Original Research Article.

NEW PROOFS OF THE EQUIVALENT STATEMENT OF THE DIRICHLET ETA FUNCTION AND OF THE RIEMANN HYPOTHESIS

ABSTRACT. We will present two new proofs (and some of our old results) for the "Dirichlet eta" function $S(s) = \sum_{n\geq 1} \frac{(-1)^n}{n^s}$ which would lead us to announce some new conjectures equivalent to that of the Riemann hypothesis.

The first conjecture announced: In the band s $(s = r + ic)$ a complex such that its real part is strictly between 0 and 1 $(0 < r < 1)$, we have the real part of the Dirchlet function $(S(s))$ can only be zero in the straight line "the real part of s is equal to 0.5" $(r = 0, 5)$. While the second conjecture informs us about what the zeros can be in the straight line $r = 0.5$.

Keywords: Riemann, hypothesis, dirchlet,conjecture,

1. INTRODUCTION

The Riemann Hypothesis is a conjecture formulated in 1859 by the mathematician Bernhard Riemann, according to which the nontrivial zeros of the Riemann zeta function are infinite and all have a real part equal to 1/2. see [5] [2]

His proof would improve knowledge of the distribution of prime numbers and open up new areas of mathematics. Riemann's article (see [3]) on the distribution of prime numbers is his only text dealing with number theory. He develops the properties of the zeta function $C(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ and proves the prime number theorem by admitting several results, including what is now called the Riemann Hypothesis. Hardy then demonstrated that there are infinitely many zeros on the critical line (Hardy's Theorem: see [4] [7], [8]), which gives us hope that the RH might be true...

This paper is a continuation of our last "A Contemporary Conjecture for the Riemann Hypothesis" work already published (see [1]).

Let

$S(s) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s} = -\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = -\eta(s)$

so

$S(s) = \rho(s) e^{i\theta(s)} S(1-s)$

Remark 1. *(Functional equation of Hardy)*

We have $\forall s \in \mathbb{C}$ *such* $\mathrm{Re}(s) \in]0, 1[$

$$S(s) = \varphi(s) S(1-s)$$

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with $\varphi(s) = 2\frac{1-2^{s-1}}{1-2^{s}}\pi^{s-1}\sin\left(\frac{s}{2}\pi\right)\Gamma(1-s) = \rho(s)\,e^{i\theta(s)}$.
see [7] *&* [8]

2. Preliminary

2.1. Analytic extension of the function ζ and its auxiliary functions C_1 and C_2:

Theorem 1. *There exists a unique function, denoted* ζ *(Riemann zeta function), verifying:*
• ζ *is meromorphic on the entire* $\mathbb{C}$*, holomorphic outside a simple pole at* $s = 1$*,with residue* 1*;*
• $\zeta(s) = L(1, s)$*, if* $\operatorname{Re}(s) > 1$

see [10] page 412.

Proposition 1. *Let* $s = r + ic$*, so*

$$S = \sum_{n=1}^{+\infty}(-1)^n\frac{e^{-i\ln(n)c}}{n^r}$$
$$S = \sum_{n=1}^{+\infty}(-1)^n\frac{e^{i\alpha_n}}{n^r}$$

The serie S *is convergent for strictly positive real* s*, by application of the alternating series criterion; it is in fact the same for* $\operatorname{Re}(s) > 0$*, which is demonstrated using Abel's lemma (we can also show more simply the absolute convergence of the serie* $\sum_{n=1}^{+\infty}\frac{(2n)^s-(2n-1)^s}{(2n)^s(2n-1)^s}$*)*

And The Riemann ζ *function is a meromorphic complex analytic function defined, for any complex number* s *such that*
$\operatorname{Re}(s) > 1$*, by the Riemann serie:*

$$\zeta = \sum_{n=1}^{+\infty}\frac{1}{n^s} = \sum_{n=1}^{+\infty}\frac{e^{-i\ln(n)c}}{(2n)^r} = \sum_{n=1}^{+\infty}\frac{e^{i\alpha_n}}{(2n)^r}$$
$$\zeta = \frac{S(s)}{2^{1-s}-1} = \frac{\eta(s)}{1-2^{1-s}}$$

with $\alpha_n = -\ln(n)\,c$*.*

According to the theory of Dirichlet series, we deduce that the function thus defined is analytic over its domain of convergence. The series does not converge at $s = 1$ *because we have* $\sum_{n=1}^{m}\frac{1}{n} \geq \int_1^{m+1}\frac{dx}{x} = \ln(m+1)$*.*

If $\operatorname{Re}(s) > 1$*,* $\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s} = 2\sum_{n=1}^{+\infty}\frac{1}{(2n)^s} - S(s) = -S(s) + \frac{2}{2^s}\zeta(s)$*, so* $\zeta(s) = \frac{S(s)}{2^{1-s}-1}$

This thus realizes the extension of the ζ *function over* $\operatorname{Re}(s) > 0$*, except for* $s = 1 + \frac{2k\pi}{\ln(2)}i$*,* $k \in \mathbb{Z}$*.*

