016).

236

#### 237 Table 6. The magnetic flux density versus electrical conductivity.

238

|  |  |  |
| --- | --- | --- |
| 𝜎 (S/m) | Gaussian Quadrature Approximation (n=5) | Romberg Approximation R15,15 |
| .27 | 3.01044489 | 2.11628609 |
| .42 | 3.01046853 | 2.11630237 |
| .57 | 3.01049217 | 2.11631864 |
| .62 | 3.01050005 | 2.11632407 |
| .87 | 3.01053945 | 2.11635119 |

239

240 The variables were 𝜀 = 67.7 H , 𝜇 = 4𝜋10−7 H , 𝐼 = 38.589 amps, 𝑡 = 0.75 hr, 𝜔 = 2120 rad , and 𝑎 = 0.1 m2. There was

m m s

241

242

243

244

245

246

minimal change to the magnetic flux density when changing electrical conductivity.

#### Table 7. The magnetic flux density versus electrical current.

|  |  |  |
| --- | --- | --- |
| 𝐼  (amps) | Gaussian Quadrature Approximation (n=5) | Romberg Approximation R15,15 |
| 34.589 | 3.01048715 | 2.11631362 |
| 36.589 | 3.01048966 | 2.11631613 |
| 38.589 | 3.01049217 | 2.11631864 |
| 40.589 | 3.01049469 | 2.11632116 |
| 42.589 | 3.01049720 | 2.11632367 |

The variables were 𝜀 = 67.7 H , 𝜎 = 0.57 S , 𝜇 = 4𝜋10−7 H , 𝑡 = 0.75 hr, 𝜔 = 2120 rad , and 𝑎 = 0.1 m2. There was minimal

m m m s

247

248

249

250

251

252

253

254

255

256

257

258

259

260

261

262

263

264

265

266

267

268

269

270

271

272

273

274

change to the magnetic flux density when changing the electrical current.

Clearly, the Gaussian Quadrature method is far superior to the Romberg’s integral method when solving the Maxwell integral equation for a space such as the human brain. Compared to the Gaussian quadrature method, the Romberg integral method did not properly converge.

# DISCUSSION

Our method was able to obtain convergence results between 3 and 7 Tesla. The fitted equations allow for interpolation and modifications of the approximations with any value in our selected ranges for each variable. For extrapolation, there is some uncertainty outside of the given bounds. Additionally, the current approximations do not use definitive vacuum permeability. Published literature has not shown a consistent permeability for gray matter, so we are estimating it around

the value for free space, 4𝜋10−7 H. Therefore, this analysis is somewhat sensitive to this value. The approximations were

m

found using the Gaussian Quadrature method and Romberg’s method. For both methods, the approximations were compared to achieve the best convergence results. For the Gaussian Quadrature method, a convergence of 10-8 was achieved with 5 nodes, while Romberg’s method did not converge at R15,15. The Romberg results vary significantly from the Gaussian Quadrature approximations and can be interpreted as less reliable. The inability to converge comes from the fact that Romberg’s method uses a combination of rectangles and trapezoids to achieve the approximation, while the human brain is not a rectangular region.

The current method investigates using an electric field to approximate the magnetic flux density. Future work could be done to get the electric field given the magnetic flux density, and see how multiple variables interact with each other.

# CONCLUSION

Previous literature mentions an optimal value of 3 Tesla for an MRI brain scan, but other researchers would like an optimal value closer to 7 Tesla (Formica & Silvestri, 2004). The improved magnetic flux density should yield better

1. images. As shown in table 3 the optimal time a patient should spend in an MRI scan for a brain image is between 45
2. minutes to 48 minutes.