Also we will realize the extension of the C_1 *and* C_2 *such*

$$C_1 = \sum_{n=1}^{+\infty} \frac{1}{(2n)^s} = \frac{1}{2^s}\zeta(s) = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n)c}}{(2n)^r} = \sum_{n=1}^{+\infty} \frac{e^{i\alpha_{2n}}}{(2n)^r} = R_1 + iI_1$$

$$C_2 = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^s} = \zeta - C_1 = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n-1)c}}{(2n-1)^r} = \sum_{n=1}^{+\infty} \frac{e^{i\alpha_{2n-1}}}{(2n-1)^r} = R_2 + iI_2$$

and

$$\begin{aligned}
R_1 &= \sum_{n=1}^{+\infty} \frac{\cos(\alpha_{2n})}{(2n)^r},\ I_1 = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_{2n})}{(2n)^r},\ R_2 = \sum_{n=1}^{+\infty} \frac{\cos(\alpha_{2n-1})}{(2n-1)^r},\ I_2 = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_{2n-1})}{(2n-1)^r} \\
R &= \sum_{n=1}^{+\infty} \frac{\cos(\alpha_n)}{(2n)^r} = R_1 + R_2,\ I = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_n)}{(2n)^r} = I_1 + I_2 \\
R' &= \sum_{n=1}^{+\infty} (-1)^n \frac{\cos(\alpha_n)}{n^r}\ ,\ I' = \sum_{n=1}^{+\infty} (-1)^n \frac{\sin(\alpha_n)}{n^r},\ R' = R_1 - R_2,\ I' = I_1 - I_2 \\
S &= C_1 - C_2 = R' + iI'\ ,\ \ \zeta = C_1 + C_2 = R + iI
\end{aligned}$$

Proposition 2. *Let* $s = r + ic = r + i\frac{\alpha}{\ln(2)}$ *(since* $\alpha = \ln(2)\,c$*)*

$$\begin{aligned}
C_1 &= \frac{e^{-i\alpha}}{2^r}\zeta \\
C_2 &= \left(1 - \frac{e^{-i\alpha}}{2^r}\right)\zeta \\
S &= \left(2^{1-r}e^{-i\alpha} - 1\right)\zeta
\end{aligned}$$

Proof. $\alpha = \ln(2)\,c \Rightarrow e^{-i\ln(2)c} = e^{-i\alpha}$

Therefore,

$$C_1 = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n)c}}{(2n)^r} = \frac{e^{-i\ln(2)c}}{2^r} \sum_{n=1}^{+\infty} \frac{e^{-i\ln(n)c}}{n^r}$$

$$C_1 = \frac{e^{-i\ln(2)c}}{2^r}\zeta = \frac{e^{-i\alpha}}{2^r}\zeta$$

$C_2 = \zeta - C_1 = \zeta - \frac{e^{-i\alpha}}{2^r}\zeta \Rightarrow$

$$C_2 = \left(1 - \frac{e^{-i\alpha}}{2^r}\right)\zeta$$

and

$S = C_1 - C_2 = \frac{e^{-i\alpha}}{2^r}\zeta - \left(1 - \frac{e^{-i\alpha}}{2^r}\right)\zeta \Longrightarrow$

$$S = \left(2^{1-r}e^{-i\alpha} - 1\right)\zeta$$

□

3. Our previous contributions

3.1. **Introduction.** All the results we will cited here are taken and slightly distorted from others already cited and demonstrated for a long time by other researchers and books. To demonstrate Lemma 1 we used the classic course of Complex Analysis like [11] and [6].

Definition 1. *(differentiability and holomorphy)*

A function $f : U \to C$ on an open set of the complex plane is said to be complex differentiable at the point $z_0 \in U$ if the limit $f'(z_0) = lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ exists. The function f is said to be holomorphic on U if it is complex differentiable at every point of U, and it is said to be holomorphic on a set A if there exists an open set V containing A on which f is defined and holomorphic. A holomorphic function on the entire $\mathbb{C}$ is called an entire function.

Definition 2. *(analytic functions)*

We say that a function $f(z)$ defined on an open set U of the complex plane $\mathbb{C}$ is analytic on U if, at every point z_0, there exists a $\delta(z_0) > 0$ and complex numbers $c_n(z_0)$ such that for $|z - z_0| < \delta(z_0)$, we have $f(z) = \sum_{n=0}^{\infty} c_n(z_0)(z - z_0)^n$.

Theorem 2. *Every analytic function is holomorphic, and indeed infinitely differentiable in the complex sense; moreover, all its derived functions are also analytic functions.*

Remark 2. *Cauchy's theory leads us to a fundamental theorem: every holomorphic function is an analytic function. In particular, every holomorphic function is automatically infinitely differentiable in the complex sense.*

Theorem 3. *1- If f and g are holomorphic on U and coincide on a set with a non-isolated point, they are equal.*

2- If the set of points of a domain U where two analytic functions f and g on U take the same values has an accumulation point in U, then in fact the two functions f and g take the same values everywhere in U. They are, in fact, the same functions.

Theorem 4. *(Adherent Point and Closure)*

Let X be a topological space and $A \subseteq X$ be a subset. A point $x \in X$ is said to be an adherent point (Closure point) of A if every open neighborhood of x intersects A. The closure of A, denoted by $\overline{A}$, consists of all adherent points of A.

Theorem 5. *(Adherent Point and Existence of Convergent Sequences)*

Let X be a topological space and $A \subseteq X$ be a subset. A point $x \in X$ is an adherent point (Closure point) of A if and only if there exists a sequence x_n in A such that

$$\lim_{n\to+\infty} x_n = x$$

3.2. **The first announcement.**

Lemma 1. *Assuming that there exists an s_1 with $r_1 = \mathrm{Re}[s_1] \in \left]0, \frac{1}{2}\right[$ and $c = \mathrm{Im}[s_1] > 0$ such that $S^2(s_1) \in IR$, so*

(i) $\exists V(s_1) \subset \mathbb{C}$ such $\forall s \in V(s_1) - \{s_1\}$, $S^2(s) \notin IR$

(ii) $\exists u_n \in V(s_1) - \{s_1\}$ such $\lim u_n = s_1$ (since $s_1 \in \overline{V(s_1) - \{s_1\}}$ with $\overline{A}$ is the adherant of A).

Proof. Obvious.

(i) Reasoning by the absurd. Suppose

$\forall V(s_1) \subset \mathbb{C}$, $\exists s \in V(s_1) - \{s_1\}$ such $S^2(s) \in IR$ $(*)$

Let $s(0) \neq s_1$ such $s(0)$ very close to s_1, and $V_0(s_1)$ such $s(0) \in V_0(s_1)$

$\Longrightarrow \exists s(1) \in V_0(s_1) - \{s_1\}$ such $S^2[s(1)] \in IR$

$s(1) \neq s_1 \Longrightarrow \exists V_1(s_1)$ such $s(1) \notin V_1(s_1)$ and $V_1(s_1) \subsetneq V_0(s_1)$

$\Longrightarrow \exists s(2) \in V_1(s_1) - \{s_1\}$ such $S^2[s(2)] \in IR$

...

and so on

if

$\exists V_{n-1}(s_1)$ such $s(n-1) \notin V_{n-1}(s_1)$ and $V_{n-1}(s_1) \subsetneq V_{n-2}(s_1)$

and $\exists s(n) \in V_{n-1}(s_1) - \{s_1\}$ such $S^2[s(n)] \in IR$

$\Downarrow$

$\exists V_n(s_1)$ such $s(n) \notin V_n(s_1)$ and $V_n(s_1) \subsetneq V_{n-1}(s_1)$

$\Longrightarrow \exists s(n+1) \in V_n(s_1) - \{s_1\}$ such $S^2[s(n+1)] \in IR$

so, we construct a sequence $s(n)$ which converges to s_1 such that $S^2(s(n)) \in IR$, such

$$\lim V_n(s_1) = \lim \bigcap_{k=0}^{n} V_k(s_1) = \{s_1\}$$

since $\{s_1\} \subsetneq V_n(s_1) \subsetneq V_{n-1}(s_1)$ $\forall n$ (the decrease of $V_n(s_1)$)

consequently

$S^2(s(n)) - \overline{S^2(s(n))} = 0$

S^2 and $\overline{S^2}$ are analytic and holomorphic functions,

it suffices to see that if f satisfies the Cauchy-Riemann equations then $\overline{f}$ also satisfies it (Using the Cauchy-Riemann equations and Schwarz's theorem).

so $S^2(s) - \overline{S^2(s)}$ is analytic and holomorphic function,

let $U = \{s(n)/n \in IN\} \cup \{s_1\}$

$S^2(s) - \overline{S^2(s)}$ and 0 are two analytic functions take the same values on U and U has an accumulation point (in U)

according to the theorem we have

$S^2(s) - \overline{S^2(s)} = 0$

$\Longrightarrow S^2(s) \in IR$ absurd!!

(ii) Using (i) and the last theorem. □

Lemma 2. *Let* $D_1 = \left\{z \in \mathbb{C}/\operatorname{Re}(z) \in \left]0,1\right[, \operatorname{Re}(z) \neq \frac{1}{2} \text{ and } \operatorname{Im}(z) \neq 0\right\}$, *so* $\forall s \in D_1$

$$S^2(s) \in IR \Leftrightarrow S^2(1-s) \in IR$$

Proof. Since the first lemma:

Assuming that there exists an s_1 with $r_1 = \operatorname{Re}[s_1] \in \left]0, \frac{1}{2}\right[$ and $c > 0$ such that $S^2(s_1) \in IR$, so

(i) $\exists V(s_1) \subset \mathbb{C}$ such $\forall s \in V(s_1) - \{s_1\}$, $S^2(s) \notin IR$

(ii) $\exists u_n \in V(s_1) - \{s_1\}$ such $\lim u_n = s_1$

(since $s_1 \in \overline{V(s_1) - \{s_1\}}$ with $\overline{A}$ is the adherant of A).

$\lim u_n = s_1 \Longrightarrow \exists N \in IN$ such $\forall n \geq N$, $u_n \in V(s_1) - \{s_1\}$

$\Rightarrow \exists N \in IN$ such $\forall n \geq N$, $S^2(u_n) \notin IR$

$\Rightarrow \left(S(u_n), \overline{S(u_n)}\right)$ is a basis of $\mathbb{C}$

$\Rightarrow \exists! (a_n, b_n) \in IR^2$ such $S(1-u_n) = a_n S(u_n) + b_n \overline{S(u_n)}$