277

278 This research only looked at the gray matter of a healthy adult brain. In the future, further using this method, one might be

279 able to compare the different electrical fields of white matter versus grey matter, which would be significant for detecting

280 early signs of Dementia or Alzheimer's. Research has shown reduced gray matter counts in human brains with

281 Alzheimer's (Thompson et al., 2003). Additionally, the model has the possibility for finding the electric field given the

282 magnetic flux density. When applying the method to other brain tissues for an unhealthy brain, one would need extra

283 Gaussian quadrature nodes to obtain good convergence results, while the Romberg integral method would not be

284 appropriate.

285

# 286 ACKNOWLEDGEMENTS

287

288 We would like to thank the Roger Williams University Provost Fund for allowing us to present a preliminary report of this

289 research at the Joint Mathematics Meeting American Mathematical Society poster session in San Francisco, CA, U.S.A

290 and Seattle, WA, U.S.A.).

291

# 292 AUTHORS’ CONTRIBUTIONS

293

294 Yajni Warnapala designed the theoretical framework of the study. Sam Bielawa performed the numerical calculations for

295 the study and managed the literature searches. Both authors managed the analysis of the study and read and approved

296 the final manuscript.

297Disclaimer (Artificial intelligence)

Option 1:

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

Option 2:

Author(s) hereby declare that generative AI technologies such as Large Language Models, etc. have been used during the writing or editing of manuscripts. This explanation will include the name, version, model, and source of the generative AI technology and as well as all input prompts provided to the generative AI technology

Details of the AI usage are given below:

1.

2.

3.

298 **REFERENCES**

299

300 Johns Hopkins Medicine. (2019). Magnetic resonance imaging (MRI). Johns Hopkins Medicine.

301 https[://www.hopkinsmedicine.org/health/treatment-tests-and-therapies/magnetic-resonance-imaging-mri](http://www.hopkinsmedicine.org/health/treatment-tests-and-therapies/magnetic-resonance-imaging-mri)

302

303 Abbosh, A., Ireland, D., Smith, M., Crozier, S., Zhang, Y., & Mudge, S. (2024). Clinical electromagnetic brain scanner.

304 Scientific Reports, 14(1), 5760. https://doi.org/10.1038/s41598-024-55360-7

305

306 Radiopaedia. (2024). Normal brain MRI. https://radiopaedia.org/cases/normal-brain-mri-6

307

308 Colton, D. L., & Kress, R. (2013). *Inverse acoustic and electromagnetic scattering theory* (3rd ed.). Springer.

309

1. Schmidt, R., & Webb, A. (2016). A new approach for electrical properties estimation using a global integral equation and
2. improvements using high permittivity materials. *Journal of Magnetic Resonance, 262*, 8–14.
3. https://doi.org/10.1016/j.jmr.2015.11.002

313

314 ChatGPT. (2025). Image generated from a text prompt. OpenAI. https://platform.openai.com/

315

1. Formica, D., & Silvestri, S. (2004). Biological effects of exposure to magnetic resonance imaging: An overview.
2. *Biomedical Engineering Online, 3*, 11. https://doi.org/10.1186/1475-925X-3-11

318

1. Thompson, P. M., Hayashi, K. M., de Zubicaray, G. I., Janke, A. L., Rose, S. E., Semple, J., et al. (2003). Dynamics of
2. gray matter loss in Alzheimer’s disease. *The Journal of Neuroscience, 23*(3), 994–1005.
3. <https://doi.org/10.1523/jneurosci.23-03-00994.2003>

Albani, M., & Bernardi, P. (1974). A numerical method based on the discretization of Maxwell equations in integral form (short papers). IEEE Transactions on Microwave Theory and Techniques, 22(4), 446-450.

322

Yla-Oijala, P., Markkanen, J., Jarvenpaa, S., & Kiminki, S. P. (2014). Surface and volume integral equation methods for time-harmonic solutions of Maxwell's equations. Progress in electromagnetics Research, 149, 15-44.

Borel, S., Levadoux, D. P., & Alouges, F. (2005). A new well-conditioned integral formulation for Maxwell equations in three dimensions. IEEE transactions on antennas and propagation, 53(9), 2995-3004.