$S(s)=\varphi(s)\,S(1-s)$
$\Rightarrow S(1-u_n)=a_n\varphi(u_n)\,S(1-u_n)+b_n\overline{\varphi(u_n)S(1-u_n)}$
$\Rightarrow [1-a_n\varphi(u_n)]\,S(1-u_n)=\left[b_n\overline{\varphi(u_n)}\right]\overline{S(1-u_n)}$
$\Rightarrow \left[1-a_n\overline{\varphi(u_n)}\right]\overline{S(1-u_n)}=[b_n\varphi(u_n)]\,S(1-u_n)$
$\Rightarrow b_n\varphi(u_n)\,S^2(1-u_n)=\left[1-a_n\overline{\varphi(u_n)}\right]|S(1-u_n)|^2$
$\varphi(s)\,\varphi(1-s)=1\ \forall s$
$\Rightarrow b_nS^2(1-u_n)=\varphi(1-u_n)\left[1-a_n\overline{\varphi(u_n)}\right]|S(1-u_n)|^2$
$\Rightarrow b_nS^2(1-u_n)=\left[\varphi(1-u_n)-a_n\overline{\varphi(u_n)}\varphi(1-u_n)\right]|S(1-u_n)|^2$
$\varphi(s)\,\varphi(1-s)=1\Rightarrow|\varphi(s)|^2\,\varphi(1-s)=\overline{\varphi(s)}$
$\Rightarrow \overline{\varphi(s)}=\rho^2\varphi(1-s)$
$\Rightarrow$

$$\begin{aligned} b_nS^2(1-u_n) &= |S(1-u_n)|^2\left[\varphi(1-u_n)-a_n\rho_n^2\varphi^2(1-u_n)\right]\\ &= |S(1-u_n)|^2\,\varphi(1-u_n)\left[1-a_n\rho_n^2\varphi(1-u_n)\right]\end{aligned}$$

$\Rightarrow |b_n|=|\varphi(1-u_n)|\left|1-a_n\rho_n^2\varphi(1-u_n)\right|$ or $|S(1-u_n)|=0$
we have $S^2(u_n)\notin IR\Rightarrow S(u_n)\neq 0\Rightarrow |S(1-u_n)|\neq 0$
$\Rightarrow |b_n|=|\varphi(1-u_n)|\left|1-a_n\rho_n^2\varphi(1-u_n)\right|$
also $\rho_n\neq 0$ since $|S(u_n)|=\rho_n\,|S(1-u_n)|$
$\Rightarrow |b_n|=\frac{1}{\rho_n}\left|1-a_n\rho_n^2\varphi(1-u_n)\right|$
$\Rightarrow b_n^2\rho_n^2=\left|1-a_n\rho_n^2\varphi(1-u_n)\right|^2$
$\Rightarrow$

$$b_n^2\rho_n^2=1+a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\qquad (1)$$

$S(1-u_n)=a_nS(u_n)+b_n\overline{S(u_n)}$
$\Rightarrow |S(1-u_n)|^2=\left(a_n^2+b_n^2\right)|S(u_n)|^2+a_nb_n\left(S^2(u_n)+\overline{S^2(u_n)}\right)$
$\Rightarrow \rho_n^2\,|S(1-u_n)|^2=\left(a_n^2\rho_n^2+b_n^2\rho_n^2\right)|S(u_n)|^2+a_nb_n\rho_n^2\left(S^2(u_n)+\overline{S^2(u_n)}\right)$
$\Rightarrow$
$|S(u_n)|^2=\left(a_n^2\rho_n^2+1+a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\right)|S(u_n)|^2+a_nb_n\rho_n^2\left(S^2(u_n)+\overline{S^2(u_n)}\right)$
$\Rightarrow 0=\left(2a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\right)|S(u_n)|^2+a_nb_n\rho_n^2\left(S^2(u_n)+\overline{S^2(u_n)}\right)$
$\Rightarrow a_n\rho_n\left[2\left(a_n\rho_n-\cos(\theta_n)\right)|S(u_n)|^2+b_n\rho_n\left(S^2(u_n)+\overline{S^2(u_n)}\right)\right]=0$
$\Rightarrow a_n\rho_n=0$ or $2\left[a_n\rho_n-\cos(\theta_n)\right]|S(u_n)|^2+b_n\rho_n\left[S^2(u_n)+\overline{S^2(u_n)}\right]=0$
$2\left[a_n\rho_n-\cos(\theta_n)\right]|S(u_n)|^2+b_n\rho_n\left[S^2(u_n)+\overline{S^2(u_n)}\right]=0\Longrightarrow$
$b_n\rho_n\left[S^2(u_n)+\overline{S^2(u_n)}\right]=-2\left[a_n\rho_n-\cos(\theta_n)\right]|S(u_n)|^2$
$\Rightarrow b_n^2\rho_n^2\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=4\left[a_n\rho_n-\cos(\theta_n)\right]^2|S(u_n)|^4$
$\Rightarrow \left(1+a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\right)\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=4\left[a_n\rho_n-\cos(\theta_n)\right]^2|S(u_n)|^4$
$\Rightarrow \left[\left(a_n\rho_n-\cos(\theta_n)\right)^2+\sin^2(\theta_n)\right]\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=4\left[a_n\rho_n-\cos(\theta_n)\right]^2|S(u_n)|^4$
$\Rightarrow$
$\sin^2(\theta_n)\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=\left[a_n\rho_n-\cos(\theta_n)\right]^2\left[4\,|S(u_n)|^4-\left(S^2(u_n)+\overline{S}^2(u_n)\right)^2\right]$
$\Rightarrow$

$\sin^2(\theta_n)\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=[a_n\rho_n-\cos(\theta_n)]^2\left[2|S(u_n)|^4-S^4(u_n)-\overline{S}^4(u_n)\right]$
$\Rightarrow \sin^2(\theta_n)\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=-[a_n\rho_n-\cos(\theta_n)]^2\left[S^2(u_n)-\overline{S^2(u_n)}\right]^2$
$\Rightarrow [i\sin(\theta_n)]^2\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=[a_n\rho_n-\cos(\theta_n)]^2\left[S^2(u_n)-\overline{S^2(u_n)}\right]^2$
$\Rightarrow$

$$[a_n\rho_n-\cos(\theta_n)]\left[S^2(u_n)-\overline{S^2(u_n)}\right]=\pm i\sin(\theta_n)\left[S^2(u_n)+\overline{S^2(u_n)}\right]$$

$\Rightarrow$
$[a_n\rho_n-\cos(\theta_n)\mp i\sin(\theta_n)]\,S^2(u_n)=[a_n\rho_n-\cos(\theta_n)\pm i\sin(\theta_n)]\,\overline{S^2(u_n)}=Z$
as $Z=\overline{Z}$ so $Z=[a_n\rho_n-\cos(\theta_n)\mp i\sin(\theta_n)]\,S^2(u_n)\in\mathbb{R}$
$\Rightarrow$

$$\left(a_n\rho_n-e^{\pm i\theta_n}\right)S^2(u_n)\in\mathbb{R}$$

$\Rightarrow$

$$S^2(u_n)=K_n\left(a_n\rho_n-e^{\mp i\theta_n}\right)$$

with $K_n\in\mathbb{R}$
$\Rightarrow$

$$\operatorname{Im}\left[S^2(u_n)\right]=\pm K_n\sin(\theta_n)$$

since $\lim u_n=s_1$ & $S^2(s_1)\in IR^*$ so
$\lim[\sin(\theta_n)]=\sin(\theta(s_1))=0$
$\Rightarrow$

$$\theta(s_1)\equiv 0\ \ [\pi]$$

as $S(s)=\varphi(s)\,S(1-s)$ & $\varphi(s)=\rho(s)\,e^{i\theta(s)}$
$\Rightarrow S(s_1)=\rho(s_1)\,e^{i\theta(s_1)}S(1-s_1)$
$\Rightarrow S^2(s_1)=\rho^2(s_1)\,e^{2i\theta(s_1)}S^2(1-s_1)$
$\theta(s_1)\equiv 0\ \ [\pi]\Rightarrow S^2(s_1)=\rho^2(s_1)\,S^2(1-s_1)$
so
$S^2(s)\in IR\Leftrightarrow S^2(1-s)\in IR$
Another proof :
If $\lim a_n=a$ and $\lim b_n=b$, and as we have

$$b_nS^2(1-u_n)=|S(1-u_n)|^2\left[\varphi(1-u_n)-a_n\rho_n^2\varphi^2(1-u_n)\right]$$

with $\rho_n=|\varphi(u_n)|$
where $n\longrightarrow+\infty$ we would have
$bS^2(1-s_1)=|S(1-s_1)|^2\left[\varphi(1-s_1)-a\overline{\varphi(s_1)}\varphi(1-s_1)\right]$
$\Rightarrow$

$$bS^2(1-s_1)=|S(1-s_1)|^2\left[\varphi(1-s_1)-a\rho^2\varphi^2(1-s_1)\right]$$

with $S(s_1)=\varphi(s_1)\,S(1-s_1)=\rho e^{i\theta(s_1)}S(1-s_1)$
$\Rightarrow S^2(1-s_1)=\rho^{-2}e^{-i2\theta(s_1)}S^2(s_1)$ (or $\rho=0$)
$(S(s_1)\neq 0\Leftrightarrow\rho\neq 0)$

$$\Rightarrow\begin{cases} b\rho^{-2}e^{-i2\theta(s_1)}S^2(s_1)=|S(1-s_1)|^2\left[\varphi(1-s_1)-a\rho^2\varphi^2(1-s_1)\right]\\ \text{or}\\ S^2(s_1)=S^2(1-s_1)=0\end{cases}$$

Moreover
$S^2(1-s_1)=\rho^{-2}e^{-i2\theta(s_1)}S^2(s_1)$ and $S^2(s_1)\in IR\Rightarrow|S(1-s_1)|^2=\pm\rho^{-2}S^2(s_1)$
$\Rightarrow\pm be^{-i2\theta(s_1)}=\varphi(1-s_1)-a\rho^2\varphi^2(1-s_1)$ or $S^2(s_1)=S^2(1-s_1)=0$
$\varphi(s_1)=\rho e^{i\theta(s_1)}$ and $\varphi(s_1)\,\varphi(1-s_1)=1$
$\Rightarrow\pm b\rho^2=\varphi(s_1)-a\rho^2$ or $S^2(s_1)=S^2(1-s_1)=0$

$\Rightarrow (a \pm b)\,\rho^2 = \varphi(s_1)$ or $S^2(s_1) = S^2(1-s_1) = 0$
$(a \pm b)\,\rho^2 = \varphi(s_1) = \rho e^{i\theta(s_1)} \Rightarrow$

$$(|a \pm b|\,\rho = 1 \text{ and } \theta(s_1) \equiv 0\,[\pi]) \text{ or } \rho = 0$$

$\Rightarrow S^2(1-s_1) = \rho^{-2} e^{-i2\theta(s_1)} S^2(s_1) = \rho^{-2} S^2(s_1) \in IR$ or $S(s_1) = 0$
Conclusion: $S^2(s_1) \in IR \Rightarrow S^2(1-s_1) \in IR$ or $S^2(s_1) = S^2(1-s_1) = 0$ □

Remark 3. *If $s \in IR$, $S(s) = \sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^s} \in IR$ and $S(1-s) = \sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^{1-s}} \in IR$.*

Lemma 3. *Let $D = \{z \in \mathbb{C} / \operatorname{Re}(z) \in]0,1[\}$, so $\forall s \in D$*

$$S^2(s) \in IR \Leftrightarrow S^2(1-s) \in IR$$

Proof. Since $D - D_1 = \left\{z \in \mathbb{C} / \operatorname{Re}(z) = \frac{1}{2} \text{ and } \operatorname{Im}(z) \neq 0\right\}$
$\operatorname{Re}(s) = \frac{1}{2} \Rightarrow 1 - s = \overline{s}$
so $S(1-s) = S(\overline{s}) = \overline{S(s)}$
and $S^2(s) \in IR \Leftrightarrow S^2(1-s) \in IR$ □

Claim 1. *Let $D = \{z \in \mathbb{C} / \operatorname{Re}(z) \in]0,1[\}$, so $\forall s \in D$*

$$\begin{aligned} S(s) &\in IR \Leftrightarrow S(1-s) \in IR \\ S(s) &\in iIR \Leftrightarrow S(1-s) \in iIR \end{aligned}$$

Proof. $S(s) \in IR$ or $S(s) \in iIR \Rightarrow S^2(s) \in IR \Rightarrow \theta(s) \equiv 0\,[\pi]$
$S(s) = \varphi(s)\, S(1-s) = \rho e^{i\theta(s)} S(1-s_1) = \pm\rho S(1-s_1)$ □

3.3. **Other results.** Let be $S = S(s)$ and $S' = S(1-s)$such $S^2(s) \in IR$
so $C_1 = \frac{e^{-i\alpha}}{2^r}\zeta$, $C_2 = \left(1 - \frac{e^{-i\alpha}}{2^r}\right)\zeta$ and $S = \left(2^{1-r}e^{-i\alpha} - 1\right)\zeta$
with $C_2 = \zeta - C_1$, $S = C_1 - C_2$
&
$C_1' = \frac{e^{i\alpha}}{2^{1-r}}\zeta'$, $C_2' = \left(1 - \frac{e^{i\alpha}}{2^{1-r}}\right)\zeta'$ and $S' = \left(2^r e^{i\alpha} - 1\right)\zeta'$
with $C_2' = \zeta' - C_1'$, $S' = C_1' - C_2'$
as $1 - s = r' + ic' = 1 - r - ic$, $\alpha = \ln(2)\, c \Rightarrow r' = 1 - r$, $c' = -c \Rightarrow r' = 1 - r$, $\alpha' = -\alpha$

Remark 4. *Let be $S = S(s)$ such $S^2(s) \in IR$, so*
1) $2C_1C_1' = \zeta\zeta'$
2) $C_1\overline{\zeta}' = 2^{1-2r}\zeta\overline{C}_1'$
3) $2C_1\overline{C}_1' = e^{-i2\alpha}\zeta\overline{\zeta}'$
4) $2C_1C_2' = S\zeta'$
5) $2C_2C_2' = SS' \in IR$
6) $SC_1' = \zeta C_2'$ (& $S'C_1 = \zeta' C_2$)

Proof. 1) $C_1C_1' = \frac{e^{-i\alpha}}{2^r}\zeta \frac{e^{i\alpha}}{2^{1-r}}\zeta' = \frac{\zeta\zeta'}{2}$
2) $C_1' = \frac{e^{i\alpha}}{2^{1-r}}\zeta' \Rightarrow \zeta' = 2^{1-r}e^{-i\alpha}C_1'$
$C_1\overline{\zeta}' = \left(\frac{e^{-i\alpha}}{2^r}\zeta\right)\left(\overline{2^{1-r}e^{-i\alpha}C_1'}\right)$
$C_1\overline{\zeta}' = \left(\frac{e^{-i\alpha}}{2^r}\zeta\right)\left(2^{1-r}e^{i\alpha}\overline{C_1'}\right) = 2^{1-2r}\zeta\overline{C}_1'$
3) $2C_1\overline{C}_1' = 2\left(\frac{e^{-i\alpha}}{2^r}\zeta\right)\left(\overline{\frac{e^{i\alpha}}{2^{1-r}}\zeta'}\right) = 2\frac{e^{-i\alpha}}{2^r}\zeta\frac{e^{-i\alpha}}{2^{1-r}}\overline{\zeta'}$

$2C_1\overline{C}_1' = e^{-i2\alpha}\zeta\overline{\zeta}'$
4) If $S \in IR$ we have $S = 2C_1 - \zeta = 2\overline{C}_1 - \overline{\zeta} = \overline{S} \in IR$
$\Longrightarrow 2C_1 + \overline{\zeta} = 2\overline{C}_1 + \zeta$
$\Longrightarrow 2C_1'\zeta + \overline{\zeta}\zeta' = 2\overline{C}_1'\zeta + \zeta\zeta'$
$\Longrightarrow 2C_1'\zeta + \overline{\zeta}\zeta' = 2\overline{C}_1'\zeta + 2C_1C_1'$
$\Longrightarrow 2C_1'\zeta - 2C_1C_1' = 2\overline{C}_1'\zeta - \overline{\zeta}\zeta'$
$\Longrightarrow 2C_1\left(\zeta' - C_1'\right) = \left(2\overline{C}_1 - \overline{\zeta}\right)\zeta'$
$\Longrightarrow 2C_1C_2' = \overline{S}\zeta' = S\zeta'$.
If $S \in iIR$ we have $S = 2C_1 - \zeta = \overline{\zeta} - 2\overline{C}_1 = -\overline{S} \in iIR$
$\Longrightarrow 2C_1 - \overline{\zeta} = -2\overline{C}_1 + \zeta$
$\Longrightarrow 2C_1\zeta' - \overline{\zeta}\zeta' = -2\overline{C}_1\zeta' + \zeta\zeta'$
$\Longrightarrow 2C_1\zeta' - \overline{\zeta}\zeta' = -2\overline{C}_1\zeta' + 2C_1C_1'$
$\Longrightarrow 2C_1\zeta' - 2C_1C_1' = -2\overline{C}_1'\zeta + \overline{\zeta}\zeta'$
$\Longrightarrow 2C_1\left(\zeta' - C_1'\right) = \left(-2\overline{C}_1 + \overline{\zeta}\right)\zeta'$
$\Longrightarrow 2C_1C_2' = -\overline{S}\zeta' = S\zeta'$.
5) $2C_1C_2' = S\zeta' \Rightarrow 2\left(S + C_2\right)C_2' = S\zeta'$
$\Rightarrow 2SC_2' + 2C_2C_2' = S\zeta' \Rightarrow S\left(\zeta' - S'\right) + 2C_2C_2' = S\zeta'$
$\Rightarrow S\zeta' - SS' + 2C_2C_2' = S\zeta'$
$\Rightarrow SS' = 2C_2C_2'$.
6) is 4)+5) □

Lemma 4.

$$S^2\left(s\right) \in IR \Rightarrow C_2\left(s\right)C_2\left(1-s\right) \in IR$$

Proof. Since the last Claim $\forall s \in D = \{z \in \mathbb{C}/\operatorname{Re}(z) \in]0,1[\}$

$$\begin{aligned} S\left(s\right) &\in IR \Leftrightarrow S\left(1-s\right) \in IR \\ S\left(s\right) &\in iIR \Leftrightarrow S\left(1-s\right) \in iIR \end{aligned}$$

$\Longrightarrow SS' = S\left(s\right)S\left(1-s\right) \in IR$
and from the last Remark 5)
$2C_2C_2' = SS' \in IR$. □

3.4. The second announcement.

Claim 2.

$$\exists s_0/S\left(s_0\right) \in iIR \Longleftrightarrow \exists s_1 \in \left(r_0, s_0\right]/S\left(s_1\right) = 0$$

such $r_0 = \operatorname{Re}\left(s_0\right) \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$ *and* $\left(r_0, s_0\right] = \{r_0 + ic \in \mathbb{C}/0 \prec c \leq c_0\}$

Proof. Assuming that
$\exists s' = r' + ic'$ such $S\left(s'\right) \in iIR^*$
with $r' \in \left]0, \frac{1}{2}\right[$
Let

$$c_0 = \min\left\{c \in IR^+/\exists n \in IN^*, S^n\left(r' + ic\right) \in iIR^*\right\} \qquad (*)$$

so $\exists m \in IN^*, S^m\left(s_0\right) \in iIR^*$ with $s_0 = r' + ic_0$ $\left(c_0 \leq c'\right)$
$\Rightarrow S^{2m}\left(s_0\right) \in IR_*^-$
without forgetting $S^{2m}\left(r'\right) \in IR_*^+$,
since

$$S\left(r\right) = \sum_{n\geq 1} \frac{\left(-1\right)^n}{n^r} \in IR, \ \forall r \in IR_*^+$$

Let now $S^{2m}(s) = R(s) + iI(s)$
we have $R(s_0) \prec 0$ and $R(r') \succ 0$, so
$\exists s_1 = r' + ic_1 \in (r', s_0)$ such $R(s_1) = 0$
$\Rightarrow S^{2m}(s_1) \in iIR$ with $0 \prec c_1 \prec c_0$
$0 \prec c_1 \prec c_0 \Rightarrow S^{2m}(s_1) \notin iIR_*$ (since $(*)$)

$$\begin{aligned} S^{2m}(s_1) &\in iIR \,\&\, S^{2m}(s_1) \notin iIR_* \Rightarrow S^{2m}(s_1) = 0 \\ &\Rightarrow S(s_1) = 0 \end{aligned}$$

Conclusions:
- $\exists s_0$ such $S(s_0) \in iIR \Rightarrow S(s_0) \in iIR^*$ or $S(s_0) = 0 \Rightarrow \exists s_1 \in (r_0, s_0]$ such $S(s_1) = 0$.
- The other implication is obvious:
$\exists s_1 \in (r_0, s_0]$ such $S(s_1) = 0 \Rightarrow \exists s_1 \in (r_0, s_1]$ such $S(s_1) = 0$
with $s_1 = s_0$ $S(s_0) = S(s_1) = 0 \Rightarrow \exists s_0$ such $S(s_0) \in iIR$. □

4. News contributions

Corollary 1.

$$\begin{aligned} (\exists m_0 \,\&\, \exists s_0 \text{ such } S^{m_0}(s_0) &\in iIR) \Longleftrightarrow \exists s_1 \in (r_0, s_0] \text{ such } S(s_1) = 0 \\ &\Longleftrightarrow \exists c \in \left]0, c_0\right] \text{ such } S(r_0 + ic) = 0 \end{aligned}$$

such $r_0 = \mathrm{Re}(s_0) \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$ *and* $(r_0, s_0] = \{r_0 + ic \in \mathbb{C}/0 \prec c \leq c_0\}$

Proof. Same proof as the previous one. □

Conjecture 1. $\forall r \in]0, 1[$ *and* $s = r + ic$

$$\begin{aligned} r &\neq \frac{1}{2} \Rightarrow \mathrm{Re}[S(s)] \neq 0 \\ \mathrm{Re}[S(s)] &= 0 \Longrightarrow r = \frac{1}{2} \\ \text{so } \sum_{n=1}^{+\infty} (-1)^n \frac{\cos[\ln(n)\, c]}{n^r} &= 0 \Longrightarrow r = \frac{1}{2} \end{aligned}$$

Proof. $\mathrm{Re}[S(s)] = 0 \Longrightarrow S(s) \in iIR \Longrightarrow \exists s_1 \in (r, s]$ such $S(s_1) = 0$ (according to the last Claim)
and according to the Riemann hypothesis:
$S(s_1) = 0 \Longrightarrow \mathrm{Re}(s_1) = \frac{1}{2}$
$s_1 \in (r, s] \Longrightarrow s_1 = r + ic_1$ such $0 \prec c_1 \leq c$
since $s_1 \in (r, s]$ & $s = r + ic$
$\Longrightarrow \mathrm{Re}(s_1) = r = \frac{1}{2}$. □

Conjecture 2.

$$\left(r = \frac{1}{2} \,\&\, \theta\left(\frac{1}{2} + ic\right) = (2k+1)\pi \right) \Longrightarrow S\left(\frac{1}{2} + ic\right) = 0$$

Proof. Assuming that
$\exists s = \frac{1}{2} + ic$ such $\theta\left(\frac{1}{2} + ic\right) = (2k+1)\pi$ & $S\left(\frac{1}{2} + ic\right) \neq 0$
$S(s) = \varphi(s) S(1-s)$, $1 - s = \overline{s}$, $\varphi(s) = \rho e^{i\theta(s)} = -1$
$\Longrightarrow S(s) = -\overline{S(s)}$
$\Longrightarrow S(s) \in iIR^*$
$\Longrightarrow S^2(s) \in IR_*^-$

$\forall r \in \left]0, \frac{1}{2}\right[, \; S(r) \in IR^* \Longrightarrow S^2(r) \in IR_*^+$
Let now $S^2 = R + iI$
we have $R(s) \prec 0$ and $R(r) \succ 0$, so
$\exists s_0 = r_0 + ic_0 \in (r, s)$ such $R(s_0) = 0$
$\Longrightarrow \exists s_0 = r_0 + ic_0$ such $r_0 \in \left]0, \frac{1}{2}\right[$ & $S^2(s_0) \in iIR$
since $r_0 + ic_0 \in \left(r, \frac{1}{2} + ic\right) \Longrightarrow r < r_0 < \frac{1}{2}$
we have seen in the last corollary that
$(\exists m_0$ & $\exists s_0$ such $S^{m_0}(s_0) \in iIR) \Longrightarrow \exists s_1 \in (r_0, s_0] / S(s_1) = 0$
with $m_0 = 2$ & $s_0 = r_0 + ic_0$
$s_1 \in (r_0, s_0] \Longrightarrow \mathrm{Re}(s_1) = r_0 \in \left]0, \frac{1}{2}\right[$
Absurd according to the Riemann hypothesis.
$\Longrightarrow S\left(\frac{1}{2} + ic\right) = 0.$ □

5. Conclusions

We have two new conjectures based on the Riemann hypothesis, and so this is a new way to see if this hypothesis is correct, and if not, we also have a useful new method for determining a counterexample.

The first conjecture announced: In the band s $(s = r + ic)$ a complex such that its real part is strictly between 0 and 1 $(0 < r < 1)$, we have the real part of the Dirchlet function $(S(s))$ can only be zero in the straight line "the real part of s is equal to 0.5" $(r = 0, 5)$. So instead of studying and finding the zeros of the Dirchlet function $(S(s)$ or $\eta(s))$, we just need to study its real part which is equivalent to studying and finding the zeros of the real function $R' = \sum_{n=1}^{+\infty} (-1)^n \frac{\cos[\ln(n)c]}{n^r}$ with two variables (r, c).

While the second conjecture informs us about what the zeros can be $(\theta\left(\frac{1}{2} + ic\right) = (2k+1)\pi)$ in the straight line $r = 0, 5$.

Competing Interests
As the sole author of this work, I declare that I have no conflict of interest with anyone.

Authors' Contributions
I am the sole author and contributor to the writing of this article. I have read and approved the final manuscript.

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